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Smallest vertex-transitive graphs
of given degree and diameter

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Abstract

For every $d$ and $k$ we determine the smallest order of a vertex-transitive graph of
degree $d$ and diameter $k$, and in each such case we show that this order is achieved by
a Cayley graph.

1 Introduction

Determination of the largest order $N(d,k)$ of a graph of maximum degree $d$ and diameter
$k$, known as the degree/diameter problem, has become a classical open question in extremal
graph theory. The problem has generated considerable activity since its 1960 statement in
the pioneering paper [8]; for a survey we refer to [12]. A spanning tree argument gives the
Moore bound $N(d,k) \leq M(d,k)$ where $M(d,1) = 1 + d$ for any $d \geq 1$ and
$M(d,k) = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$ for every $d, k \geq 2$. Apart from the trivial cases when
$d \leq 2$ or $k = 1$, the Moore bound is known to be achieved only when $k = 2$ and $d = 3, 7,$
and possibly 57, by results of [8] for diameters 2 and 3, and [1, 5] for all larger diameters.
For all the remaining pairs $(d,k)$ we have the inequality $N(d,k) \leq M(d,k) - 2$, proved for
$k = 2$ in [6] and for all $k \geq 3$ in [2]. It has been conjectured in [7] that $N(d,k) \leq M(d,k) - 3$
for every $d,k \geq 4$; for results towards this conjecture and other related facts see [7] and
references therein. More general improvements of the Moore bound currently appear to be
beyond reach.

Motivated by the methods of generating large graphs of given degree and diameter [11],
there has been growing interest in determining or at least estimating the largest order $V(d,k)$

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of a vertex-transitive graph of degree \( d \) and diameter \( k \), cf. [12]. Indeed, about a half of the current largest orders of graphs of a given maximum degree and diameter, kept in online tables [16], have been found by searching over Cayley graphs or over lifts of small-order quotients resulting in vertex-transitive graphs, cf. [11]. A focus on such a restriction is also natural in the light of the fact that, except the case of diameter 2 and degree 57 (due to a result of Higman, see [4]), all the extremal graphs establishing the equality \( N(d, k) = M(d, k) \) are vertex-transitive. A new impetus for research in this direction is the paper [9] where it was shown that vertex-transitivity can indeed keep the number of graphs arbitrarily far away from the Moore bound in a considerable number of cases by proving that for every fixed integers \( d \geq 3 \) and \( \delta \geq 1 \), the set of positive integers \( k \) for which \( V(d, k) \leq M(d, k) - \delta \) has positive lower asymptotic density.

In extremal graph theory it is customary to consider both extremes of any given scalar parameter of graphs. Here, however, one has to be careful about the formulation of the problem. Asking for the smallest order of a graph of maximum degree \( d \) and diameter \( k \) would not be interesting, as the answer is trivial: An extremal graph in this situation would have to contain an induced path on \( k + 1 \) vertices and, leaving the trivial case of diameter 1 aside, for \( k \geq 2 \) one just needs to attach \( d - 2 \) pendant edges to some inner vertex of the path, which yields the smallest order \( d + k - 1 \) if \( d, k \geq 2 \). Asking for the smallest order \( n(d, k) \) of a graph of minimum degree \( d \) and diameter \( k \) changes the answer but one just needs a little extra effort to obtain it. Indeed, let \( v \) be a vertex of eccentricity \( k \) in an extremal graph and let \( x_i \) \((0 \leq i \leq k)\) denote the number of vertices at distance \( i \) from \( v \). The quantity \( n(d, k) \) is then equal to the minimum of \( x_0 + x_1 + \ldots + x_k \) over positive integers \( x_0, x_1, \ldots, x_k \) with \( x_0 = 1 \) subject to the conditions that all of \( x_0 + x_1, x_{k-1} + x_k \), and \( x_{i-1} + x_i + x_{i+1}, 1 \leq i \leq k - 1 \), be at least \( d + 1 \), since these conditions are also sufficient for the existence of a corresponding graph. An evaluation gives \( n(d, k) = (d - 2)[k/3] + d + k + 1 \) for any \( d \geq 3 \) and \( k \geq 3 \), with similar formulae for the remaining cases. Almost the same values are obtained when asking for the smallest order of a regular graph with degree \( d \) and diameter \( k \), see [10], where the most interesting part is the construction of extremal graphs.

Asking about the smallest order \( v(d, k) \) of a vertex-transitive graph of degree \( d \) and diameter \( k \), however, is a much more interesting question, both per se as well as in the context of the ‘opposite end’ of the vertex-transitive version of the degree-diameter problem. In this paper we solve the problem of determining \( v(d, k) \) completely and our main result can be formulated in the following way:

**Main Result.** For all \( d \geq 2 \) and \( k \geq 4 \) we have \( v(d, k) = 2\lceil\frac{d+1}{3}\rceil k - \delta \), where \( \delta = 4 \) if \( d \in \{3, 6, 9\} \); \( \delta = 2 \) if \( d = 4 \) or if \( d \in \{12, 24\} \) and \( k = 4 \); and \( \delta = 0 \) otherwise.

In particular, in terms of the product \( dk \), the value of \( v(d, k) \) is only about two times that of \( n(d, k) \), which appears to be rather surprising.

The paper is organized as follows. We begin by treating small values of \( d \) and \( k \) in Section 2. To be able to handle the remaining values, in Section 3 we prove a few auxiliary results
on vertex-transitive graphs. Section 4 then deals with large values of \(d\) and \(k\) and the final Section 5 contains a number of remarks.

2  Small values of the degree and the diameter

In the paper, \(G\) will always denote a vertex-transitive graph. For any vertex \(x\) of \(G\) and for any \(i \geq 0\) we let \(N_i(x)\) be the set of vertices of \(G\) at distance \(i\) from \(x\); formally, \(N_i(x) = \{y \in V(G); \text{dist}_G(x, y) = i\}\). By vertex-transitivity, for every \(i \geq 0\) the quantity \(n_i = |N_i(x)|\) is independent of the choice of \(x\). Dependence of the sets \(N_i(x)\) and the numbers \(n_i\) on \(G\) will be automatically assumed throughout.

We begin with an observation which will be used frequently. By connectivity of a graph we will always mean vertex-connectivity.

**Lemma 2.1.** Let \(G\) be a vertex-transitive graph of order \(n\), degree \(d \geq 2\), diameter \(k \geq 3\) and connectivity \(\kappa\). Then \(n \geq 2d + 2 + (k-3)\kappa\).

**Proof.** Let \(x\) and \(y\) be vertices of \(G\) at distance \(k\). The set \(N_0(x) \cup N_1(x)\) contains \(x\) and all the vertices adjacent to \(x\), and \(N_{k-1}(x) \cup N_k(x)\) contains \(y\) and all the vertices adjacent to \(y\). For every \(i\), \(2 \leq i \leq k - 2\), we have \(n_i \geq \kappa\) since \(N_i(x)\) is a cut set of \(G\). It follows that \(n = \sum_{i=0}^{k} n_i \geq (d + 1) + (k-3)\kappa + (d + 1)\).

