Smallest vertex-transitive graphs of given degree and diameter

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Abstract

For every d and k we determine the smallest order of a vertex-transitive graph of degree d and diameter k, and in each such case we show that this order is achieved by a Cayley graph.

1 Introduction

Determination of the largest order $N(d,k)$ of a graph of maximum degree d and diameter k, known as the *degree/diameter problem*, has become a classical open question in extremal graph theory. The problem has generated considerable activity since its 1960 statement in the pioneering paper [8]; for a survey we refer to [12]. A spanning tree argument gives the *Moore bound* $N(d,k) \leq M(d,k)$ where $M(d,1) = 1 + d$ for any $d \geq 1$ and $M(d,k) = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$ for every $d, k \geq 2$. Apart from the trivial cases when $d \leq 2$ or $k = 1$, the Moore bound is known to be achieved only when $k = 2$ and $d = 3$, 7, and possibly 57, by results of [8] for diameters 2 and 3, and [1, 5] for all larger diameters. For all the remaining pairs (d,k) we have the inequality $N(d,k) \leq M(d,k) - 2$, proved for $k = 2$ in [6] and for all $k \geq 3$ in [2]. It has been conjectured in [7] that $N(d,k) \leq M(d,k) - 3$ for every $d, k \geq 4$; for results towards this conjecture and other related facts see [7] and references therein. More general improvements of the Moore bound currently appear to be beyond reach.

Motivated by the methods of generating large graphs of given degree and diameter [11], there has been growing interest in determining or at least estimating the largest order $V(d,k)$

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of a *vertex-transitive* graph of degree d and diameter k, cf. [12]. Indeed, about a half of the current largest orders of graphs of a given maximum degree and diameter, kept in online tables [16], have been found by searching over Cayley graphs or over lifts of small-order quotients resulting in vertex-transitive graphs, cf. [11]. A focus on such a restriction is also natural in the light of the fact that, except the case of diameter 2 and degree 57 (due to a result of Higman, see [4]), all the extremal graphs establishing the equality $N(d,k) = M(d,k)$ are vertex-transitive. A new impetus for research in this direction is the paper [9] where it was shown that vertex-transitivity can indeed keep the number $V(d,k)$ arbitrarily far away from the Moore bound in a considerable number of cases by proving that for every fixed integers $d \geq 3$ and $\delta \geq 1$, the set of positive integers k for which $V(d,k) \leq M(d,k) - \delta$ has positive lower asymptotic density.

In extremal graph theory it is customary to consider both extremes of any given scalar parameter of graphs. Here, however, one has to be careful about the formulation of the problem. Asking for the smallest order of a graph of maximum degree d and diameter k would not be interesting, as the answer is trivial: An extremal graph in this situation would have to contain an induced path on $k+1$ vertices and, leaving the trivial case of diameter 1 aside, for $k \geq 2$ one just needs to attach $d-2$ pendant edges to some inner vertex o f the path, which yields the smallest order $d+k-1$ if $d, k \geq 2$. Asking for the smallest order $n(d,k)$ of a graph of *minimum* degree d and diameter k changes the answer but one just needs a little extra effort to obtain it. Indeed, let v be a vertex of eccentricity k in an extremal graph and let x_i ($0 \leq i \leq k$) denote the number of vertices at distance i from v. The quantity $n(d,k)$ is then equal to the minimum of $x_0 + x_1 + \ldots + x_k$ over positive integers $x_0, x_1, \ldots, x_k$ with $x_0 = 1$ subject to the conditions that all of $x_0 + x_1$, $x_{k-1} + x_k$, and $x_{i-1} + x_i + x_{i+1}$, $1 \leq i \leq k-1$, be at least $d+1$, since these conditions are also sufficient for the existence of a corresponding graph. An evaluation gives $n(d,k) = (d-2)\lfloor k/3 \rfloor + d + k + 1$ for any $d \geq 3$ and $k \geq 3$, with similar formulae for the remaining cases. Almost the same values are obtained when asking for the smallest order of a *regular* graph with degree d and diameter k, see [10], where the most interesting part is the construction of extremal graphs.

Asking about the smallest order $v(d,k)$ of a *vertex-transitive* graph of degree d and diameter k, however, is a much more interesting question, both per se as well as in the context of the 'opposite end' of the vertex-transitive version of the degree-diameter problem. In this paper we solve the problem of determining $v(d,k)$ completely and our main result can be formulated in the following way:

Main Result. *For all $d \geq 2$ and $k \geq 4$ we have $v(d,k) = 2\lceil \frac{d+1}{3} \rceil k - \delta$, where $\delta = 4$ if $d \in \{3,6,9\}$; $\delta = 2$ if $d = 4$ or if $d \in \{12, 24\}$ and $k = 4$; and $\delta = 0$ otherwise.*

In particular, in terms of the product dk, the value of $v(d,k)$ is only about two times that of $n(d,k)$, which appears to be rather surprising.

The paper is organized as follows. We begin by treating small values of d and k in Section 2. To be able to handle the remaining values, in Section 3 we prove a few auxiliary results on vertex-transitive graphs. Section 4 then deals with large values of d and k and the final Section 5 contains a number of remarks.

2 Small values of the degree and the diameter

In the paper, G will always denote a vertex-transitive graph. For any vertex x of G and for any $i \geq 0$ we let $N_i(x)$ be the set of vertices of G at distance i from x; formally, $N_i(x) = \{y \in V(G);\ \mathrm{dist}_G(x, y) = i\}$. By vertex-transitivity, for every $i \geq 0$ the quantity $n_i = |N_i(x)|$ is independent of the choice of x. Dependence of the sets $N_i(x)$ and the numbers n_i on G will be automatically assumed throughout.

We begin with an observation which will be used frequently. By *connectivity* of a graph we will always mean vertex-connectivity.

Lemma 2.1. *Let G be a vertex-transitive graph of order n, degree $d \geq 2$, diameter $k \geq 3$ and connectivity κ. Then $n \geq 2d + 2 + (k-3)\kappa$.*

Proof. Let x and y be vertices of G at distance k. The set $N_0(x) \cup N_1(x)$ contains x and all the vertices adjacent to x, and $N_{k-1}(x) \cup N_k(x)$ contains y and all the vertices adjacent to y. For every i, $2 \leq i \leq k-2$, we have $n_i \geq \kappa$ since $N_i(x)$ is a cut set of G. It follows that $n = \sum_{i=0}^{k} n_i \geq (d+1) + (k-3)\kappa + (d+1)$. □

If H is a complete graph or a complete bipartite graph with equal parts, we let $H^{(-1)}$ and $H^{(-2)}$ denote the graph obtained from H by removing all edges of a perfect matching and a Hamilton cycle, respectively. For small diameters we have the following statement.

