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Line graph operation and small worlds

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Abstract

Complex networks, such as small world networks, are the focus of recent interest because of their potential as models for the interaction networks of complex systems. Most of the well-known models of small world networks are stochastic. The randomness makes it more difficult to gain a visual understanding of how networks are shaped, and how different vertices relate to each other. In this paper, we present and study a method for constructing deterministic small worlds using the line graph operator. This operator introduces cliques at every vertex of the original graph, which may imply larger clustering coefficients. On the other hand, this operator can increase the diameter at most by one and assure the small world property.

Keywords: large network, small world, line graph, diameter

1 Introduction

The Neural networks, transportation systems, biological and chemical systems, social networks, the Internet and the World Wide Web, are only a few examples of systems composed of a large number of highly interconnected dynamic units. A widely used approach for capturing global properties of large networks is to model them as graphs, whose vertices represent the objects or individuals and whose edges describe pairwise connections. Of course, this is a restrictive representation, since the interaction between two objects or individuals depends also on time, space and many other factors. From a practical point of view, such a representation provides a simple but still very informative model of the real network.

In this representation, real networks are characterized by correlations in the vertex degrees, by having relatively short paths between any two vertices, and by the presence of a large number of short cycles or specific motifs. This feature of having a relatively short path between any two vertices within a network, despite of its large size, is known as the *small world property.* It was first investigated, in the social context, by Milgram [12] in the 1960s in a series of experiments to estimate the actual number of steps in a chain of acquaintances.

The small world property has been observed in a variety of other real networks, including biological and technological ones, and is an obvious mathematical property in some network models, e.g., in random graphs. In contrast to random graphs, the small world property in real networks is often associated with the presence of clustering, indicated by high values of the clustering coefficient. For this reason, Watts and Strogatz [18] proposed to define *small world networks* as networks having both a short diameter, like random graphs, and a high clustering coefficient, like regular lattices. Thus, their model of a large network is situated between an ordered finite lattice and a random graph, presenting the small world property and high clustering coefficient. Soon after the appearance of [18], Barthélémy and Amaral [4] studied the origins of the small world behaviour, while Barrat and Weigt [3] addressed analytically as well as numerically the structure properties of the Watts–Strogatz model. Since then the study of complex networks, including small world networks, has experienced considerable progress as an interdisciplinary subject. Several excellent general reviews and books are available and we refer to them for the reader who would like to obtain more information on the topic [1, 2, 5, 13, 15, 16, 17]. In 2000, Kleinberg [10] extended the Watts–Strogatz model by explaining another important aspect of small world networks. He showed that the short paths not only exist but can be found using a simple greedy strategy with limited local information only. However, in our work we concentrate strictly on the basic properties of the Watts-Strogatz model and we leave these further improvements for future research.

Most well-known models of small world networks are stochastic. But deterministic models have the strong advantage that it is often possible to compute analytically their properties, for example, degree distribution, clustering coefficient, average path length, diameter, etc. Deterministic networks can be created by various techniques. We can modify regular graphs [6], or we can use standard graph operations such as the addition or the product of graphs [7], one can use recursive or iterative techniques based on the existence of cliques in a given network [8, 19, 20], and other mathematical methods.

In this paper, we focus on the small world network topology generated in a deterministic way, using the line graph operator. This deterministic approach enables one to obtain the relevant network parameters: degree distribution, clustering coefficient and diameter. We show that a network obtained in this way has strong clustering and a small diameter.

2 Definitions and notations

In this section we briefly introduce the important terms underlying our work and three axioms that must be satisfied by every Watts–Strogatz model of a small world network.

We consider only simple undirected connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We set $n = |V(G)|$ and $m = |E(G)|$. A *line graph* $L(G)$ has vertices corresponding to edges of G. That is, for every edge $e \in E(G)$ we have a vertex $v_e \in V(L(G))$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G share a common vertex. Denote $n' = |V(L(G))|$ and $m' = |E(L(G))|$. In the sequel we often use the fact that $n' = m$. We remark that the number of edges of $L(G)$ depends on the degree distribution in G.

The *diameter* of G is the greatest distance between two vertices in G:

$$\operatorname{diam}(G) = \max_{u,v \in V(G)} d(u, v). \tag{1}$$

Recall that the distance $d(u, v)$ is the number of edges in a shortest path starting at u and terminating at v. Regarding the diameter of line graphs, we will use the following statement given in [14]:

Theorem 1. *Let G be a connected graph with at least one edge. Then,*

$$\mathrm{diam}(G) - 1 \leq \mathrm{diam}(L(G)) \leq \mathrm{diam}(G) + 1.$$

A *clustering coefficient* is a measure of the degree to which vertices in a graph tend to cluster together and its value is always between 0 and 1. We can define a local and a global clustering coefficient. The *(local) clustering coefficient* of a vertex v of G, $\mathrm{CC}_G(v)$, is the ratio of the total number of existing connections between the neighbours $N_G(v)$ of v and the number of all possible connections between them. (Since G has no loops, $v \notin N_G(v)$.) We remark that if v has degree 0 or 1, then we set $\mathrm{CC}_G(v) = 0$. A *(global) clustering coefficient* can then be obtained by averaging the local clustering coefficients of all vertices of G, that is,

$$\mathrm{CC}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \mathrm{CC}_G(v). \tag{2}$$

There are two more definitions we need to include here. An edge is an (a, b)-edge if it has one endvertex of degree a and the other of degree b. An edge is *good* if it either has at least one endvertex of degree at least 3, or it lies in a triangle. Otherwise, it is a *bad edge*.