If \(H\) is a complete graph or a complete bipartite graph with equal parts, we let \(H^{(-1)}\) and \(H^{(-2)}\) denote the graph obtained from \(H\) by removing all edges of a perfect matching and a Hamilton cycle, respectively. For small diameters we have the following statement.

**Theorem 2.2.** For every \(d \geq 2\) we have the following:

(i) if \(k = 1\) then \(v(d, k) = d + 1\) and \(K_{d+1}\) is an extremal graph;

(ii) if \(k = 2\) and \(d\) is even then \(v(d, k) = d + 2\) and \(K_{d+2}^{(-1)}\) is an extremal graph;

(iii) if \(k = 2\) and \(d\) is odd then \(v(d, k) = d + 3\) and \(K_{d+3}^{(-2)}\) is an extremal graph;

(iv) if \(k = 3\) then \(v(d, k) = 2d + 2\) and \(K_{d+1,d+1}^{(-1)}\) is an extremal graph.

**Proof.** The case \(k = 1\) is obvious since \(K_{d+1}\) is the unique graph of diameter 1 and degree \(d\).

Let \(k = 2\). As in the proof of Lemma 2.1 one can show that \(n_0 + n_1 \geq d + 1\) and \(n_2 \geq 1\). Thus, \(v(d, 2) \geq d + 2\). But there is no graph of odd degree on an odd number of vertices, and therefore \(v(d, 2) \geq d + 3\) if \(d\) is odd. Since \(K_{d+2}^{(-1)}\) and \(K_{d+3}^{(-2)}\) are vertex-transitive graphs of diameter 2 and degree \(d\) for \(d\) even and odd, respectively, the result follows.

It remains to deal with the value \(k = 3\). By Lemma 2.1 we have \(v(d, 3) \geq 2d+2\). The graph \(K_{d+1,d+1}^{(-1)}\) is vertex-transitive and has degree \(d\) and diameter 3, implying \(v(d, 3) = 2d + 2\). \(\square\)
We now turn our attention to small values of the degree.

**Lemma 2.3.** Let \( G \) be a vertex-transitive graph of connectivity \( \kappa \) and degree \( d \), where \( d \in \{3, 4, 6\} \). Then \( \kappa = d \).

**Proof.** The case \( d = 3 \) can be derived from [15, Theorem 3] and the cases \( d = 4 \) and \( d = 6 \) follow from [15, Corollary 3B]. \( \square \)

We will, however, need more detailed information regarding the parameters \( n_i \) in cubic graphs, that is, graphs of degree \( d = 3 \).

**Lemma 2.4.** Let \( G \) be a cubic vertex-transitive graph of diameter \( k \geq 4 \). Then \( n_i \geq 4 \) for every \( i \) such that \( 2 \leq i \leq k - 2 \).

**Proof.** For each vertex \( x \) of \( G \) we let \( N_{<j}(x) = \bigcup_{i<j}N_i(x) \). By way of contradiction, suppose that there is some \( t \), \( 2 \leq t \leq k - 2 \), such that \( n_t \leq 3 \). By Lemma 2.3 we have \( n_t = 3 \). Let \( a_x, b_x \) and \( c_x \) be the three vertices of \( N_t(x) \); obviously, each of these has a neighbour in \( N_{t-1}(x) \). But each of \( a_x, b_x \) and \( c_x \) must also have a neighbour in \( N_{t+1}(x) \), since otherwise there is a 2-cut in \( G \), which contradicts Lemma 2.3.

In what follows we fix one vertex \( v \) of \( G \). Let \( u \) be a neighbour of \( v \) lying on a shortest \( v \to a_v \) path. Then all of \( a_u, b_u, c_u \) belong to \( N_{t-1}(v) \cup N_t(v) \cup N_{t+1}(v) \). Since \( a_v \) has a neighbour in \( N_{t+1}(v) \) and this neighbour lies also in \( N_t(u) \), we may assume that \( a_u \in N_{t+1}(v) \) and \( a_v a_u \) is an edge of \( G \).

Observe that \( N_t(u) \subset \bigcup_{i \geq t} N_i(v) \), since otherwise for every \( x \in N_{<t}(v) \cup \{a_v\} \) there would be a \( v \to x \) path avoiding \( N_t(u) \) and this would mean that \( |N_t(u)| \geq |N_{<t}(v)| + 1 \), a contradiction. Thus, there is a vertex of \( N_t(u) \), say, \( c_u \), such that \( c_u \in N_{t-1}(v) \). We distinguish three cases, depending on the position of \( b_u \) in our graph.

**Case 1:** \( b_u \in N_{t+1}(v) \). If neither \( b_v \) nor \( c_v \) were reachable from \( v \) in \( G - \{c_u, a_v\} \), then \( \{a_v, c_u\} \) would be a 2-cut, contradicting Lemma 2.3. On the other hand, if there is a vertex of \( N_t(v) \), say \( b_v \), with the property that there is a \( v \to b_v \) path in \( G - \{c_u, a_v\} \), then \( N_{<t}(u) \) would contain the sets \( N_{<t}(v) \setminus \{c_u\} \) and \( \{a_v, b_v\} \), implying that \( |N_{<t}(u)| \geq |N_{<t}(v)| + 1 \), a contradiction.

**Case 2:** \( b_u \in N_{t-1}(v) \). If \( b_v \notin N_{t-1}(u) \) and \( c_v \notin N_{t-1}(u) \), then \( b_v, c_v \in N_{t+1}(u) \). Consequently, \( N_{<t}(u) \subset (N_{<t}(v) \setminus \{b_v, c_u\}) \cup \{a_v\} \), and so \( |N_{<t}(u)| \leq |N_{<t}(v)| - 1 \), a contradiction. We may therefore assume that \( b_u \in N_{t-1}(u) \). Since all vertices of \( N_{t+1}(v) \) are contained in \( N_t(u) \cup N_{t+1}(u) \cup N_{t+2}(u) \) and \( b_u, c_u \in N_{t-1}(v) \), the vertex \( a_u \) is the unique neighbour of \( b_u \) in \( N_{t+1}(v) \). One can show in a similar way that \( a_v \) is the unique neighbour of \( a_v \) in \( N_{t+1}(v) \). Thus, \( \{a_u, c_v\} \) is a 2-cut, a contradiction.