Theorem 2.2. *For every $d \geq 2$ we have the following:*

(i) *if $k = 1$ then $v(d, k) = d + 1$ and K_{d+1} is an extremal graph;*

(ii) *if $k = 2$ and d is even then $v(d, k) = d + 2$ and $\mathrm{K}_{d+2}^{(-1)}$ is an extremal graph;*

(iii) *if $k = 2$ and d is odd then $v(d, k) = d + 3$ and $\mathrm{K}_{d+3}^{(-2)}$ is an extremal graph;*

(iv) *if $k = 3$ then $v(d, k) = 2d + 2$ and $\mathrm{K}_{d+1,d+1}^{(-1)}$ is an extremal graph.*

Proof. The case $k = 1$ is obvious since K_{d+1} is the unique graph of diameter 1 and degree d.

Let $k = 2$. As in the proof of Lemma 2.1 one can show that $n_0 + n_1 \geq d + 1$ and $n_2 \geq 1$. Thus, $v(d, 2) \geq d + 2$. But there is no graph of odd degree on an odd number of vertices, and therefore $v(d, 2) \geq d + 3$ if d is odd. Since $\mathrm{K}_{d+2}^{(-1)}$ and $\mathrm{K}_{d+3}^{(-2)}$ are vertex-transitive graphs of diameter 2 and degree d for d even and odd, respectively, the result follows.

It remains to deal with the value $k = 3$. By Lemma 2.1 we have $v(d, 3) \geq 2d+2$. The graph $\mathrm{K}_{d+1,d+1}^{(-1)}$ is vertex-transitive and has degree d and diameter 3, implying $v(d, 3) = 2d + 2$. □

We now turn our attention to small values of the degree.

Lemma 2.3. *Let G be a vertex-transitive graph of connectivity κ and degree d, where $d \in \{3, 4, 6\}$. Then $\kappa = d$.*

Proof. The case $d = 3$ can be derived from [15, Theorem 3] and the cases $d = 4$ and $d = 6$ follow from [15, Corollary 3B]. $\square$

We will, however, need more detailed information regarding the parameters n_i in cubic graphs, that is, graphs of degree $d = 3$.

Lemma 2.4. *Let G be a cubic vertex-transitive graph of diameter $k \geq 4$. Then $n_i \geq 4$ for every i such that $2 \leq i \leq k - 2$.*

Proof. For each vertex x of G we let $N_{<j}(x) = \cup_{i<j} N_i(x)$. By way of contradiction, suppose that there is some t, $2 \leq t \leq k-2$, such that $n_t \leq 3$. By Lemma 2.3 we have $n_t = 3$. Let a_x, b_x and c_x be the three vertices of $N_t(x)$; obviously, each of these has a neighbour in $N_{t-1}(x)$. But each of a_x, b_x and c_x must also have a neighbour in $N_{t+1}(x)$, since otherwise there is a 2-cut in G, which contradicts Lemma 2.3.

In what follows we fix one vertex v of G. Let u be a neighbour of v lying on a shortest $v - a_v$ path. Then all of a_u, b_u, c_u belong to $N_{t-1}(v) \cup N_t(v) \cup N_{t+1}(v)$. Since a_v has a neighbour in $N_{t+1}(v)$ and this neighbour lies also in $N_t(u)$, we may assume that $a_u \in N_{t+1}(v)$ and $a_v a_u$ is an edge of G.

Observe that $N_t(u) \not\subseteq \cup_{i \geq t} N_i(v)$, since otherwise for every $x \in N_{<t}(v) \cup \{a_v\}$ there would be a $v - x$ path avoiding $N_t(u)$ and this would mean that $|N_{<t}(u)| \geq |N_{<t}(v)| + 1$, a contradiction. Thus, there is a vertex of $N_t(u)$, say, c_u, such that $c_u \in N_{t-1}(v)$. We distinguish three cases, depending on the position of b_u in our graph.

Case 1: $b_u \in N_{t+1}(v)$. If neither b_v nor c_v were reachable from v in $G - \{c_u, a_v\}$, then $\{a_v, c_u\}$ would be a 2-cut, contradicting Lemma 2.3. On the other hand, if there is a vertex of $N_t(v)$, say b_v, with the property that there is a $v - b_v$ path in $G - \{c_u, a_v\}$, then $N_{<t}(u)$ would contain the sets $N_{<t}(v) \backslash \{c_u\}$ and $\{a_v, b_v\}$, implying that $|N_{<t}(u)| \geq |N_{<t}(v)| + 1$, a contradiction.

Case 2: $b_u \in N_{t-1}(v)$. If $b_v \notin N_{t-1}(u)$ and $c_v \notin N_{t-1}(u)$, then $b_v, c_v \in N_{t+1}(u)$. Consequently, $N_{<t}(u) \subseteq (N_{<t}(v) \setminus \{b_u, c_u\}) \cup \{a_v\}$, and so $|N_{<t}(u)| \leq |N_{<t}(v)| - 1$, a contradiction. We may therefore assume that $b_v \in N_{t-1}(u)$. Since all vertices of $N_{t+1}(v)$ are contained in $N_t(u) \cup N_{t+1}(u) \cup N_{t+2}(u)$ and $b_u, c_u \in N_{t-1}(v)$, the vertex a_u is the unique neighbour of b_v in $N_{t+1}(v)$. One can show in a similar way that a_u is the unique neighbour of a_v in $N_{t+1}(v)$. Thus, $\{a_u, c_v\}$ is a 2-cut, a contradiction.