Now we state axioms for a graph G to be a Watts–Strogatz model for a small world network, see [9, 16]:

(A1) *The graph G is sparse.* We require $|E(G)| \in O(n \lg n)$, that is, $|E(G)|/|V(G)| \in O(\lg |V(G)|)$.

(A2) *The diameter of G is small.* We require $\mathrm{diam}(G) \in O(\lg |V(G)|)$.

(A3) *The clustering coefficient* $\mathrm{CC}(G)$ *is large.* We require $\mathrm{CC}(G) \geq c$ for a positive constant c.

We remark that some authors prefer slightly different axioms. For example, they consider average distance instead of the diameter, or use Θ notation instead of our O notation, etc. In what follows, we study sufficient conditions under which these axioms are satisfied by line graphs.

3 Line graph operator and axioms of small worlds

Here we study which of the properties $(A1)$, $(A2)$ and $(A3)$ are preserved by the line graph operator. First, we consider the second axiom.

Proposition 2. *If G satisfies $(A2)$, then also $L(G)$ satisfies $(A2)$.*

Proof. Since the graph G is connected, $|V(L(G))| = m \geq n - 1$. By Theorem 1, $\operatorname{diam}(L(G)) \leq \operatorname{diam}(G) + 1$. Hence, $\operatorname{diam}(L(G)) \in O(\lg n) + 1 \subseteq O(\lg m)$. $\square$

In order to prove an analogue of Proposition 2 for $(A3)$, we first state two lemmas.

Lemma 3. *Let e be an edge in G. Then,*

(a) *If e is a bad edge, then $\mathrm{CC}_{L(G)}(v_e) = 0$;*

(b) *If e is a good edge, then $\mathrm{CC}_{L(G)}(v_e) \geq \frac{1}{3}$.*

Proof. Denote by u and v the endvertices of e. Further, denote by a (resp. b) the number of edges adjacent with e at u (resp. v). Without loss of generality we may assume that $a \leq b$. Observe that $\deg_G(u) = a + 1$ and $\deg_G(v) = b + 1$, so that e is an $(a+1, b+1)$ edge, and $\deg_{L(G)}(v_e) = a + b$. The vertex v_e is in two disjoint cliques in $L(G)$. The sizes of these cliques are $a + 1$ and $b + 1$, so there are at least $\binom{a}{2} + \binom{b}{2}$ edges among the vertices in $N_{L(G)}(v_e)$. In fact, if e is not in a triangle in G, then there are exactly $\binom{a}{2} + \binom{b}{2}$ edges among the vertices in $N_{L(G)}(v_e)$. Thus,

$$\mathrm{CC}_{L(G)}(v_e) \geq \frac{\binom{a}{2} + \binom{b}{2}}{\binom{a+b}{2}}.$$

If $a \leq 1$, $b \leq 1$ and e is not in a triangle, then the edge e is bad and $\binom{a}{2} + \binom{b}{2} = 0$. Hence, for every bad edge the clustering coefficient is zero.

On the other hand, if the endvertices of e have degrees at most 2 and e is in a triangle in G, then these endvertices have degrees exactly 2 and the two neighbours of v_e are adjacent in $L(G)$. That is, $\mathrm{CC}_{L(G)}(v_e) = 1$.

Now suppose that e has at least one endvertex of degree at least three, that is, $b \geq 2$. Since $a = 0$ and $b = 2$ implies $CC_{L(G)}(v_e) = 1$, we may assume that $a+b \geq 3$. Denote $\delta = \lfloor \frac{a+b}{2} \rfloor$. If $a < b$ then $\frac{\binom{a}{2}+\binom{b}{2}}{\binom{a+b}{2}} \geq \frac{\binom{a+1}{2}+\binom{b-1}{2}}{\binom{a+b}{2}}$. This means that

$$\mathrm{CC}_{L(G)}(v_e) \geq \begin{cases} \frac{\binom{\delta}{2}+\binom{\delta}{2}}{\binom{2\delta}{2}}, & a+b \text{ is even.} \\ \frac{\binom{\delta}{2}+\binom{\delta+1}{2}}{\binom{2\delta+1}{2}}, & a+b \text{ is odd.} \end{cases}$$

If $a+b$ is even then $a+b \geq 4$ and $\delta \geq 2$, so that $\mathrm{CC}_{L(G)}(v_e) \geq \frac{1}{3}$. On the other hand, if $a+b$ is odd then $a+b \geq 3$ and $\delta \geq 1$, and again $\mathrm{CC}_{L(G)}(v_e) \geq \frac{1}{3}$. Therefore, the clustering coefficient of good edges is always greater than or equal to $\frac{1}{3}$. $\square$

Bad edges in G can cause small $CC(L(G))$, and by Lemma 3, only vertices of degree at most 2 in G which do not lie in triangles can be the endpoints of bad edges. The next lemma uses this fact to bound the number of bad edges.

