Hyperbolic analogues of fullerenes with face-types $(6,9)$ and $(6,10)$

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Abstract

Mathematical models of fullerenes are cubic polyhedral and spherical maps of face-type $(5,6)$, that is, with pentagonal and hexagonal faces only. Any such map necessarily contains exactly 12 pentagons, and it is known that for any integer $\alpha \geq 0$ except $\alpha = 1$ there exists a fullerene map with precisely α hexagons.

In this paper we consider hyperbolic analogues of fullerenes, modelled by cubic polyhedral maps of face-type $(6,k)$, where $k \in \{9,10\}$, on orientable surface of genus at least two. The number of k-gons in this case depends on the genus but the number of hexagons is again independent of the surface. For every triple $k \in \{9,10\}$, $g \geq 2$ and $\alpha \geq 0$, we determine if there exists a cubic polyhedral map of face-type $(6,k)$ with exactly α hexagons on an orientable surface of genus g. The only unsolved cases are $k = 10$, $g = 5$ and $\alpha \leq 3$ when we are not able to say if a hyperbolic fullerene with these parameters exists.

1 Introduction

Fullerenes are carbon-cage molecules in which every atom is connected by bonds to exactly three next atoms. The well-known Buckminster fullerene C_{60} was found by Kroto et al. [14], and later confirmed by experiments by Krätchmer et al. [13] and Taylor et al. [18]. Since the discovery of C_{60}, fullerenes have attracted considerable interest of scientists all over the world, see e.g. [2, 4, 7, 15, 16].

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If one replaces the carbon atoms by vertices, bonds by edges, and fills the smallest cycles (of lengths 5 and 6) by 2-cells, then a fullerene is turned into a spherical embedding of cubic 3-connected graph, with faces bounded by cycles of lengths 5 and 6. Hence, mathematical model of a fullerene is a cubic, spherical and polyhedral map of face-type $(5, 6)$.

In this paper we study mathematical models of fullerene analogues on orientable surfaces of higher genera. By a ***hyperbolic k-gonal fullerene*** we understand any trivalent polyhedral map on some orientable surface of genus at least two, with all faces bounded by cycles of length 6 or k for some fixed $k \geq 7$, that is of ***face-type*** $(6, k)$, see [3]. The ***genus*** of the k-gonal fullerene is simply the genus of its supporting surface. Analogues of fullerenes embedded on hyperbolic surfaces have been considered earlier by a number of authors, see e.g. [5], [19] or [20] and references therein. Constructions of higher genus fullerenes with additional symmetries have been suggested in [12].

Denote by α and β the number of hexagonal and k-gonal faces, respectively, in a hyperbolic k-gonal fullerene of genus g. By Euler's formula, we have

$$\beta = 12(g-1)/(k-6), \tag{1}$$

but there is no restriction on α. Hence, we have the following problem:

Problem 1. *For every $k \geq 7$ and $g \geq 2$, characterize all α's such that there exists a hyperbolic k-gonal fullerene of genus g with exactly α hexagonal faces.*

We remark that the necessary conditions for existence of a hyperbolic k-gonal fullerene of genus $g \geq 2$ are also sufficient for large enough α, see [10]. However, it seems to be impossible to determine the corresponding bound for α just using the results of [10], and moreover, the smallest values of α are the most interesting.

As regards the analogue of Problem 1 for classical fullerenes, that is for cubic, spherical and polyhedral maps of face-type $(5, 6)$, it is well-known that (mathematical models of) these fullerenes with precisely α hexagonal faces exist for all non-negative values of α with the sole exception of $\alpha = 1$, see [6, section 13.4]. In [3], Problem 1 is solved for the cases $k = 7$ and 8 and all $g \geq 2$:

Theorem 2. *If $k \in \{7, 8\}$, $g \geq 2$ and $\alpha \geq 0$, then there exists a hyperbolic k-gonal fullerene of genus g with exactly α hexagonal faces, except for $k = 8$, $g = 2$ and $\alpha \leq 3$, where no such maps exist.*