**Case 3:** \( b_u \in N_t(v) \). Since \( a_v \in N_{t-1}(u) \), we have either \( b_u = b_v \) or \( b_u = c_v \). Without loss of generality, suppose that \( b_u = b_v \). Because of \( (N_{<t}(v) \setminus \{c_u\}) \cup \{a_v\} \subset N_{<t}(u) \) we
have $N_{ct}(u) = (N_{ct}(v) \setminus \{c_u\}) \cup \{a_v\}$, which implies that $c_v \in N_{t+1}(u)$. Since there is an edge joining a vertex of $N_{t-1}(v)$ with $c_v$, and since $N_{t-1}(v) \setminus \{c_u\} \subseteq N_{ct}(u)$, it follows that $c_v$ has no other neighbour in $N_{t-1}(v)$. From $c_v \in N_{t+1}(u)$ and $a_v \in N_{t-1}(u)$, we conclude that there is no edge in $G$ joining the vertices $a_v$ and $c_v$. By a similar argument the same conclusion applies to the vertices $a_u$ and $c_u$. But as $u, v \in N_t(b_v)$ and $uv \in E(G)$, there is an edge within $N_t(u)$ and, analogously, there is an edge within $N_t(v)$. That is, one of $a_u b_v$, $b_v c_u$ and also one of $a_v b_v$, $b_v c_v$ are edges of $G$. Consequently, one of the three neighbours of $b_v$ must lie in $N_t(v)$. As stated earlier, $b_v$ has also neighbours in both $N_{t-1}(v)$ and $N_{t+1}(v)$. If $a_u b_v \in E(G)$ then, since $a_u$ is the unique neighbour of $a_v$ in $N_{t+1}(v)$ (otherwise $N_t(u)$ would not be a cut-set), the set $\{a_u, c_u\}$ would be a 2-cut, a contradiction. The only remaining possibility is that $b_v c_u \in E(G)$. Then, since $c_u$ is the unique neighbour of $c_v$ in $N_{t-1}(v)$, the set $\{a_v, c_u\}$ would be a 2-cut in $G$. This contradiction completes the proof.

Let $P_n$ be a prism on $2n$ vertices, that is $P_n = C_n \Box K_2$, where $\Box$ stands for the box product of graphs. Further, let $A_n$ denote an antiprism on $2n$ vertices, that is $A_n = C_n^2$, where $G^2$ is obtained from $G$ by adding all edges $xy$ between vertices $x, y$ at distance 2 in $G$. For small degrees we now can state and prove the following.

**Theorem 2.5.** For any $k \geq 4$ we have:

(i) if $d = 2$ then $v(d, k) = dk$, with $C_{2k}$ as an extremal graph;

(ii) if $d = 3$ then $v(d, k) = (d+1)k - 4$, with $P_{2k-2}$ as an extremal graph;

(iii) if $d = 4$ then $v(d, k) = dk - 2$, with $A_{2k-1}$ as an extremal graph.

**Proof.** Since there are only two non-isomorphic cycles of diameter $k$, namely $C_{2k}$ and $C_{2k+1}$, the result is obvious for $d = 2$.

Let $G$ be a cubic vertex-transitive graph of order $n$ and diameter $k$, that is, we are in the case $d = 3$. As in the proof of Lemma 2.1 one can show that $n_0 + n_1 \geq d + 1 = 4$ and $n_{k-1} + n_k \geq 4$. By Lemma 2.4 for every $i$, $2 \leq i \leq k - 2$, we have $n_i \geq 4$. It follows that $n = \sum_{i=0}^{k} n_i \geq 4k - 4$. Since $P_{2k-2}$ is a cubic vertex-transitive graph of diameter $k$ and order $4k - 4$, we have $v(3, k) = 4k - 4$.

Finally, let $G$ be a vertex-transitive graph of order $n$, degree 4, diameter $k$, and connectivity $\kappa$. Lemma 2.1 implies that $n \geq 2d + 2 + (k-3)\kappa$. By Lemma 2.3 we have $\kappa = 4$, and so $n \geq 4k - 2$. Observing that $A_{2d-1}$ is a vertex-transitive graph of order $4k - 2$, degree 4 and diameter $k$, we conclude that $v(4, k) = 4k - 2$. 

3 Auxiliary results on vertex-transitive graphs

We will keep to the notation introduced earlier and study the ‘fine’ structure of vertex-transitive graphs. For a graph $G$ with connectivity $\kappa$ we let $C(G)$ denote the set of all vertex-cuts of size $\kappa$ in $G$. Let

$$p(G) = \min \{ \min \{|V(P)|; \text{ } P \text{ is a component of } G - C \}; \text{ } C \in C(G) \}. $$

Following [15], if $C \in C(G)$, a component $P$ of $G - C$ is said to be an atomic part of $G$ if $|V(P)| = p(G)$. The following theorem summarizes and extends (in the last item) some results of [15] for vertex-transitive graphs we will need in our considerations.

**Theorem 3.1.** Let $G$ be a vertex-transitive graph and let $P$ be an atomic part of $G$ corresponding to a cut $C$. Then

(i) $P$ is a vertex-transitive graph;

(ii) $G$ is isomorphic to a disjoint union of several copies of $P$ together with some edges joining them;

(iii) if $P'$ is another atomic part of $G$, then either $V(P') \subseteq C$ or $V(P') \cap C = \emptyset$;

(iv) $\kappa = t \cdot p(G)$ for some $t \geq 2$;

(v) if $G^*$ is a simple graph obtained from $G$ by contracting every atomic part into a single vertex, then $G^*$ is vertex-transitive.

**Proof.** Items (i) and (ii) are proved in [15, Theorem 2], (iii) is proved in [15, Lemma 3.5] and (iv) is proved in [15, Lemma 4.1]. In fact, in [15] the proofs are made for the case $\kappa < d$, but the case $\kappa = d$ is trivial (and useless) as then every atomic part is a single vertex.

It remains to prove (v). Let $\varphi$ be an automorphism of $G$. As the set of vertices $x \in V(G) \setminus V(P)$ adjacent to a vertex of $P$ is a cut-set in $G$, its $\varphi$-image, that is, the set of vertices $y \in V(G) \setminus V(\varphi(P))$ adjacent to a vertex of $\varphi(P)$, forms a cut-set in $G$ as well. Thus, the image of an atomic part is again an atomic part, which means that $\varphi$ induces an automorphism of $G^*$. Take two vertices, say $u_1$ and $u_2$ of $G^*$ and denote $P_1$ and $P_2$, respectively, the corresponding atomic parts in $G$. Further, choose two vertices $x_1$ and $x_2$ in $G$ such that $x_1 \in V(P_1)$ and $x_2 \in V(P_2)$. Since $G$ is vertex-transitive, there is an automorphism in $G$ mapping $x_1$ to $x_2$, and this automorphism induces an automorphism of $G^*$ mapping $u_1$ onto $u_2$. Hence, $G^*$ is a vertex-transitive graph. 

Throughout the rest of this paper, $G^*$ and $t = t(G)$ will denote the objects associated with $G$ as introduced in Theorem 3.1; note that $t$ is the degree of $G^*$.