Case 3: $b_u \in N_t(v)$. Since $a_v \in N_{t-1}(u)$, we have either $b_u = b_v$ or $b_u = c_v$. Without loss of generality, suppose that $b_u = b_v$. Because of $(N_{<t}(v) \setminus \{c_u\}) \cup \{a_v\} \subseteq N_{<t}(u)$ we

have $N_{<t}(u) = (N_{<t}(v) \setminus \{c_u\}) \cup \{a_v\}$, which implies that $c_v \in N_{t+1}(u)$. Since there is an edge joining a vertex of $N_{t-1}(v)$ with c_v, and since $N_{t-1}(v) \setminus \{c_u\} \subseteq N_{<t}(u)$, it follows that $c_u c_v \in E(G)$ and, moreover, c_v has no other neighbour in $N_{t-1}(v)$. From $c_v \in N_{t+1}(u)$ and $a_v \in N_{t-1}(u)$, we conclude that there is no edge in G joining the vertices a_v and c_v. By a similar argument the same conclusion applies to the vertices a_u and c_u. But as $u, v \in N_t(b_v)$ and $uv \in E(G)$, there is an edge within $N_t(u)$ and, analogously, there is an edge within $N_t(v)$. That is, one of $a_u b_v$, $b_v c_u$ and also one of $a_v b_v$, $b_v c_v$ are edges of G. Consequently, one of the three neighbours of b_v must lie in $N_t(v)$. As stated earlier, b_v has also neighbours in both $N_{t-1}(v)$ and $N_{t+1}(v)$. If $a_u b_v \in E(G)$ then, since a_u is the unique neighbour of a_v in $N_{t+1}(v)$ (otherwise $N_t(u)$ would not be a cut-set), the set $\{a_u, c_v\}$ would be a 2-cut, a contradiction. The only remaining possibility is that $b_v c_u \in E(G)$. Then, since c_u is the unique neighbour of c_v in $N_{t-1}(v)$, the set $\{a_v, c_u\}$ would be a 2-cut in G. This contradiction completes the proof. □

Let P_n be a prism on $2n$ vertices, that is $\mathrm{P}_n = \mathrm{C}_n \square \mathrm{K}_2$, where $\square$ stands for the box product of graphs. Further, let A_n denote an antiprism on $2n$ vertices, that is $\mathrm{A}_n = \mathrm{C}_{2n}^2$, where G^2 is obtained from G by adding all edges xy between vertices x, y at distance 2 in G. For small degrees we now can state and prove the following.

Theorem 2.5. *For any $k \geq 4$ we have:*

(i) *if $d = 2$ then $v(d,k) = dk$, with C_{2k} as an extremal graph;*

(ii) *if $d = 3$ then $v(d,k) = (d+1)k - 4$, with P_{2k-2} as an extremal graph;*

(iii) *if $d = 4$ then $v(d,k) = dk - 2$, with A_{2k-1} as an extremal graph.*

Proof. Since there are only two non-isomorphic cycles of diameter k, namely C_{2k} and C_{2k+1}, the result is obvious for $d = 2$.

Let G be a cubic vertex-transitive graph of order n and diameter k, that is, we are in the case $d = 3$. As in the proof of Lemma 2.1 one can show that $n_0 + n_1 \geq d + 1 = 4$ and $n_{k-1} + n_k \geq 4$. By Lemma 2.4 for every i, $2 \leq i \leq k-2$, we have $n_i \geq 4$. It follows that $n = \sum_{i=0}^{k} n_i \geq 4k - 4$. Since P_{2k-2} is a cubic vertex-transitive graph of diameter k and order $4k - 4$, we have $v(3,k) = 4k - 4$.

Finally, let G be a vertex-transitive graph of order n, degree 4, diameter k, and connectivity κ. Lemma 2.1 implies that $n \geq 2d + 2 + (k-3)\kappa$. By Lemma 2.3 we have $\kappa = 4$, and so $n \geq 4k - 2$. Observing that A_{2d-1} is a vertex-transitive graph of order $4k - 2$, degree 4 and diameter k, we conclude that $v(4,k) = 4k - 2$. □

3 Auxiliary results on vertex-transitive graphs

We will keep to the notation introduced earlier and study the 'fine' structure of vertex-transitive graphs. For a graph G with connectivity κ we let $C(G)$ denote the set of all vertex-cuts of size κ in G. Let

$$p(G) = \min\{\min\{|V(P)|;\ P \text{ is a component of } G - C\};\ C \in C(G)\}.$$

Following [15], if $C \in C(G)$, a component P of $G - C$ is said to be an *atomic part* of G if $|V(P)| = p(G)$. The following theorem summarizes and extends (in the last item) some results of [15] for vertex-transitive graphs we will need in our considerations.

Theorem 3.1. *Let G be a vertex-transitive graph and let P be an atomic part of G corresponding to a cut C. Then*

(i) *P is a vertex-transitive graph;*

(ii) *G is isomorphic to a disjoint union of several copies of P together with some edges joining them;*

(iii) *if P' is another atomic part of G, then either $V(P') \subseteq C$ or $V(P') \cap C = \emptyset$;*

(iv) *$\kappa = t \cdot p(G)$ for some $t \geq 2$;*

(v) *if G^* is a simple graph obtained from G by contracting every atomic part into a single vertex, then G^* is vertex-transitive.*

Proof. Items (i) and (ii) are proved in [15, Theorem 2], (iii) is proved in [15, Lemma 3.5] and (iv) is proved in [15, Lemma 4.1]. In fact, in [15] the proofs are made for the case $\kappa < d$, but the case $\kappa = d$ is trivial (and useless) as then every atomic part is a single vertex.

It remains to prove (v). Let φ be an automorphism of G. As the set of vertices $x \in V(G) \setminus V(P)$ adjacent to a vertex of P is a cut-set in G, its φ-image, that is, the set of vertices $y \in V(G) \setminus V(\varphi(P))$ adjacent to a vertex of $\varphi(P)$, forms a cut-set in G as well. Thus, the image of an atomic part is again an atomic part, which means that φ induces an automorphism of G^*. Take two vertices, say u_1 and u_2 of G^* and denote P_1 and P_2, respectively, the corresponding atomic parts in G. Further, choose two vertices x_1 and x_2 in G such that $x_1 \in V(P_1)$ and $x_2 \in V(P_2)$. Since G is vertex-transitive, there is an automorphism in G mapping x_1 to x_2, and this automorphism induces an automorphism of G^* mapping u_1 onto u_2. Hence, G^* is a vertex-transitive graph. □

Throughout the rest of this paper, G^* and $t = t(G)$ will denote the objects associated with G as introduced in Theorem 3.1; note that t is the degree of G^*.

The next statement is a slight generalization of [15, Theorem 3].