(In fact, for $k = 8$, $g = 2$ and $\alpha = 3$, it is claimed in [3] that a corresponding hyperbolic fullerene exists, which is false. They claim that "the graph $K_9 - K_3$, the complete graph on nine vertices with three edges forming a triangle removed, can be embedded on the orientable surface with genus 3 due to Heffter [9]", while Heffter [9] only shows that K_9 has an embedding on S_3. When Youngs [21] refers to Heffter's work, it is to resolve Heawood problem for K_9, but not to show the existence of triangular embedding for $K_9 - K_3$. On the contrary, it was shown by Jungermann [11] (using computer program) that $K_9 - K_3$ has no orientable triangular embedding.)

In this paper we consider the next two values, namely $k = 9$ and 10. Analogous to the cases $k = 7$ and 8, we give a complete solution of Problem 1 for all $g \geq 2$ with the exception of cases $k = 10$, $g = 5$ and $\alpha \leq 3$, when we are not able to state if corresponding hyperbolic 10-gonal fullerenes exist.

We remark that values $k = 7$, 8, 9, 10 are ***universal*** in the sense that there is a trivalent polyhedral map of face-type $(6, k)$ for all genera $g \geq 2$. The remaining universal values are $k = 12$ and 18, see [3], and it will be interesting to investigate Problem 1 for these two values of k.

In the next section we present some preliminary results and a general construction for duals of hyperbolic k-gonal fullerenes when $k \in \{9, 10\}$. Section 3 is devoted to hyperbolic 9-gonal fullerenes, and Section 4 deals with hyperbolic 10-gonal fullerenes.

2 Preliminaries

A ***map*** is an embedding of a graph, possibly with loops or multiple edges, into a surface, such that every face is homeomorphic to an open 2-cell. A map is ***polyhedral*** if the following is true, see [17, Proposition 5.5.12] and the text below.

(p1) The underlying graph of the map is simple, that is, without loops and multiple edgs.

(p2) All facial walks are cycles, that is, no vertex appears more than once on the bondary of a face.

(p3) The intersection of any two faces is either empty, or it contains a unique vertex, or exactly two vertices and the edge joining them.

Let T be a map. If we shrink every face of T to a vertex and extend every vertex of T to a face, we obtain a ***dual map*** T^D. Then T and T^D have the same numbers of edges and every edge of T intersects exactly one edge of T^D and vice versa. By [17, Proposition 5.5.12 (b), (d)], a map is polyhedral if and only if its dual map is polyhedral.

Let T be a hyperbolic k-gonal fullerene of genus g and let T^D be its dual map. Then T^D is a polyhedral triangulation of orientable surface of genus g in which every vertex has degree either 6 or k. However, to check polyhedrality for T^D, it suffices to check (p1), as (p2) and (p3) follow. More precisely, (p2) is implied by the fact that the underlying graph for T^D does not contain loops, and (p3) is implied by the fact that this graph does not have multiple edges and the vertex degrees are greater than 2. Hence, we have the following proposition:

Proposition 3. *Let T be a triangulation of an orientable surface of genus $g \geq 2$ by a simple graph G, such that α vertices of G have degree 6 and the remaining vertices have degree k. Then the dual of T is a hyperbolic k-gonal fullerene of genus g with exactly α hexagonal faces.*

In our constructions, we do not construct hyperbolic k-gonal fullerenes directly, instead we construct their duals. By Proposition 3, this approach reasonably simplifies the check for polyhedrality. In all but finitely many cases we construct the required triangulations using triangulations of tori by 6-regular simple graphs.