The next statement is a slight generalization of [15, Theorem 3].
Theorem 3.2. Let $G$ be a vertex-transitive graph of connectivity $\kappa$ and degree $d$, and let $t = t(G)$. Then $t \geq 2$ and

$$\kappa \geq \frac{t}{t+1}(d+1).$$

Proof. By Theorem 3.1 (iv), we have $\kappa = t \cdot p(G)$, where $t \geq 2$. Any vertex in an atomic part $P$ is adjacent to at most $|V(P)| - 1$ vertices in $P$ and at most $p(G)$ vertices of every atomic part that is included in the cut-set corresponding to $P$. It follows that

$$d \leq p(G) - 1 + t \cdot p(G),$$

and so

$$\kappa = t \cdot p(G) \geq \frac{t}{t+1}(d+1).$$

We proceed with presenting a tool which will, in some cases, help reduce our problem for a given degree to a smaller one.

Lemma 3.3. Let $G$ be a vertex-transitive graph of degree $d \geq 3$ and diameter $k \geq 4$. Let $G^*$ and $t$ be as in Theorem 3.1 and let $k^*$ be the diameter of $G^*$. If $d > (2t+1)p(G)/2 - 1$, then $k^* = k$.

Proof. For any $x \in V(G)$ let $P_x$ be the atomic part containing $x$ and let $z_x$ be the corresponding vertex of $G^*$. If $z_xz_y \in E(G^*)$ then we will say that $P_x$ is adjacent to $P_y$ and vice versa. Denote by $P_{x,1}, P_{x,2}, \ldots, P_{x,t}$ the atomic parts adjacent to $P_x$. Let $(\delta_0(x); \delta_1(x), \delta_2(x), \ldots, \delta_t(x))$ be a sequence such that $\delta_0(x)$ is the degree of $x$ in $P_x$ and $\delta_i(x) = |\{xy \in E(G); y \in P_{x,i}\}|$, where $1 \leq i \leq t$. (We note that one can deduce from Theorem 3.1 that the multiset $\{\delta_1(x), \delta_2(x), \ldots, \delta_t(x)\}$ does not depend on the choice of $x \in V(G)$, but we will not use this here.) Observe that $\delta_0(x) + \delta_1(x) + \ldots + \delta_t(x) = d$, $\delta_0(x) \leq p(G) - 1$ and $\delta_i(x) \leq p(G)$, where $1 \leq i \leq t$. By our assumption, for every $x \in V(G)$ and each $i$, $1 \leq i \leq t$, we have

$$\delta_i(x) > p(G)/2.$$
Obviously, $k^* \leq k$, and we now prove the reverse inequality. Let $u, v \in V(G)$. We show that either $\text{dist}_G(u, v) \leq 3$ or $\text{dist}_G(u, v) \leq \text{dist}_{G^*}(z_u, z_v)$, distinguishing three cases.

**Case 1:** $z_u = z_v$. Let $P$ be an atomic part adjacent to $P_u$. According to (2), the vertices $u$ and $v$ have a common neighbour in $P$, and so $\text{dist}_G(u, v) \leq 2$.

**Case 2:** $z_u z_v \in E(G^*)$. By (2), there is a vertex $w \in P_x$ such that $uw \in E(G)$. Since in Case 1 we have already shown that $\text{dist}_G(w, v) \leq 2$, we have $\text{dist}_G(u, v) \leq 3$.

**Case 3:** $\text{dist}_{G^*}(z_u, z_v) \geq 2$. Let $z_u, z_{x_1}, z_{x_2}, \ldots, z_{x_{q-1}}, z_v$ be a shortest path from $z_u$ to $z_v$ in $G^*$. From (2) it follows that for every $i$, $2 \leq i \leq q-1$, there exists $w_i \in P_{x_i}$ such that $w_2, w_3, \ldots, w_{q-1}, v$ is a path in $G$. Further, by (2) the vertices $u$ and $w_2$ have a common neighbour in $P_{x_1}$. Thus, $\text{dist}_G(u, v) \leq \text{dist}_{G^*}(z_u, z_v)$.

The proof can now be completed easily. By the assumption that $k \geq 4$, there are $u, v \in V(G)$ such that $\text{dist}_G(u, v) \geq 4$. Since for all pairs $u, v \in V(G)$ such that $\text{dist}_G(u, v) \geq 4$ we have $\text{dist}_G(u, v) \leq \text{dist}_{G^*}(z_u, z_v)$, we conclude that $k \leq k^*$.

### 4 Large values of the degree and the diameter

Let $H$ and $H'$ be graphs. By $H \circ H'$ we denote the graph on the vertex set $\{(x, y); x \in V(H) \text{ and } y \in V(H')\}$, in which a pair of distinct vertices $(x_1, y_1)(x_2, y_2)$ forms an edge if either $x_1x_2 \in E(H)$ or $x_1 = x_2$ and $y_1y_2 \in E(H')$. The operation $\circ$ is known as the lexicographic product. We will need two more operations derived from $\circ$, applicable to graphs satisfying certain extra conditions. Assume that $H$ contains a pair of disjoint perfect matchings and let $M_1$ and $M_2$ be such a pair. Further, for $i \in \{1, 2\}$ let $\overrightarrow{M}_i = \{(x_1, y)(x_2, y); (x_1, y), (x_2, y) \in V(H \circ H') \text{ and } x_1x_2 \in M_i\}$. Then both $\overrightarrow{M}_1$ and $\overrightarrow{M}_2$ are perfect matchings in $H \circ H'$. By $H \circ H'$ we denote a graph obtained from $H \circ H'$ by removing the edges of $\overrightarrow{M}_1$, and by $H \circ H'$ we denote a graph obtained from $H \circ H'$ by removing the edges of both $\overrightarrow{M}_1$ and $\overrightarrow{M}_2$.

As we shall see, in almost all cases considered in the theorems of this section, the graphs $C_{2k} \circ K_j$, $C_{2k} \circ K_j$ and $C_{2k} \circ K_j$ will be examples of extremal graphs, i.e., of order $v(d, k)$ for the corresponding values of $d$ and $k$. An exception is Theorem 4.3 where we will also use powers of cycles. In general, if $H$ is a graph and $t \geq 2$, then $H^t$, the $t$-th power of $H$, is the graph with $V(H^t) = V(H)$ in which $xy \in E(H^t)$ if and only if $\text{dist}_H(x, y) \leq t$.

**Theorem 4.1.** Let $d = 3j - 1$, $j \geq 2$ and let $k \geq 4$. Then $v(d, k) = 2jk$ and $C_{2k} \circ K_j$ is an extremal graph.