Theorem 3.2. *Let G be a vertex-transitive graph of connectivity κ and degree d, and let $t = t(G)$. Then $t \geq 2$ and*

$$\kappa \geq \frac{t}{t+1}\Big(d+1\Big).$$

Proof. By Theorem 3.1 (iv), we have $\kappa = t \cdot p(G)$, where $t \geq 2$. Any vertex in an atomic part P is adjacent to at most $|V(P)| - 1$ vertices in P and at most $p(G)$ vertices of every atomic part that is included in the cut-set corresponding to P. It follows that

$$d \leq p(G) - 1 + t \cdot p(G),$$

and so

$$\kappa = t \cdot p(G) \geq \frac{t}{t+1}\Big(d+1\Big).$$

□

For any given degree one may regard Theorem 3.2 as giving a lower bound on κ in terms of a function of t. Later in the proof of Theorem 4.3 we will need the reverse inequality; since by Theorem 3.2 we have $t \leq \kappa/(d+1-\kappa)$, it follows that

$$t \leq \left\lfloor \frac{\kappa}{d+1-\kappa} \right\rfloor. \tag{1}$$

We proceed with presenting a tool which will, in some cases, help reduce our problem for a given degree to a smaller one.

Lemma 3.3. *Let G be a vertex-transitive graph of degree $d \geq 3$ and diameter $k \geq 4$. Let G^* and t be as in Theorem 3.1 and let k^* be the diameter of G^*. If $d > (2t+1)p(G)/2 - 1$, then $k^* = k$.*

Proof. For any $x \in V(G)$ let P_x be the atomic part containing x and let z_x be the corresponding vertex of G^*. If $z_x z_y \in E(G^*)$ then we will say that P_x is *adjacent* to P_y and vice versa. Denote by $P_{x,1}, P_{x,2}, \ldots, P_{x,t}$ the atomic parts adjacent to P_x. Let $(\delta_0(x); \delta_1(x), \delta_2(x), \ldots, \delta_t(x))$ be a sequence such that $\delta_0(x)$ is the degree of x in P_x and $\delta_i(x) = |\{xy \in E(G);\ y \in P_{x,i}\}|$, where $1 \leq i \leq t$. (We note that one can deduce from Theorem 3.1 that the multiset $\{\delta_1(x), \delta_2(x), \ldots, \delta_t(x)\}$ does not depend on the choice of $x \in V(G)$, but we will not use this here.) Observe that $\delta_0(x) + \delta_1(x) + \ldots + \delta_t(x) = d$, $\delta_0(x) \leq p(G) - 1$ and $\delta_i(x) \leq p(G)$, where $1 \leq i \leq t$. By our assumption, for every $x \in V(G)$ and each i, $1 \leq i \leq t$, we have

$$\delta_i(x) > p(G)/2. \tag{2}$$

Obviously, $k^* \le k$, and we now prove the reverse inequality. Let $u, v \in V(G)$. We show that either $\mathrm{dist}_G(u,v) \le 3$ or $\mathrm{dist}_G(u,v) \le \mathrm{dist}_{G^*}(z_u, z_v)$, distinguishing three cases.

Case 1: $z_u = z_v$. Let P be an atomic part adjacent to P_u. According to (2), the vertices u and v have a common neighbour in P, and so $\mathrm{dist}_G(u,v) \le 2$.

Case 2: $z_u z_v \in E(G^*)$. By (2), there is a vertex $w \in P_v$ such that $uw \in E(G)$. Since in Case 1 we have already shown that $\mathrm{dist}_G(w,v) \le 2$, we have $\mathrm{dist}_G(u,v) \le 3$.

Case 3: $\mathrm{dist}_{G^*}(z_u, z_v) \ge 2$. Let $z_u, z_{x_1}, z_{x_2}, \dots, z_{x_{q-1}}, z_v$ be a shortest path from z_u to z_v in G^*. From (2) it follows that for every i, $2 \le i \le q-1$, there exists $w_i \in P_{x_i}$ such that $w_2, w_3, \dots, w_{q-1}, v$ is a path in G. Further, by (2) the vertices u and w_2 have a common neighbour in P_{x_1}. Thus, $\mathrm{dist}_G(u,v) \le \mathrm{dist}_{G^*}(z_u, z_v)$.

The proof can now be completed easily. By the assumption that $k \ge 4$, there are $u, v \in V(G)$ such that $\mathrm{dist}_G(u,v) \ge 4$. Since for all pairs $u, v \in V(G)$ such that $\mathrm{dist}_G(u,v) \ge 4$ we have $\mathrm{dist}_G(u,v) \le \mathrm{dist}_{G^*}(z_u, z_v)$, we conclude that $k \le k^*$. □

4 Large values of the degree and the diameter

Let H and H' be graphs. By $H \circ H'$ we denote the graph on the vertex set $\{(x,y);\ x \in V(H) \text{ and } y \in V(H')\}$, in which a pair of distinct vertices $(x_1,y_1)(x_2,y_2)$ forms an edge if either $x_1x_2 \in E(H)$ or $x_1 = x_2$ and $y_1y_2 \in E(H')$. The operation $\circ$ is known as the *lexicographic product*. We will need two more operations derived from $\circ$, applicable to graphs satisfying certain extra conditions. Assume that H contains a pair of disjoint perfect matchings and let M_1 and M_2 be such a pair. Further, for $i \in \{1,2\}$ let $\overline{M}_i = \{(x_1,y)(x_2,y);\ (x_1,y),(x_2,y) \in V(H \circ H') \text{ and } x_1x_2 \in M_i\}$. Then both $\overline{M}_1$ and $\overline{M}_2$ are perfect matchings in $H \circ H'$. By $H \circ^{-} H'$ we denote a graph obtained from $H \circ H'$ by removing the edges of $\overline{M}_1$, and by $H \circ^{=} H'$ we denote a graph obtained from $H \circ H'$ by removing the edges of both $\overline{M}_1$ and $\overline{M}_2$.

As we shall see, in almost all cases considered in the theorems of this section, the graphs $\mathrm{C}_{2k} \circ \mathrm{K}_j$, $\mathrm{C}_{2k} \circ^{-} \mathrm{K}_j$ and $\mathrm{C}_{2k} \circ^{=} \mathrm{K}_j$ will be examples of extremal graphs, i.e., of order $v(d,k)$ for the corresponding values of d and k. An exception is Theorem 4.3 where we will also use powers of cycles. In general, if H is a graph and $t \ge 2$, then H^t, the t-th power of H, is the graph with $V(H^t) = V(H)$ in which $xy \in E(H^t)$ if and only if $\mathrm{dist}_H(x,y) \le t$.