For $k = 9$, take two toroidal triangulations by 6-regular simple graphs. In each of them, cut out two adjacent facial triangles together with the edge joining them. This leaves a 4-hole in each surface. These 4-holes are bounded by 4-cycles with four vertices having degrees 6 and 5 distributed alternatively around the cycle. Hence, if we glue these holes together ***properly***, that is, we identify the boundary cycles so that in every case a vertex of degree 6 will be identified with a vertex of degree 5 and we identify these cycles in the opposite way (so that the resulting surface is orientable when making more gluing of this type), we obtain a triangulation of an orientable surface in which the vertex degrees are 6 and 9 only, see Figure 1.

Figure 1: Gluing the holes to obtain a dual of hyperbolic 9-gonal fullerene.

For $k = 10$, the process is analogous but we cut out only one facial triangle from each of the two toroidal triangulations by 6-regular graphs. This leaves one 3-hole in each surface. Now if we glue these holes together ***properly***, that is, we identify the boundary cycles in the opposite way, we obtain a triangulation of an orientable surface in which the vertex degrees are 6 and 10 only.

Of course, there may be more vertex-disjoint $(13 - k)$-holes and more toroidal triangulations. We will glue these holes properly in pairs to obtain a single surface without a hole. The problem is that in general this process does not yield a triangulation by a simple graph. The following lemma will handle this issue.

Lemma 4. *Let $T_1, T_2, \dots, T_n$ be n triangulations of tori by 6-regular simple graphs, and $k \in \{9, 10\}$. Cut out in total $2t$ vertex-disjoint $(13-k)$-holes from $T_1, T_2, \dots, T_n$, and glue them in pairs properly to obtain a map T, on a single orientable surface where the vertices of the holes get degree k. Then T is a triangulation of an orientable surface of genus $t+1$ and the embedded graph has vertex degrees 6 and k only. Moreover, if the holes are glued so that for every pair T_i and T_j, $1 \le i < j \le n$, there is at most one hole in T_i and one in T_j which are glued together, and for every i', $1 \le i' \le n$, there is no pair of holes in $T_{i'}$ which are glued together, then the underlying graph of the resulting map is simple.*

Proof. In the first part of lemma it remains to show that the genus of the resulting surface S is $t+1$. Note that T has exactly $(13-k)t$ vertices of degree k while all the other vertices are of degree 6. Denote by α the number of vertices of degree 6 in T. Then T has $(13-k)t+\alpha$ vertices, $((13-k)tk+6\alpha)/2$ edges and $((13-k)tk+6\alpha)/3$ faces. So the Euler characteristics of S is

$$(13-k)t+\alpha+\frac{(13-k)tk+6\alpha}{3}-\frac{(13-k)tk+6\alpha}{2}=t(13-k)\frac{6-k}{6}=-2t,$$

where the last equation is true since $k \in \{9, 10\}$. Hence, the genus of S is $t+1$.

Now suppose that if two holes are glued together then they come from T_i and T_j, where $1 \le i < j \le n$, and for given i and j there is at most one pair of such holes. Observe that in the process of obtaining T from $T_1, T_2, \dots, T_n$, we did not add a single edge. We only identified some of them and some of the vertices. Moreover, $T_1, T_2, \dots, T_n$ are triangulations by simple graphs. Let G be the underlying graph of T. We show that G has neither loops nor multiple edges.

Suppose G has a loop. It means the end vertices of an edge from some T_i have been identified. But the vertices that are identified come from different T_i's, a contradiction.

Now suppose G has a multiple edge. Denote by u and v vertices such that there are two edges joining u with v in G. Since we did not add edges in the process of obtaining T from $T_1, T_2, \dots, T_n$, the two parallel edges must come from different edges which are already present in $T_1 \cup T_2 \cup \dots \cup T_n$. If these edges come both from one of $T_1, T_2, \dots, T_n$, say they come from T_i, then since the holes are vertex-disjoint, there must be two holes in T_i which are glued together, a contradiction. Thus, we may assume that one of these edges comes from T_i and the other comes from T_j, $1 \le i < j \le n$. Then u (v) is obtained by identifying two vertices, say u_1 and u_2 (v_1 and v_2), where u_1v_1 is an edge of T_i and u_2v_2 is an edge of T_j.