**Proof.** By Theorem 3.2 and Theorem 3.1 (iv), we have $\kappa \geq \frac{1}{t+1}(d+1)$, where $t$ is an integer such that $t \geq 2$. Thus, $\kappa \geq \frac{2}{3}(d+1) = 2j$. By Lemma 2.1 we obtain $v(d, k) \geq 2d + 2 + (k-3)2j = 2jk$. On the other hand, it is easy to see that $C_{2k} \circ K_j$ is a vertex-transitive graph of degree $3j - 1$, diameter $k$ and order $2jk$. 

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Theorem 4.2. Let \( d = 3j - 2 \), \( j \geq 3 \) and let \( k \geq 4 \). Then \( v(d, k) = 2jk \) and \( C_{2k} \circ K_j \) is an extremal graph.

Proof. It is easy to see that \( C_{2k} \circ K_j \) is a vertex-transitive graph of degree \( 3j - 2 \), diameter \( k \) and order \( 2jk \), implying that \( v(d, k) \leq 2jk \). But by Theorem 3.2 and Theorem 3.1 (iv) we have \( \kappa \geq \frac{t}{t+1}(d+1) \) for some integer \( t \geq 2 \). Thus, \( \kappa \geq \frac{2}{3}(d+1) = 2j - \frac{2}{3} \), and so \( \kappa \geq 2j \). By Lemma 2.1 we obtain \( v(d, k) \geq 2d + 2 + (k-3)2j = 2jk - 2 \). In the rest of the proof we show that there is no vertex-transitive graph of degree \( d \) and diameter \( k \) on \( 2jk - 2 \) or \( 2jk - 1 \) vertices.

By way of contradiction, suppose that there is a vertex-transitive graph \( G \) of degree \( d \) and diameter \( k \) of order \( 2j - 2 \) or \( 2j - 1 \). If \( \kappa \geq 2j + 2 \), then by Lemma 2.1 \( |V(G)| \geq 2d + 2 + (k-3)(2j + 2) = 2jk + 2(k-4) \geq 2jk \), a contradiction. Hence, \( \kappa \leq 2j + 1 \).

If the degree of \( G^* \) is 2, that is, if \( t = 2 \), then \( \kappa = 2p(G) \) and so \( p(G) = j \) and \( \kappa = 2j \). By Theorem 3.1 (ii), \( j \) divides \( |V(G)| \). Since \( j \geq 3 \), the graph \( G \) cannot have \( 2jk - 2 \) or \( 2jk - 1 \) vertices. Hence, \( t \geq 3 \).

Suppose that \( \kappa = 2j + 1 \). Then \( |V(G)| \geq 2d + 2 + (k-3)(2j + 1) = 2jk + (k - 5) \), and so necessarily \( k = 4 \) and \( |V(G)| = 2jk - 1 \). Since there is no regular graph of odd degree on an odd number of vertices, \( d \) is even and consequently \( j \) is even. If \( j \geq 8 \) then \( \kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-1}{2} > 2j + 1 \), a contradiction. Thus, \( j \leq 6 \). We distinguish two cases.

Case 1: \( j = 6 \). Then, \( \kappa = 2j + 1 = 13 \). But as \( t \geq 3 \) and \( \kappa = t \cdot p(G) = 13 \), we obtain \( t = 13 \) and \( p(G) = 1 \). Hence, every atomic part consists of a single vertex, which means that \( \kappa = d \). Since \( d = 3j - 2 = 16 \), this is impossible.

Case 2: \( j = 4 \). It follows that \( \kappa = 2j + 1 = 9 \) and from \( d = 3j - 2 = 10 \) we have \( t = p(G) = 3 \). The facts that \( |V(G)| = 2jk - 1 = 31 \) and \( 3 \not| 31 \) now contradict Theorem 3.1 (ii).

It remains to consider the situation when \( \kappa = 2j \). If \( j \geq 4 \) then \( \kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-3}{4} > 2j \), a contradiction. Thus, \( j = 3 \), and then \( \kappa = 2j = 6 \). Since \( d = 7 \) and \( \kappa = t \cdot p(G) \) for some \( t \geq 3 \), we have \( t = 3 \) and \( p(G) = 2 \). From \( d = 7 > 6 = (2t+1)p(G)/2 - 1 \) and Lemma 3.3 we have \( \text{diam}(G^*) = \text{diam}(G) \). Since \( G^* \) is a cubic vertex-transitive graph of diameter \( k \), by Theorem 2.5 it has at least \( 4(k-1) \) vertices. The inequality \( k \geq 4 \) finally implies \( |V(G)| \geq 2 \cdot 4(k-1) = 6k + 2(k-4) \geq 6k = 2jk \). \( \square \)

Theorem 4.3. Let \( d = 3j - 3 \), \( j \geq 3 \), and let \( k \geq 4 \). We have:

(i) if \( d = 6 \) then \( v(d, k) = 2jk - 4 \) and \( C_{6k-4}^3 \) is an extremal graph;

(ii) if \( d = 9 \) then \( v(d, k) = 2jk - 4 \) and \( A_{2k-1} \circ K_2 \) is an extremal graph;

(iii) if \( d = 12 \) and \( k = 4 \) then \( v(d, k) = 2jk - 2 \) and \( C_{6}^{18} \) is an extremal graph;

(iv) if \( d = 24 \) and \( k = 4 \) then \( v(d, k) = 2jk - 2 \) and \( A_7 \circ K_5 \) is an extremal graph.

In all the remaining cases we have \( v(d, k) = 2jk \) and \( C_{2k} \circ K_j \) is an extremal graph.
Proof. It is easy to see that $C_{2k} \circ K_j$ is a vertex-transitive graph of degree $3j - 3$, diameter $k$ and order $2jk$. Thus, $v(d, k) \leq 2jk$. On the other hand, by Theorem 3.2 and Theorem 3.1 (iv) we have $\kappa \geq \frac{2}{3}(d+1)$, where $t \geq 2$. It follows that $\kappa \geq \frac{2}{3}(d+1) = 2j - \frac{4}{3} > 2j - 2$, and so $\kappa \geq 2j - 1$. But if $t = 2$, then $\kappa = 2p(G)$ and so $\kappa$ has to be even, while if $t \geq 3$ then $\kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-6}{4} > 2j - 1$ since $j \geq 3$. We conclude that $\kappa \geq 2j$. By Lemma 2.1 we have $v(d, k) \geq 2d + 2 + (k-3)2j = 2jk - 4$. Since $C_{6k-4}^3$ is a 6-regular graph of diameter $k$ on $6k - 4$ vertices, we have $v(6, k) = 6k - 4$, which proves the theorem for $j = 3$. In the remaining (and longer) part of the proof we will assume that there is a vertex transitive graph $G$ of degree $d$ and diameter $k$ with $2jk - 4 \leq |V(G)| < 2jk$. The result for $j = 3$ has already been shown and so we assume $j \geq 4$.