Theorem 4.1. *Let $d = 3j-1$, $j \ge 2$ and let $k \ge 4$. Then $v(d,k) = 2jk$ and $\mathrm{C}_{2k} \circ \mathrm{K}_j$ is an extremal graph.*

Proof. By Theorem 3.2 and Theorem 3.1 (iv), we have $\kappa \ge \frac{t}{t+1}(d+1)$, where t is an integer such that $t \ge 2$. Thus, $\kappa \ge \frac{2}{3}(d+1) = 2j$. By Lemma 2.1 we obtain $v(d,k) \ge 2d + 2 + (k-3)2j = 2jk$. On the other hand, it is easy to see that $\mathrm{C}_{2k} \circ \mathrm{K}_j$ is a vertex-transitive graph of degree $3j-1$, diameter k and order $2jk$. □

Theorem 4.2. *Let $d = 3j - 2$, $j \geq 3$ and let $k \geq 4$. Then $v(d,k) = 2jk$ and $\mathrm{C}_{2k} \circ^{-} \mathrm{K}_j$ is an extremal graph.*

Proof. It is easy to see that $\mathrm{C}_{2k} \circ^{-} \mathrm{K}_j$ is a vertex-transitive graph of degree $3j-2$, diameter k and order $2jk$, implying that $v(d,k) \leq 2jk$. But by Theorem 3.2 and Theorem 3.1 (iv) we have $\kappa \geq \frac{t}{t+1}(d+1)$ for some integer $t \geq 2$. Thus, $\kappa \geq \frac{2}{3}(d+1) = 2j - \frac{2}{3}$, and so $\kappa \geq 2j$. By Lemma 2.1 we obtain $v(d,k) \geq 2d + 2 + (k-3)2j = 2jk - 2$. In the rest of the proof we show that there is no vertex-transitive graph of degree d and diameter k on $2jk-2$ or $2jk-1$ vertices.

By way of contradiction, suppose that there is a vertex-transitive graph G of degree d and diameter k of order $2j-2$ or $2j-1$. If $\kappa \geq 2j+2$, then by Lemma 2.1 $|V(G)| \geq 2d + 2 + (k-3)(2j+2) = 2jk + 2(k-4) \geq 2jk$, a contradiction. Hence, $\kappa \leq 2j+1$.

If the degree of G^* is 2, that is, if $t = 2$, then $\kappa = 2p(G)$ and so $p(G) = j$ and $\kappa = 2j$. By Theorem 3.1 (ii), j divides $|V(G)|$. Since $j \geq 3$, the graph G cannot have $2jk-2$ or $2jk-1$ vertices. Hence, $t \geq 3$.

Suppose that $\kappa = 2j+1$. Then $|V(G)| \geq 2d + 2 + (k-3)(2j+1) = 2jk + (k-5)$, and so necessarily $k = 4$ and $|V(G)| = 2jk - 1$. Since there is no regular graph of odd degree on an odd number of vertices, d is even and consequently j is even. If $j \geq 8$ then $\kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-3}{4} > 2j+1$, a contradiction. Thus, $j \leq 6$. We distinguish two cases.

Case 1: $j = 6$. Then, $\kappa = 2j+1 = 13$. But as $t \geq 3$ and $\kappa = t \cdot p(G) = 13$, we obtain $t = 13$ and $p(G) = 1$. Hence, every atomic part consists of a single vertex, which means that $\kappa = d$. Since $d = 3j-2 = 16 > 13$, this is impossible.

Case 2: $j = 4$. It follows that $\kappa = 2j+1 = 9$ and from $d = 3j-2 = 10$ we have $t = p(G) = 3$. The facts that $|V(G)| = 2jk-1 = 31$ and $3 \nmid 31$ now contradict Theorem 3.1 (ii).

It remains to consider the situation when $\kappa = 2j$. If $j \geq 4$ then $\kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-3}{4} > 2j$, a contradiction. Thus, $j = 3$, and then $\kappa = 2j = 6$. Since $d = 7$ and $\kappa = t \cdot p(G)$ for some $t \geq 3$, we have $t = 3$ and $p(G) = 2$. From $d = 7 > 6 = (2t+1)p(G)/2 - 1$ and Lemma 3.3 we have $\mathrm{diam}(G^*) = \mathrm{diam}(G)$. Since G^* is a cubic vertex-transitive graph of diameter k, by Theorem 2.5 it has at least $4(k-1)$ vertices. The inequality $k \geq 4$ finally implies $|V(G)| \geq 2 \cdot 4(k-1) = 6k + 2(k-4) \geq 6k = 2jk$. $\square$

Theorem 4.3. *Let $d = 3j-3$, $j \geq 3$, and let $k \geq 4$. We have:*

(i) *if $d = 6$ then $v(d,k) = 2jk - 4$ and C^3_{6k-4} is an extremal graph;*

(ii) *if $d = 9$ then $v(d,k) = 2jk - 4$ and $\mathrm{A}_{2k-1} \circ \mathrm{K}_2$ is an extremal graph;*

(iii) *if $d = 12$ and $k = 4$ then $v(d,k) = 2jk - 2$ and C^6_{38} is an extremal graph;*

(iv) *if $d = 24$ and $k = 4$ then $v(d,k) = 2jk - 2$ and $\mathrm{A}_7 \circ \mathrm{K}_5$ is an extremal graph.*

In all the remaining cases we have $v(d,k) = 2jk$ and $\mathrm{C}_{2k} \circ^{=} \mathrm{K}_j$ is an extremal graph.

Proof. It is easy to see that $\mathrm{C}_{2k} \circ\!\!= \mathrm{K}_j$ is a vertex-transitive graph of degree $3j-3$, diameter k and order $2jk$. Thus, $v(d,k) \leq 2jk$. On the other hand, by Theorem 3.2 and Theorem 3.1 (iv) we have $\kappa \geq \frac{t}{t+1}(d+1)$, where $t \geq 2$. It follows that $\kappa \geq \frac{2}{3}(d+1) = 2j - \frac{4}{3} > 2j-2$, and so $\kappa \geq 2j-1$. But if $t=2$, then $\kappa = 2p(G)$ and so κ has to be even, while if $t \geq 3$ then $\kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-6}{4} > 2j-1$ since $j \geq 3$. We conclude that $\kappa \geq 2j$. By Lemma 2.1 we have $v(d,k) \geq 2d+2+(k-3)2j = 2jk-4$. Since C^3_{6k-4} is a 6-regular graph of diameter k on $6k-4$ vertices, we have $v(6,k) = 6k-4$, which proves the theorem for $j=3$. In the remaining (and longer) part of the proof we will assume that there is a vertex transitive graph G of degree d and diameter k with $2jk-4 \leq |V(G)| < 2jk$. The result for $j=3$ has already been shown and so we assume $j \geq 4$.