If both u_1 and v_1 are in one hole in T_i, then u_2 and v_2 are in one hole in T_j, since the holes are vertex-disjoint. But then either u_1v_1 is an edge bounding this hole which means that after the gluing process the edge u_1v_1 is identified with u_2v_2 and no parallel edges occur, or $k=9$ and u_1 and v_1 are opposite vertices in a 4-hole which means that in the process of cutting out holes one of u_1v_1 and u_2v_2 was deleted, see Figure 1. Thus we may assume that u_1 and v_1 are in two distinct holes in T_i. But then also u_2 and v_2 come from two distinct holes of T_j, which means that there are at least two holes in T_i which we glue with two holes in T_j, a contradiction. $\square$

Suppose that we provide the gluing process on n toroidal triangulations $T_1, T_2, \dots, T_n$ by 6-regular graphs. That is, we cut out d_i $(13-k)$-holes in T_i, $1 \le i \le n$, and glue them in pairs properly. Then the resulting map T is triangular with vertex degrees 6 and k only. Construct an auxiliary graph G_T with vertices $T_1, T_2, \dots, T_n$. Whenever there is a hole in T_i which is glued to a hole in T_j, add to G_T an edge connecting T_i with T_j. So if there are ℓ holes in T_i glued to ℓ holes in T_j, then we have ℓ parallel edges connecting T_i with T_j in G_T. The graph G_T is called an *associated graph* for T. Observe that the degree of T_i is d_i in G_T. If we denote by m the number of edges of G_T, then $m = \frac{1}{2}(d_1 + d_2 + \dots + d_n)$ and T is a map on an orientable surface of genus $m+1$, by Lemma 4.

In our constructions, it will be not important which hole of T_i is glued to which hole in T_j. Only the structure of G_T will be important. By Lemma 4, if G_T contains neither parallel edges nor loops, then the underlying graph for T is simple.

Let Γ be a group and let S be a set of elements that generates Γ and such that $a^{-1} \in S$ whenever $a \in S$. Then a *Cayley graph* $G = \mathrm{Cay}(\Gamma, S)$ has $V(G) = \Gamma$ and $xy \in E(G)$ if and only if $y = xa$ for some $a \in S$. A *Cayley map* is an embedding of a Cayley graph on a surface such that every face is homeomorphic to an open disc and there is an ordering $(a_1, a_2, \dots, a_s)$ of the s elements of S such that when traversing around $x \in V(G)$ on a small circle, we intersect edges with one vertex x and the other $(xa_1, xa_2, \dots, xa_s)$, always in this circular order. To simplify the notation, we denote the Cayley map as $\mathrm{Cay}(\Gamma, (a_1, a_2, \dots, a_s))$.

In our proof, we use Cayley maps to give construction for fullerenes. In a few cases, for small values of α, we have used the program `CGF` ([8], available from [1]) that allows to enumerate maps, polyhedral or not, of a fixed genus with a fixed combination of face sizes.

3 Hyperbolic fullerenes of face-type $(6, 9)$

Theorem 5. *If $g = 2$ and $\alpha \le 5$ or $g = 3$ and $\alpha \le 2$, then there are no hyperbolic 9-gonal fullerenes of genus g having exactly α hexagonal faces. On the other hand, if $g = 2$ and $\alpha \ge 6$ or $g = 3$ and $\alpha \ge 3$ or $g \ge 4$ and $\alpha \ge 0$, then there exists a hyerbolic 9-gonal fullerene of genus g having exactly α hexagonal faces.*

Proof. First we prove the negative results. If $g = 2$ then a (hyperbolic) 9-gonal fullerene has exactly four 9-gonal faces, by (1). By polyhedral properties (p2 and p3), every 9-gon is bounded by another nine distinct faces, and so the number of hexagonal faces satisfies $\alpha \ge 6$. Analogously, if $g = 3$ then a nonagonal fullerene has exactly eight 9-gons and so $\alpha \ge 2$.