If $\kappa \geq 2j + 4$ then by Lemma 2.1 $|V(G)| \geq 2d + 2 + (k-3)(2j+4) \geq 2jk + 4(k-4) \geq 2jk$, because $k \geq 4$; therefore $\kappa \leq 2j + 3$. Suppose that the degree of $G^*$ is 2, that is, $t = 2$. Then $\kappa = 2j + 2$ or $\kappa = 2j$. We consider the two cases separately.

Case 1: $\kappa = 2j + 2$. Then $|V(G)| \geq 2d + 2 + (k-3)(2j+2) \geq 2jk + 2(k-5) \geq 2jk$ if $k \geq 5$, implying that $k = 4$ and $|V(G)| \geq 2jk - 2 = 8j - 2$. Since $\kappa = 2j + 2 = 2p(G)$, we have $p(G) = j + 1$. Consequently, $(j+1)$ divides $|V(G)|$ by Theorem 3.1 (ii).

If $|V(G)| = 2jk - 2 = 8j - 2$, then $(j+1)$ divides $8j - 2$, and so $8j - 2 = i(j+1)$ for some $i \leq 7$. If $i = 7$, then $8j - 2 = 7j + 7$ which gives $j = 9$ and $d = 24$. This is exactly the case (iv) and later we show that there exists a graph $G$ of order $2jk - 2$ with the required properties. If $i = 6$, then $8j - 2 = 6j + 6$ which gives $j = 4$. Then $d = 9$ and $\kappa = 10$, which is impossible. If $i \leq 5$, then $j < 4$, a contradiction.

Now suppose that $|V(G)| = 2jk - 1$. Then $|V(G)| = 8j - 1$ and $(j+1)$ divides $8j - 1$, that is, $8j - 1 = i(j+1)$ for some $i \leq 7$. If $i = 7$, then $8j - 1 = 7j + 7$ which gives $j = 8$. Then $d = 21$ and $|V(G)| = 63$, which is impossible since both $d$ and $|V(G)|$ cannot be odd. If $i \leq 6$ then $j < 4$, a contradiction again.

Case 2: $\kappa = 2j$. We have already seen that $|V(G)| \geq 2jk - 4$. Since $\kappa = 2j = 2p(G)$, we have $p(G) = j$ and, by Theorem 3.1 (ii), $j$ divides $|V(G)|$. Since $j \geq 4$, we obtain $|V(G)| = 2jk - 4$ and $j = 4$, which means that $d = 9$. This is exactly the case (ii) and later we show that there is a required graph $G$ with $|V(G)| = 2jk - 4$.

Thus, from now on we assume that $t \geq 3$. We already derived $2j \leq \kappa \leq 2j + 3$, so let $\kappa = 2j + \ell$, where $0 \leq \ell \leq 3$. Since $\kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-6}{4}$, we have $j \leq 4\ell + 6$. This implies the following.

- If $\kappa = 2j + 3$ then $j \leq 18$ and $|V(G)| \geq 2d + 2 + (k-3)(2j+3) \geq 2jk + 3(k-4) - 1$. Thus, $|V(G)| \geq 2jk$ if $k \geq 5$ and so $k = 4$ and $|V(G)| = 2jk - 1$. Since $|V(G)|$ is odd, $d$ must be even, which means that $j$ must be odd.
- If $\kappa = 2j + 2$ then $j \leq 14$ and $|V(G)| \geq 2jk + 2(k-5)$. Thus, $k = 4$ and $|V(G)| \geq 2jk - 2$.
- If $\kappa = 2j + 1$ then $j \leq 10$ and $|V(G)| \geq 2jk + (k-7)$. Thus, $k \leq 6$ and $|V(G)| \geq 2jk - 3$.
• If $\kappa = 2j$ then $j \leq 6$, $|V(G)| \geq 2jk - 4$ and there is no bound on $k$.

In what follows we consider every value of $j$, $j \leq 18$, separately. As noted above, the cases $j = 18$ and $j = 16$ are impossible due to parity restrictions.

**Case 1:** $j = 17$. Then $\kappa = 2j + 3 = 37$. Since $\kappa = t \cdot p(G)$ and $t \geq 3$, we have $t = 37$. However, by (1) we have $t \leq \left\lfloor \frac{2j+3}{j-5} \right\rfloor = 3$, a contradiction.

**Case 2:** $j = 15$. Then $\kappa = 2j + 3 = 33 = t \cdot p(G)$, where $t \geq 3$. By (1) we have $t \leq \left\lfloor \frac{2j+3}{j-5} \right\rfloor = 3$, and so $t = 3$, $p(G) = 11$ and $d = 42$. Since $d = 42 > 7 \cdot 11/2 - 1 = (2t+1)p(G)/2 - 1$, we have $\text{diam}(G^*) = \text{diam}(G)$, by Lemma 3.3. From $k = 4$ we obtain $|V(G^*)| \geq 12$ by Theorem 2.5, which implies that $|V(G)| \geq 11 \cdot 12 > 120 = 2jk$.

**Case 3:** $j = 14$. Since $d$ is odd, the case $|V(G)| = 2jk - 1$ is impossible and therefore $|V(G)| = 2jk - 2$ and $\kappa = 2j + 2 = 30 = t \cdot p(G)$. By (1), $t \leq \left\lfloor \frac{2j+2}{j-4} \right\rfloor = 3$, and so $t = 3$ and $p(G) = 10$. From $d = 39 > 7 \cdot 10/2 - 1 = (2t+1)p(G)/2 - 1$ it follows that $\text{diam}(G^*) = \text{diam}(G)$, by Lemma 3.3. Since $k = 4$, we have $|V(G^*)| \geq 12$ by Theorem 2.5, and so $|V(G)| \geq 10 \cdot 12 > 112 = 2jk$.

**Case 4:** $j = 13$. If $\kappa = 2j + 3 = 29$, then $t \geq 3$ and $t \leq \left\lfloor \frac{2j+3}{j-5} \right\rfloor = 3$ by (1), a contradiction. Thus, $\kappa = 2j + 2 = 28$. By (1) we have $t \leq \left\lfloor \frac{2j+2}{j-4} \right\rfloor = 3$, a contradiction.

**Case 5:** $j = 12$. Since $d$ is odd, we have $\kappa = 2j + 2 = 26$. By (1), $t \leq \left\lfloor \frac{2j+2}{j-4} \right\rfloor = 3$, which is impossible.

**Case 6:** $j = 11$. If $\kappa = 2j + 3 = 25$ then $t \leq \left\lfloor \frac{2j+3}{j-5} \right\rfloor = 4$, which is impossible as neither 3 nor 4 divides 25. Therefore $\kappa(G) = 2j + 2 = 24$. By (1), $t \leq \left\lfloor \frac{2j+2}{j-4} \right\rfloor = 3$, and so $t = 3$ and $p(G) = 8$. Since $d = 30 > 7 \cdot 8/2 - 1 = (2t+1)p(G)/2 - 1$, we have $\text{diam}(G^*) = \text{diam}(G)$, by Lemma 3.3. Since $k = 4$, we have $|V(G^*)| \geq 12$ by Theorem 2.5, and so $|V(G)| \geq 8 \cdot 12 > 88 = 2jk$.