If $\kappa \geq 2j+4$ then by Lemma 2.1 $|V(G)| \geq 2d+2+(k-3)(2j+4) \geq 2jk+4(k-4) \geq 2jk$, because $k \geq 4$; therefore $\kappa \leq 2j+3$. Suppose that the degree of G^* is 2, that is, $t=2$. Then $\kappa = 2j+2$ or $\kappa = 2j$. We consider the two cases separately.

Case 1: $\kappa = 2j+2$. Then $|V(G)| \geq 2d+2+(k-3)(2j+2) \geq 2jk+2(k-5) \geq 2jk$ if $k \geq 5$, implying that $k=4$ and $|V(G)| \geq 2jk-2 = 8j-2$. Since $\kappa = 2j+2 = 2p(G)$, we have $p(G) = j+1$. Consequently, $(j+1)$ divides $|V(G)|$ by Theorem 3.1 (ii).

If $|V(G)| = 2jk-2 = 8j-2$, then $(j+1)$ divides $8j-2$, and so $8j-2 = i(j+1)$ for some $i \leq 7$. If $i=7$, then $8j-2 = 7j+7$ which gives $j=9$ and $d=24$. This is exactly the case (iv) and later we show that there exists a graph G of order $2jk-2$ with the required properties. If $i=6$, then $8j-2 = 6j+6$ which gives $j=4$. Then $d=9$ and $\kappa=10$, which is impossible. If $i \leq 5$, then $j<4$, a contradiction.

Now suppose that $|V(G)| = 2jk-1$. Then $|V(G)| = 8j-1$ and $(j+1)$ divides $8j-1$, that is, $8j-1 = i(j+1)$ for some $i \leq 7$. If $i=7$, then $8j-1 = 7j+7$ which gives $j=8$. Then $d=21$ and $|V(G)| = 63$, which is impossible since both d and $|V(G)|$ cannot be odd. If $i \leq 6$ then $j<4$, a contradiction again.

Case 2: $\kappa = 2j$. We have already seen that $|V(G)| \geq 2jk-4$. Since $\kappa = 2j = 2p(G)$, we have $p(G) = j$ and, by Theorem 3.1 (ii), j divides $|V(G)|$. Since $j \geq 4$, we obtain $|V(G)| = 2jk-4$ and $j=4$, which means that $d=9$. This is exactly the case (ii) and later we show that there is a required graph G with $|V(G)| = 2jk-4$.

Thus, from now on we assume that $t \geq 3$. We already derived $2j \leq \kappa \leq 2j+3$, so let $\kappa = 2j+\ell$, where $0 \leq \ell \leq 3$. Since $\kappa \geq \frac{3}{4}(d+1) = 2j + \frac{j-6}{4}$, we have $j \leq 4\ell+6$. This implies the following.

- If $\kappa = 2j+3$ then $j \leq 18$ and $|V(G)| \geq 2d+2+(k-3)(2j+3) \geq 2jk+3(k-4)-1$. Thus, $|V(G)| \geq 2jk$ if $k \geq 5$ and so $k=4$ and $|V(G)| = 2jk-1$. Since $|V(G)|$ is odd, d must be even, which means that j must be odd.

- If $\kappa = 2j+2$ then $j \leq 14$ and $|V(G)| \geq 2jk+2(k-5)$. Thus, $k=4$ and $|V(G)| \geq 2jk-2$.

- If $\kappa = 2j+1$ then $j \leq 10$ and $|V(G)| \geq 2jk+(k-7)$. Thus, $k \leq 6$ and $|V(G)| \geq 2jk-3$.

- If $\kappa = 2j$ then $j \le 6$, $|V(G)| \ge 2jk - 4$ and there is no bound on k.

In what follows we consider every value of j, $j \le 18$, separately. As noted above, the cases $j = 18$ and $j = 16$ are impossible due to parity restrictions.

Case 1: $j = 17$. Then $\kappa = 2j + 3 = 37$. Since $\kappa = t \cdot p(G)$ and $t \ge 3$, we have $t = 37$. However, by (1) we have $t \le \lfloor \frac{2j+3}{j-5} \rfloor = 3$, a contradiction.

Case 2: $j = 15$. Then $\kappa = 2j + 3 = 33 = t \cdot p(G)$, where $t \ge 3$. By (1) we have $t \le \lfloor \frac{2j+3}{j-5} \rfloor = 3$, and so $t = 3$, $p(G) = 11$ and $d = 42$. Since $d = 42 > 7 \cdot 11/2 - 1 = (2t+1)p(G)/2 - 1$, we have $\mathrm{diam}(G^*) = \mathrm{diam}(G)$, by Lemma 3.3. From $k = 4$ we obtain $|V(G^*)| \ge 12$ by Theorem 2.5, which implies that $|V(G)| \ge 11 \cdot 12 > 120 = 2jk$.

Case 3: $j = 14$. Since d is odd, the case $|V(G)| = 2jk - 1$ is impossible and therefore $|V(G)| = 2jk - 2$ and $\kappa = 2j + 2 = 30 = t \cdot p(G)$. By (1), $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 3$, and so $t = 3$ and $p(G) = 10$. From $d = 39 > 7 \cdot 10/2 - 1 = (2t+1)p(G)/2 - 1$ it follows that $\mathrm{diam}(G^*) = \mathrm{diam}(G)$, by Lemma 3.3. Since $k = 4$, we have $|V(G^*)| \ge 12$ by Theorem 2.5, and so $|V(G)| \ge 10 \cdot 12 > 112 = 2jk$.

Case 4: $j = 13$. If $\kappa = 2j+3 = 29$, then $t \ge 3$ and $t \le \lfloor \frac{2j+3}{j-5} \rfloor = 3$ by (1), a contradiction. Thus, $\kappa = 2j + 2 = 28$. By (1) we have $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 3$, a contradiction.