Now suppose that $g = 3$ and $\alpha = 2$. If there exists a required 9-gonal fullerene, then it contains $(8 \cdot 9 + 2 \cdot 6)/2 = 42$ edges. Since there are $\binom{8}{2} = 28$ pairs of 9-gons and each pair

1	:	2 3 4	7	:	3 15 11	13	:	6 21 22	19	:	10 29 30	25	:	15 24 16
2	:	1 5 6	8	:	3 16 17	14	:	6 23 24	20	:	12 28 27	26	:	15 23 22
3	:	1 7 8	9	:	4 12 18	15	:	7 25 26	21	:	13 30 17	27	:	16 30 20
4	:	1 9 10	10	:	4 19 11	16	:	8 27 25	22	:	13 18 26	28	:	17 29 20
5	:	2 11 12	11	:	5 10 7	17	:	8 21 28	23	:	14 26 29	29	:	19 23 28
6	:	2 13 14	12	:	5 9 20	18	:	9 22 24	24	:	14 18 25	30	:	19 21 27

Table 1: A rotation scheme for a hyperbolic 9-gonal fullerene of genus 2 with $\alpha = 3$.

shares at most one edge, there are at least $42 - 28 = 14$ edges which are bounding hexagonal faces. But since $\alpha = 2$, the number of edges bounding hexagonal faces cannot exceed 12, a contradiction. Hence, there is no 9-gonal fullerene of genus 3 with α hexagons if $\alpha \le 2$.

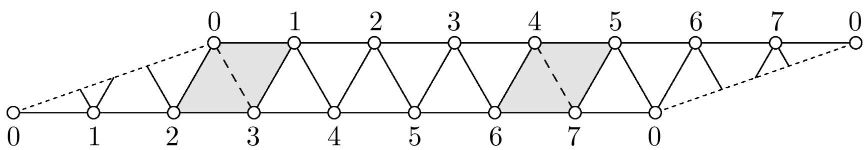

Figure 2: The graph M_8 with two 4-holes $[0, 1, 3, 2]$ and $[4, 5, 7, 6]$.

Now we construct the required hyperbolic fullerenes. We use a triangulation of torus by Cayley map $M_n = \mathrm{Cay}(\mathbb{Z}_n, (1, 3, 2, -1, -3, -2))$, $n \ge 7$, see Figure 2 for M_8. Another way of obtaining this triangulation is to cut out a parallelogram from a plane tesselation by regular triangles and join opposite sides to obtain a torus. It is obvious that the underlying graph for M_n is simple for $n \ge 7$. In M_n we cut out several 4-holes. A hole containing edges ab, bc, cd, da, from which we cut out the edge ac, will be denoted by $[a, b, c, d]$, see Figure 2 for the 4-holes $[0, 1, 3, 2]$ and $[4, 5, 7, 6]$.

We distinguish three cases.

Case 1: $g = 2$ *and* $\alpha \ge 6$. Take two maps, M_7 and $M_{\alpha+1}$, and cut out the 4-hole $[0, 1, 3, 2]$ in each. Then glue these holes properly. By Lemma 4, the resulting map T triangulates orientable surface of genus 2 and has α vertices of degree 6 while the remaining vertices have degree 9. Since the underlying graphs for both M_7 and $M_{\alpha+1}$ are simple (recall that $\alpha \ge 6$) and the associated graph for T is a path of length 1, the underlying graph for T is simple, by Lemma 4. Hence, the dual of T is a 9-gonal fullerene of genus 2 with exactly α hexagons, by Proposition 3.

Case 2: $g = 3$ *and* $\alpha \ge 3$. In the case $\alpha = 3$, the `CGF` program ([1]) found three polyhedral maps. A rotation scheme for one of them is given in Table 1.