**Case 7:** $j = 10$. Oddness of $d$ implies that $\kappa \leq 2j+2$. If $\kappa = 2j + 2 = 22$, then $t \leq \left\lfloor \frac{2j+2}{j-4} \right\rfloor = 3$, a contradiction. Thus, $\kappa = 2j + 1 = 21$. By (1), $t \leq \left\lfloor \frac{2j+1}{j-3} \right\rfloor = 3$, and so $t = 3$ and $p(G) = 7$. Since $d = 27 > 7 \cdot 7/2 - 1 = (2t+1)p(G)/2 - 1$, we have $\text{diam}(G^*) = \text{diam}(G)$, by Lemma 3.3. Theorem 2.5 now shows that $|V(G^*)| \geq 4(k-1)$, and so $|V(G)| \geq 20k + 4(2k-7) > 2jk$ as $k \geq 4$.

**Case 8:** $j = 9$. If $\kappa = 2j + 1 = 19$, then $t \leq \left\lfloor \frac{2j+1}{j-3} \right\rfloor = 3$ yields a contradiction, while if $\kappa = 2j + 3$, then $k = 4$ and $|V(G)| \geq 2jk - 1$. We therefore have to consider the case $\kappa = 2j + 2$. Then $k = 4$, $d = 24$ and $|V(G)| \geq 2jk - 2$. But $A_7 \circ K_5$ is a vertex-transitive graph of degree 24, diameter 4 and order 70 = 2jk - 2, showing that $v(24, 4) = 2jk - 2$.

**Case 9:** $j = 8$. If $\kappa = 2j + 1 = 17$, then $t \leq \left\lfloor \frac{2j+1}{j-3} \right\rfloor = 3$, which is impossible. The case $\kappa = 2j + 3$ is impossible since $d$ is odd; therefore $\kappa = 2j + 2 = 18$. By (1), $t \leq \left\lfloor \frac{2j+2}{j-4} \right\rfloor = 4$, giving $t = 3$ and $p(G) = 6$. Since $d = 21 > 7 \cdot 6/2 - 1 = (2t+1)p(G)/2 - 1$, Lemma 3.3 implies that $\text{diam}(G^*) = \text{diam}(G)$. From $k = 4$ we obtain $|V(G^*)| \geq 12$ by Theorem 2.5, and hence $|V(G)| \geq 6 \cdot 12 > 64 = 2jk$. 

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Case 10: \( j = 7 \). If \( \kappa = 2j + 3 = 17 \), then \( t \leq \left\lfloor \frac{2j+3}{j-3} \right\rfloor = 8 \), a contradiction. Suppose that \( \kappa = 2j + 2 = 16 \). By (1) we have \( t \leq \left\lfloor \frac{2j+2}{j-3} \right\rfloor = 5 \), and so \( t = 4 \) and \( p(G) = 4 \). From \( d = 18 > 9 \cdot 4/2 - 1 = (2t+1)p(G)/2 - 1 \), we deduce that \( \text{diam}(G^*) = \text{diam}(G) \), by Lemma 3.3. Since \( k = 4 \), Theorem 2.5 shows that \( |V(G^*)| \geq 14 \), and then \( |V(G)| \geq 4 \cdot 14 \geq 56 = 2jk \).

Suppose now that \( \kappa = 2j + 1 = 15 \). By (1) we have \( t \leq \left\lfloor \frac{2j+1}{j-3} \right\rfloor = 3 \), and therefore \( t = 3 \) and \( p(G) = 5 \). The facts that \( d = 18 > 7 \cdot 5/2 - 1 = (2t+1)p(G)/2 - 1 \) together with Lemma 3.3 imply that \( \text{diam}(G^*) = \text{diam}(G) \). Thus, \( |V(G^*)| \geq 4k - 4 \), by Theorem 2.5, which shows that \( |V(G)| \geq 14k + 2(3k-10) > 2jk \) since \( k \geq 4 \).

Case 11: \( j = 6 \). Since \( d \) is odd, we have \( \kappa \leq 2j+2 \). If \( \kappa = 2j+1 = 13 \), then \( t \leq \left\lfloor \frac{2j+1}{j-3} \right\rfloor = 4 \), which is impossible as neither 3 nor 4 divides 13. Suppose that \( \kappa = 2j + 2 = 14 \). By (1), \( t \leq \left\lfloor \frac{2j+2}{j-3} \right\rfloor = 7 \), and so \( t = 7 \) and \( p(G) = 2 \). From \( d = 15 > 15 \cdot 2/2 - 1 = (2t+1)p(G)/2 - 1 \) and from Lemma 3.3 we have \( \text{diam}(G^*) = \text{diam}(G) \). As \( k = 4 \), we have \( |V(G^*)| \geq 24 \) by Theorem 4.2, and so \( |V(G)| \geq 2 \cdot 24 \geq 48 = 2jk \). Now suppose that \( \kappa = 2j = 12 \). By (1), \( t \leq \left\lfloor \frac{2j}{j-3} \right\rfloor = 3 \), and so \( t = 3 \) and \( p(G) = 4 \). Lemma 3.3 and the inequality \( d = 15 > 7 \cdot 4/2 - 1 = (2t+1)p(G)/2 - 1 \) imply \( \text{diam}(G^*) = \text{diam}(G) \). Theorem 2.5 then shows that \( |V(G^*)| \geq 4k - 4 \), and from \( k \geq 4 \) we obtain \( |V(G)| \geq 12k + 4(k-4) \geq 2jk \).

Case 12: \( j = 5 \). If \( \kappa = 2j = 10 \), then \( t \leq \left\lfloor \frac{2j}{j-3} \right\rfloor = 3 \), a contradiction. Similarly, if \( \kappa = 2j + 1 = 11 \), then \( t \leq \left\lfloor \frac{2j+1}{j-3} \right\rfloor = 5 \), which is impossible. Thus, \( \kappa \geq 2j + 2 \), and then \( k = 4 \), \( d = 12 \) and \( |V(G)| \geq 2jk - 2 \). However, \( C_{38}^6 \) is a vertex-transitive graph of degree 12, diameter 4 and order 38 = 2jk – 2, which shows that \( v(12,4) = 2jk - 2 \).

Case 13: \( j = 4 \). Then, \( d = 9 \). Since \( A_{2k-1} \circ K_2 \) is a vertex-transitive graph of degree 9 and diameter \( k \) on \( 2jk - 4 \) vertices, we have \( v(9,k) = 2jk - 4 \).