Case 5: $j = 12$. Since d is odd, we have $\kappa = 2j + 2 = 26$. By (1), $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 3$, which is impossible.

Case 6: $j = 11$. If $\kappa = 2j + 3 = 25$ then $t \le \lfloor \frac{2j+3}{j-5} \rfloor = 4$, which is impossible as neither 3 nor 4 divides 25. Therefore $\kappa(G) = 2j + 2 = 24$. By (1), $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 3$, and so $t = 3$ and $p(G) = 8$. Since $d = 30 > 7 \cdot 8/2 - 1 = (2t+1)p(G)/2 - 1$, we have $\mathrm{diam}(G^*) = \mathrm{diam}(G)$, by Lemma 3.3. Since $k = 4$, we have $|V(G^*)| \ge 12$ by Theorem 2.5, and so $|V(G)| \ge 8 \cdot 12 > 88 = 2jk$.

Case 7: $j = 10$. Oddness of d implies that $\kappa \le 2j+2$. If $\kappa = 2j+2 = 22$, then $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 3$, a contradiction. Thus, $\kappa = 2j+1 = 21$. By (1), $t \le \lfloor \frac{2j+1}{j-3} \rfloor = 3$, and so $t = 3$ and $p(G) = 7$. Since $d = 27 > 7 \cdot 7/2 - 1 = (2t+1)p(G)/2 - 1$, we have $\mathrm{diam}(G^*) = \mathrm{diam}(G)$, by Lemma 3.3. Theorem 2.5 now shows that $|V(G^*)| \ge 4(k-1)$, and so $|V(G)| \ge 20k + 4(2k-7) > 2jk$ as $k \ge 4$.

Case 8: $j = 9$. If $\kappa = 2j + 1 = 19$, then $t \le \lfloor \frac{2j+1}{j-3} \rfloor = 3$ yields a contradiction, while if $\kappa = 2j + 3$, then $k = 4$ and $|V(G)| \ge 2jk - 1$. We therefore have to consider the case $\kappa = 2j + 2$. Then $k = 4$, $d = 24$ and $|V(G)| \ge 2jk - 2$. But $\mathrm{A}_7 \circ \mathrm{K}_5$ is a vertex-transitive graph of degree 24, diameter 4 and order $70 = 2jk - 2$, showing that $v(24, 4) = 2jk - 2$.

Case 9: $j = 8$. If $\kappa = 2j + 1 = 17$, then $t \le \lfloor \frac{2j+1}{j-3} \rfloor = 3$, which is impossible. The case $\kappa = 2j + 3$ is impossible since d is odd; therefore $\kappa = 2j + 2 = 18$. By (1), $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 4$, giving $t = 3$ and $p(G) = 6$. Since $d = 21 > 7 \cdot 6/2 - 1 = (2t+1)p(G)/2 - 1$, Lemma 3.3 implies that $\mathrm{diam}(G^*) = \mathrm{diam}(G)$. From $k = 4$ we obtain $|V(G^*)| \ge 12$ by Theorem 2.5, and hence $|V(G)| \ge 6 \cdot 12 > 64 = 2jk$.

Case 10: $j = 7$. If $\kappa = 2j + 3 = 17$, then $t \le \lfloor \frac{2j+3}{j-5} \rfloor = 8$, a contradiction. Suppose that $\kappa = 2j + 2 = 16$. By (1) we have $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 5$, and so $t = 4$ and $p(G) = 4$. From $d = 18 > 9 \cdot 4/2 - 1 = (2t+1)p(G)/2 - 1$, we deduce that $\mathrm{diam}(G^*) = \mathrm{diam}(G)$, by Lemma 3.3. Since $k = 4$, Theorem 2.5 shows that $|V(G^*)| \ge 14$, and then $|V(G)| \ge 4 \cdot 14 \ge 56 = 2jk$. Suppose now that $\kappa = 2j + 1 = 15$. By (1) we have $t \le \lfloor \frac{2j+1}{j-3} \rfloor = 3$, and therefore $t = 3$ and $p(G) = 5$. The facts that $d = 18 > 7 \cdot 5/2 - 1 = (2t+1)p(G)/2 - 1$ together with Lemma 3.3 imply that $\mathrm{diam}(G^*) = \mathrm{diam}(G)$. Thus, $|V(G^*)| \ge 4k - 4$, by Theorem 2.5, which shows that $|V(G)| \ge 14k + 2(3k-10) > 2jk$ since $k \ge 4$.

Case 11: $j = 6$. Since d is odd, we have $\kappa \le 2j+2$. If $\kappa = 2j+1 = 13$, then $t \le \lfloor \frac{2j+1}{j-3} \rfloor = 4$, which is impossible as neither 3 nor 4 divides 13. Suppose that $\kappa = 2j + 2 = 14$. By (1), $t \le \lfloor \frac{2j+2}{j-4} \rfloor = 7$, and so $t = 7$ and $p(G) = 2$. From $d = 15 > 15 \cdot 2/2 - 1 = (2t+1)p(G)/2 - 1$ and from Lemma 3.3 we have $\mathrm{diam}(G^*) = \mathrm{diam}(G)$. As $k = 4$, we have $|V(G^*)| \ge 24$ by Theorem 4.2, and so $|V(G)| \ge 2 \cdot 24 \ge 48 = 2jk$. Now suppose that $\kappa = 2j = 12$. By (1), $t \le \lfloor \frac{2j}{j-2} \rfloor = 3$, and so $t = 3$ and $p(G) = 4$. Lemma 3.3 and the inequality $d = 15 > 7 \cdot 4/2 - 1 = (2t+1)p(G)/2 - 1$ imply $\mathrm{diam}(G^*) = \mathrm{diam}(G)$. Theorem 2.5 then shows that $|V(G^*)| \ge 4k - 4$, and from $k \ge 4$ we obtain $|V(G)| \ge 12k + 4(k-4) \ge 2jk$.

Case 12: $j = 5$. If $\kappa = 2j = 10$, then $t \le \lfloor \frac{2j}{j-2} \rfloor = 3$, a contradiction. Similarly, if $\kappa = 2j + 1 = 11$, then $t \le \lfloor \frac{2j+1}{j-3} \rfloor = 5$, which is impossible. Thus, $\kappa \ge 2j + 2$, and then $k = 4$, $d = 12$ and $|V(G)| \ge 2jk - 2$. However, C_{38}^6 is a vertex-transitive graph of degree 12, diameter 4 and order $38 = 2jk - 2$, which shows that $v(12, 4) = 2jk - 2$.