Now suppose that $\alpha \ge 4$. Take two maps, M_8 and $M_{\alpha+8}$. To distinguish the vertices, denote those of $M_{\alpha+8}$ by $\{0', 1', \ldots, (\alpha+7)'\}$. Cut out 4-holes $[0, 1, 3, 2]$ and $[4, 5, 7, 6]$ from M_8, and $[0', 1', 3', 2']$ and $[6', 7', 9', 8']$ from $M_{\alpha+8}$. Then glue pairs of these holes properly so that the following identifications are made: $0 \equiv 2'$, $1 \equiv 3'$, $3 \equiv 1'$, $2 \equiv 0'$, $4 \equiv 7'$, $5 \equiv 6'$, $7 \equiv 8'$ and $6 \equiv 9'$. By Lemma 4, the resulting map T triangulates an orientable surface of genus 3 and has α vertices of degree 6 while the remaining vertices have degree 9. Since in $M_{\alpha+8}$ the only possible edges between the vertices of the holes are $(3', 6')$ and $(0', 9')$, and since neither $(1, 5)$ nor $(2, 6)$ is an edge in M_8, the underlying graph for T is simple. By Proposition 3, the dual of T is a 9-gonal fullerene of genus 3 with exactly α hexagons.

Case 3: $g \geq 4$ *and* $\alpha \geq 0$. Take $g-2$ copies of M_8 and one copy of $M_{\alpha+8}$. Then in each of the triangulations cut out two 4-holes, namely $[0, 1, 3, 2]$ and $[4, 5, 7, 6]$. Glue these holes in pairs properly so that the associated graph is a $(g-1)$-cycle and denote the resulting map by T. By Lemma 4, T is a triangulation of an orientable surface of genus g by a simple graph, in which there are α vertices of degree 6 while the remaining vertices have degree 9. Hence, the dual of T is a 9-gonal fullerene of genus g with exactly α hexagons, by Proposition 3. $\square$

4 Hyperbolic fullerenes of face-type $(6, 10)$

Lemma 6. *If $g = 2$ and $\alpha \leq 7$ or $g = 3$ and $\alpha \leq 6$ or $g = 4$ and $\alpha \leq 2$, then there are no hyperbolic 10-gonal fullerenes of genus g with exactly α hexagonal faces.*

Proof. We prove the statement for the dual embeddings. Let G be a simple graph with vertices of degree 6 or 10 which triangulates an orientable surface of genus g. Denote by α the number of vertices of degree 6 in G.

If $g = 2$, then G has 3 vertices of degree 10, see (1). Thus every vertex of degree 10 in G must be adjacent to at least 8 vertices of degree 6, and so $\alpha \geq 8$.

If $g = 3$, then G has 6 vertices of degree 10. Thus in G, every vertex of degree 10 must be adjacent to at least 5 vertices of degree 6, and hence $\alpha \geq 5$. Suppose $\alpha = 5$. This implies that in G, every vertex of degree 10 is adjacent to every vertex of degree 6, and so G must be isomorphic to $K_{11} - K_5$. But it is known that $K_{11} - K_5$ has no orientable triangular embedding, see [11]. It only remains to show that there is no hyperbolic 10-gonal fullerene of genus 3 with exactly 6 hexagons. The `CGF` program ([1]) exhausted all the possibilities, and showed that there is no required 10-gonal fullerene.

If $g = 4$, then there are 9 vertices of degree 10 in G. Thus in G, every vertex of degree 10 must be adjacent to at least two vertices of degree 6, and hence there are at least 9×2 edges in G that connect a vertex of degree 10 to a vertex of degree 6. This implies that there must be at least 3 vertices of degree 6 in G. $\square$

Theorem 7. *If $g = 2$ and $\alpha \geq 8$ or $g = 3$ and $\alpha \geq 7$ or $g = 4$ and $\alpha \geq 3$ or $g = 5$ and $\alpha \geq 4$ or $g \geq 6$ and $\alpha \geq 0$, then there exists a hyperbolic 10-gonal fullerene of genus g with exactly α hexagonal faces.*

Proof. In the constructions we use the Cayley map $M_n = \mathrm{Cay}(\mathbb{Z}_n, (1, 3, 2, -1, -3, -2))$ analogously as in the proof of Theorem 5.