5 Remarks

In the computer assisted generation of large graphs of a given degree and diameter described in [11] it turned out that some of the record large vertex-transitive graphs found as lifts of small quotients turned out to be Cayley graphs. Here we have the case that among our examples of smallest vertex-transitive graphs of a given degree and diameter there is always at least one Cayley graph. In the Table below we give, for each such case (and with a reference to the relevant theorem) an example of the corresponding Cayley graph of order \( v(d,k) \) by listing a group and a generating set that is closed under taking inverse elements and does not contain the unit element of the group. We use the standard additive notation for cyclic groups \( Z_n \) of order \( n \). There is one appearance of a dihedral group \( D_{2k} \) of order \( 2k \) in the Table, presented in the form \( \langle \alpha, \beta; \alpha^2 = \beta^2 = (\alpha \beta)^k = e \rangle \), with \( e \) being the unit element. If a variable in the fifth column is not quantified, it means that it can have any value in the corresponding group. The star in the bottom left cell is included to make the reader cautious.
that there are exceptions to the set of degrees listed in this cell, formed by the degrees referred to in Theorem 4.3 (i) – (iv).

<table>
<thead>
<tr>
<th>Degree</th>
<th>Diameter</th>
<th>$v(d, k)$</th>
<th>Group</th>
<th>Generating set</th>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>even $d$</td>
<td>$d + 1$</td>
<td>$Z_{d+1}$</td>
<td>$Z_{d+1} \setminus {0}$</td>
<td>2.2 (i)</td>
<td></td>
</tr>
<tr>
<td>odd $d$</td>
<td>$d + 3$</td>
<td>$Z_{d+3}$</td>
<td>$Z_{d+3} \setminus {0, \pm 1}$</td>
<td>2.2 (ii)</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>$2d + 2$</td>
<td>$Z_{d+1} \times Z_2$</td>
<td>${(a, 1); a \neq 0}$</td>
<td>2.2 (iv)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$k \geq 4$</td>
<td>$2k$</td>
<td>$Z_{2k}$</td>
<td>${\pm 1}$</td>
<td>2.5 (i)</td>
</tr>
<tr>
<td>3</td>
<td>$k \geq 4$</td>
<td>$4k - 4$</td>
<td>$Z_{2k-2} \times Z_2$</td>
<td>${\pm 1, 0}$</td>
<td>2.5 (ii)</td>
</tr>
<tr>
<td>4</td>
<td>$k \geq 4$</td>
<td>$4k - 2$</td>
<td>$Z_{4k-2}$</td>
<td>${\pm 1, \pm 2}$</td>
<td>4.1 (iii)</td>
</tr>
<tr>
<td>$3j - 1$, $j \geq 2$</td>
<td>$k \geq 4$</td>
<td>$2jk$</td>
<td>$Z_{2k} \times Z_j$</td>
<td>${(0, x), (\pm 1, y); x \neq 0}$</td>
<td>4.2 (i)</td>
</tr>
<tr>
<td>$3j - 2$, $j \geq 3$</td>
<td>$k \geq 4$</td>
<td>$2jk$</td>
<td>$D_{2k} \times Z_j$</td>
<td>${x, y, (\alpha, y), (\beta, z); x, y \neq 0}$</td>
<td>4.3 (ii)</td>
</tr>
<tr>
<td>6</td>
<td>$k \geq 4$</td>
<td>$6k - 4$</td>
<td>$Z_{6k-4}$</td>
<td>${\pm 1, \pm 2, \pm 3}$</td>
<td>4.3 (i)</td>
</tr>
<tr>
<td>9</td>
<td>$k \geq 4$</td>
<td>$8k - 4$</td>
<td>$Z_{4k-2} \times Z_2$</td>
<td>${(0, 1), (\pm 1, x), (\pm 2, y)}$</td>
<td>4.3 (ii)</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>38</td>
<td>$Z_{38}$</td>
<td>${\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6}$</td>
<td>4.3 (i)</td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>70</td>
<td>$Z_{14} \times Z_5$</td>
<td>${(0, x), (\pm 1, y), (\pm 2, z); x \neq 0}$</td>
<td>4.3 (iv)</td>
</tr>
<tr>
<td>$3j - 3$, $j \geq 3^*$</td>
<td>$k \geq 4$</td>
<td>$2jk$</td>
<td>$Z_{2k} \times Z_j$</td>
<td>${(0, x), (\pm 1, y); x, y \neq 0}$</td>
<td>4.3 (v)</td>
</tr>
</tbody>
</table>

One may now introduce $c(d, k)$ to be the smallest order of a Cayley graph of degree $d$ and diameter $k$. Our finding not only imply that $v(d, k) = c(d, k)$ for all $d \geq 2$ and $k \geq 1$, but (leaving a few cases of small diameter aside) allow to formulate our Main result from the Introduction as follows.

**Theorem 5.1.** For every $d \geq 2$ and $k \geq 4$ we have $v(d, k) = c(d, k) = 2\left\lceil \frac{d+1}{3}\right\rceil k - \delta$, with $\delta = 4$ if $d \in \{3, 6, 9\}$, $\delta = 2$ if either $d = 4$ or $(d, k) \in \{(12, 4), (24, 4)\}$, and $\delta = 0$ otherwise.

This appears to be in a sharp contrast with the degree-diameter problem for vertex-transitive graphs. Along the parameter $V(d, k)$ mentioned in the Introduction one also studies the largest order $C(d, k)$ of a Cayley graph of degree $d$ and diameter $k$; see again [12] for a survey of results in this direction. Obviously, $V(d, k) \geq C(d, k)$, with an obvious equality if $d \leq 2$ or $k = 1$. In the nontrivial cases, however, that is, for $d \geq 3$ and $k \geq 2$, the only known cases of equality between the two parameters are $C(4, 2) = 4$, $C(3, 3) = V(3, 3) = 14$, $C(3, 5) = V(3, 5) = 60$, $C(3, 7) = V(3, 7) = 168$ and $C(3, 8) = V(3, 8) = 300$; the last three follow from the census [13] of cubic vertex-transitive graphs of order up to 1280. As regards inequalities, it is well known that $8 = C(3, 2) < V(3, 2) = 10$, and the census [13] implies that $24 = C(3, 4) < V(3, 4) = 30$, $72 = C(3, 6) < V(3, 6) = 82$ and $36 = C(7, 2) < V(7, 2) = 50$. In fact, for $d \geq 3$ and $k \geq 2$ there are just 21 pairs $(d, k)$ for which the exact value of $C(d, k)$ is known and these were determined earlier by M. Conder, see [17]. The situation is even worse for $V(d, k)$, and for $d \geq 3$ and $k \geq 2$ the only known exact
values seem to be the ones listed above. We believe that the inequality $C(d,k) < V(d,k)$ holds for infinitely many pairs $(d,k)$, although the current evidence is slim. As a further example we mention that running the Magma system [3] on the data of [14] one can show that $V(4,3) \geq 35$, while by [17] one has $C(4,3) = 30$.

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