Case 13: $j = 4$. Then, $d = 9$. Since $\mathrm{A}_{2k-1} \circ \mathrm{K}_2$ is a vertex-transitive graph of degree 9 and diameter k on $2jk - 4$ vertices, we have $v(9, k) = 2jk - 4$. □

5 Remarks

In the computer assisted generation of large graphs of a given degree and diameter described in [11] it turned out that *some* of the record large vertex-transitive graphs found as lifts of small quotients turned out to be Cayley graphs. Here we have the case that among our examples of smallest vertex-transitive graphs of a given degree and diameter there is *always* at least one Cayley graph. In the Table below we give, for each such case (and with a reference to the relevant theorem) an example of the corresponding Cayley graph of order $v(d, k)$ by listing a group and a generating set that is closed under taking inverse elements and does not contain the unit element of the group. We use the standard additive notation for cyclic groups Z_n of order n. There is one appearance of a dihedral group D_{2k} of order $2k$ in the Table, presented in the form $\langle \alpha, \beta;\ \alpha^2 = \beta^2 = (\alpha\beta)^k = \varepsilon \rangle$, with ε being the unit element. If a variable in the fifth column is not quantified, it means that it can have any value in the corresponding group. The star in the bottom left cell is included to make the reader cautious that there are exceptions to the set of degrees listed in this cell, formed by the degrees referred to in Theorem 4.3 (i) – (iv).

Degree	Diameter	$v(d,k)$	Group	Generating set	Theorem
d	1	$d+1$	Z_{d+1}	$Z_{d+1}\backslash\{0\}$	2.2 (i)
even d	2	$d+2$	Z_{d+2}	$Z_{d+2}\backslash\{0,(d+2)/2\}$	2.2 (ii)
odd d	2	$d+3$	Z_{d+3}	$Z_{d+3}\backslash\{0,\pm 1\}$	2.2 (iii)
d	3	$2d+2$	$Z_{d+1}\times Z_2$	$\{(a,1);\ a\neq 0\}$	2.2 (iv)
2	$k\geq 4$	$2k$	Z_{2k}	$\{\pm 1\}$	2.5 (i)
3	$k\geq 4$	$4k-4$	$Z_{2k-2}\times Z_2$	$\{(\pm 1,0),(0,1)\}$	2.5 (ii)
4	$k\geq 4$	$4k-2$	Z_{4k-2}	$\{\pm 1,\pm 2\}$	2.5 (iii)
$3j{-}1,\ j{\geq}2$	$k\geq 4$	$2jk$	$Z_{2k}\times Z_j$	$\{(0,x),(\pm 1,y);\ x\neq 0\}$	4.1
$3j{-}2,\ j{\geq}3$	$k\geq 4$	$2jk$	$D_{2k}\times Z_j$	$\{(\varepsilon,x),(\alpha,y),(\beta,z);\ x,y\neq 0\}$	4.2
6	$k\geq 4$	$6k-4$	Z_{6k-4}	$\{\pm 1,\pm 2,\pm 3\}$	4.3 (i)
9	$k\geq 4$	$8k-4$	$Z_{4k-2}\times Z_2$	$\{(0,1),(\pm 1,x),(\pm 2,y)\}$	4.3 (ii)
12	4	38	Z_{38}	$\{\pm 1,\pm 2,\pm 3,\pm 4,\pm 5,\pm 6\}$	4.3 (iii)
24	4	70	$Z_{14}\times Z_5$	$\{(0,x),(\pm 1,y),(\pm 2,z);\ x\neq 0\}$	4.3 (iv)
$3j{-}3,\ j{\geq}3^*$	$k\geq 4$	$2jk$	$Z_{2k}\times Z_j$	$\{(0,x),(\pm 1,y);\ x,y\neq 0\}$	4.3 (v)

One may now introduce $c(d,k)$ to be the smallest order of a Cayley graph of degree d and diameter k. Our finding not only imply that $v(d,k)=c(d,k)$ for all $d\geq 2$ and $k\geq 1$, but (leaving a few cases of small diameter aside) allow to formulate our Main result from the Introduction as follows.

Theorem 5.1. *For every $d\geq 2$ and $k\geq 4$ we have $v(d,k)=c(d,k)=2\lceil\frac{d+1}{3}\rceil k-\delta$, with $\delta=4$ if $d\in\{3,6,9\}$, $\delta=2$ if either $d=4$ or $(d,k)\in\{(12,4),(24,4)\}$, and $\delta=0$ otherwise.*

This appears to be in a sharp contrast with the degree-diameter problem for vertex-transitive graphs. Along the parameter $V(d,k)$ mentioned in the Introduction one also studies the largest order $C(d,k)$ of a Cayley graph of degree d and diameter k; see again [12] for a survey of results in this direction. Obviously, $V(d,k)\geq C(d,k)$, with an obvious equality if $d\leq 2$ or $k=1$. In the nontrivial cases, however, that is, for $d\geq 3$ and $k\geq 2$, the only known cases of equality between the two parameters are $C(4,2)=V(4,2)=13$, $C(3,3)=V(3,3)=14$, $C(3,5)=V(3,5)=60$, $C(3,7)=V(3,7)=168$ and $C(3,8)=V(3,8)=300$; the last three follow from the census [13] of cubic vertex-transitive graphs of order up to 1280. As regards inequalities, it is well known that $8=C(3,2)<V(3,2)=10$, and the census [13] implies that $24=C(3,4)<V(3,4)=30$, $72=C(3,6)<V(3,6)=82$ and $36=C(7,2)<V(7,2)=50$. In fact, for $d\geq 3$ and $k\geq 2$ there are just 21 pairs (d,k) for which the exact value of $C(d,k)$ is known and these were determined earlier by M. Conder, see [17]. The situation is even worse for $V(d,k)$, and for $d\geq 3$ and $k\geq 2$ the only known exact

values seem to be the ones listed above. We believe that the inequality $C(d,k) < V(d,k)$ holds for infinitely many pairs (d,k), although the current evidence is slim. As a further example we mention that running the `Magma` system [3] on the data of [14] one can show that $V(4,3) \geq 35$, while by [17] one has $C(4,3) = 30$.

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