Case 1: $g = 2$ *and* $\alpha \geq 8$. Take two maps, M_7 and $M_{\alpha-1}$, and cut out the 3-hole $[0, 1, 3]$ in each. Then glue these holes properly and denote the resulting map by T. By Lemma 4, T

triangulates an orientable surface of genus 2 and has α vertices of degree 6 while the remaining vertices have degree 10. Since the underlying graphs for both M_7 and $M_{\alpha+1}$ are simple and the associated graph for T is a path of length 1, the underlying graph for T is simple, by Lemma 4. Hence, the dual of T is a 10-gonal fullerene of genus 2 with exactly α hexagons, by Proposition 3.

Case 2: *$g = 3$ and $\alpha \geq 7$.* Take two maps, M_8 and $M_{\alpha+4}$. To distinguish the vertices, denote those of $M_{\alpha+4}$ by $\{0', 1', \ldots, (\alpha{+}3)'\}$. Cut out the 3-holes $[0, 1, 3]$ and $[4, 5, 7]$ from M_8, and $[0', 1', 3']$ and $[8', 7', 5']$ from $M_{\alpha+4}$. Then glue pairs of these holes properly so that the following identifications are made: $0 \equiv 0'$, $1 \equiv 3'$, $3 \equiv 1'$, $4 \equiv 8'$, $5 \equiv 5'$ and $7 \equiv 7'$. The resulting map T triangulates an orientable surface of genus 3 and has α vertices of degree 6 while the remaining vertices have degree 10. Since in $M_{\alpha+4}$ the only edges between the vertices of the holes are $(3', 5')$ and $(0', 8')$ (the latter only in the case $\alpha = 7$), and since neither $(1, 5)$ nor $(0, 4)$ is an edge in M_8, the underlying graph for T is simple. By Proposition 3, the dual of T is a 10-gonal fullerene of genus 3 with exactly α hexagons.

Case 3: *$g \in \{4, 5\}$ and $\alpha \geq g - 1$.* Take $g - 2$ copies of M_7 and one copy of $M_{\alpha+8-g}$. Then in each of the triangulations cut out two 3-holes, namely $[0, 1, 3]$ and $[2, 5, 4]$. Glue these holes in pairs properly so that the associated graph is a $(g-1)$-cycle and denote the resulting map by T. By Lemma 4, T is a triangulation of an orientable surface of genus g by a simple graph in which there are α vertices of degree 6 while the degrees of the remaining vertices are 10. Hence, the dual of T is a 10-gonal fullerene of genus g with exactly α hexagons, by Proposition 3.

Case 4: $g \geq 6$ and $\alpha \geq 0$. Take $M = M_{6(g-1)+\alpha}$ and cut out $2(g{-}1)$ 3-holes $[0, 1, 3]$, $[2, 5, 4]$, $[6, 7, 9]$, $[8, 11, 10], \ldots, [6(g{-}2), 6(g{-}2) + 1, 6(g{-}2) + 3]$, $[6(g-2) + 2, 6(g{-}2) + 5, 6(g{-}2) + 4]$. Let S be the set of vertices of these 3-holes. By Lemma 4, if we glue the holes in pairs properly, then the resulting map is a triangulation of an orientable surface of genus g with α vertices of degree 6 while the remaining vertices have degree 10. To make the map simple, glue the holes so that for every i, $0 \leq i \leq g - 2$, the following identification is made: $6i \equiv 6i{+}11$, $6i{+}1 \equiv 6i{+}8$, $6i{+}3 \equiv 6i{+}10$, the arithmetic being modulo $6(g{-}1)$. Denote by T the resulting map. Observe that this gluing produces an orientable surface since the triangles $(6i, 6i{+}1, 6i{+}3)$ and $(6i{+}11, 6i{+}8, 6i{+}10)$ are oriented in the opposite way. Denote by G the underlying graph for T. We claim that G is simple. First, there is no loop in G, because every pair of vertices in S that are identified (in T) share no edge. Also, for every such pair, say u and v, there is no common neighbor in M since $|u - v| > 6$. Thus if a vertex is not in S, it can not be the end point of a multiple edge in G. This implies that if there is a multiple edge in G, then its endpoints correspond to pairs of identified vertices from S. First suppose that $\alpha = 0$. For every v in S, let S_v be the set of its 4 neighbors in M that don't lie on the same 3-hole as v. Then $S_v \subset \{v{-}3, v{-}2, v{-}1, v{+}1, v{+}2, v{+}3\}$, the arithmetics being modulo $6(g{-}1)$. Below we list S_v in a form in which the identifications are easily recognizable.

1. $S_{6i} = \{6(i{-}1) + 3, 6(i{-}2) + 10, 6(i{-}2) + 11, 6(i{-}1) + 8\}$,
   $S_{6i+11} = \{6(i{+}1) + 3, 6(i{+}2), 6(i{+}2) + 1, 6(i{+}1) + 8\}$

2. $S_{6i+1} = \{6(i-2)+10, 6(i-2)+11, 6(i-1)+8, 6(i-1)+10\}$,
   $S_{6i+8} = \{6(i-1)+11, 6(i+1), 6(i+1)+1, 6(i+1)+3\}$

3. $S_{6i+3} = \{6(i-1)+8, 6(i-1)+10, 6(i-1)+11, 6(i+1)\}$,
   $S_{6i+10} = \{6(i+1)+1, 6(i+1)+3, 6(i+2), 6(i+2)+1\}$

Since $g \geq 6$, the values $6(i-2)+j$, $6(i-1)+j$, $6i+j$, $6(i+1)+j$ and $6(i+2)+j$, for fixed i and j where $0 \leq i \leq g-2$, $0 \leq j \leq 11$, are distinct modulo $6(g-1)$. Therefore if two vertices $u, v \in S$ are identified in T, then no two vertices of S_u and S_v are identified together. To demonstrate this, let $u = 6i$ and $v = 6i+11$. Then the elements in S_u are identified with $\{6(i-1)+10, 6(i-2)+3, 6(i-2)+0, 6(i-1)+1\}$ and this set has no element in common with S_v. Similar check can be done for the other two types of vertices, which implies that there is no multiple edge in G provided that $\alpha = 0$.

Now suppose that $\alpha > 0$. Since the neighbours of v are 'consecutive' vertices $v-3, v-2, v-1, v+1, v+2, v+3$, the arithmetics being modulo $6(g-1)+\alpha$, S_v in this case is just a subset of S_v for the case $\alpha = 0$ described above. Consequently, theree is no multiple edge in T even when $\alpha > 0$. By Proposition 3, the dual of T is the required 10-gonal fullerene of genus g with exactly α hexagons. □

The cases $g = 5$ and $\alpha \in \{0, 1, 2, 3\}$ are covered by neither Lemma 6 nor Theorem 7. At the moment we are not able to determine if there is a 10-gonal fullerene for these parameters, so these four cases are left open. All of our attempts to find constructions for these cases have failed. We ran the `CGF` program ([1]) for quite a long time but no hyperbolic fullerenes were found, and to run the program exhaustively seems to be beyond our computer capacities. Observe that if the required 10-gonal fullerenes exist, then the underlying graph for the dual map is K_{12} minus a perfect matching in the case $\alpha = 0$. However, already in the case $\alpha = 1$ there are 15 possibilities for the underlying graph of the dual map and this number increases for $\alpha \in \{2, 3\}$.

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