Lemma 4. *Let S be the set of vertices of degree at most 2 in G which do not lie in a triangle. Further, let S' be the set of vertices of $L(G)$ which correspond to bad edges in G. Then, $|S'| \leq |S|$.*

Proof. For a subset T of vertices of a graph, say H, by $\langle T \rangle$ we denote the subgraph of H induced by vertices of T. Observe that the subgraph $\langle S \rangle$ of G, as well as the subgraph $\langle S' \rangle$ of $L(G)$, is a collection of paths and cycles. We will consider connected components of $\langle S' \rangle$, and we show that each such component on t vertices in $L(G)$ corresponds to t edges in G that cover at least t vertices of S in G, where these sets of covered vertices are mutually disjoint. This gives $|S'| \leq |S|$.

Let P' be a component of $\langle S' \rangle$. Since all the vertices of P' correspond to $(1,2)$-edges and $(2,2)$-edges, we have $P' = L(P)$, where $V(P) \subseteq S$. Since P is either a cycle or a path, it has at most $|V(P)|$ edges. Hence, the vertices of P' correspond to edges in G which cover at least $|V(P')|$ vertices of S. $\square$

Theorem 5. *Let G be a connected graph satisfying $(A3)$. Then, $L(G)$ satisfies $(A3)$ as well.*

Proof. Let S be the set of vertices of degree at most 2 in G which do not lie in a triangle. Obviously, $\mathrm{CC}_G(v) = 0$ if $v \in S$. Thus,

$$\mathrm{CC}(G) = \frac{1}{n} \sum_{v \in V(G)} \mathrm{CC}_G(v) \leq \frac{1}{n}|V(G) \setminus S| = 1 - \frac{1}{n}|S|.$$

Further, let S' be the set of vertices of $L(G)$ which correspond to bad edges in G. By Lemma 3, we have

$$\mathrm{CC}(L(G)) = \frac{1}{m}\sum_{e\in E(G)} \mathrm{CC}_{L(G)}(v_e) \geq \frac{1}{3m}|E(G)\setminus S'| = \frac{1}{3}\Big(1-\frac{1}{m}|S'|\Big).$$

Since G is connected, we have $m \geq n-1$. But since trees do not satisfy $(A3)$, we infer $m \geq n$. Further, $|S'| \leq |S|$ by Lemma 4. Thus,

$$\frac{1}{n}|S| \geq \frac{1}{m}|S'| \qquad \text{which gives} \qquad 1-\frac{1}{n}|S| \leq 1-\frac{1}{m}|S'|.$$

Consequently, $\mathrm{CC}(G) \leq 3\,\mathrm{CC}(L(G))$. $\square$

In what follows, we prove that no subset of the assumptions $(A1)$, $(A2)$, $(A3)$ for G guarantees that a particular property (A_j) holds in $L(G)$, except that $(A2)$ in G is a sufficient condition for $(A2)$ in $L(G)$ (Proposition 2) and $(A3)$ in G is a sufficient condition for $(A3)$ in $L(G)$ (Theorem 5).

Proposition 6. *There exist graphs satisfying* $(A1)$ *and* $(A2)$*, but their line graphs do not satisfy* $(A3)$*.*

Proof. Let G be a graph obtained from the star $K_{1,2^r/r}$ by subdividing each edge $r-1$ times. Then $|V(G)| = n = 2^r+1$ and $\mathrm{diam}(G) = 2r \in O(\lg n)$. Hence, $(A2)$ holds for G. The graph obviously satisfies $(A1)$.

Notice that the graph $L(G)$ consists of a clique of size $2^r/r$ and $2^r/r$ paths of length $r-1$ attached to vertices of the clique. Each vertex of the clique has clustering coefficient at most 1 and any vertex not in the clique has clustering coefficient 0. Since $|V(L(G))| = 2^r$, we obtain

$$\mathrm{CC}(L(G)) = \frac{1}{2^r}\sum_{v_e\in V(L(G))} \mathrm{CC}_{L(G)}(v_e) \leq \frac{1}{2^r}\cdot\frac{2^r}{r} = \frac{1}{r} \in O\Big(\frac{1}{\lg n}\Big).$$

$\square$

The next proposition gives an example which shows that the assumptions $(A1)$ and $(A3)$ for G do not assure $(A2)$ in $L(G)$.

Proposition 7. *There exist graphs satisfying* $(A1)$ *and* $(A3)$*, but their line graphs do not satisfy* $(A2)$*.*

Proof. Let $G = P_n^2$, that is, the graph G is the second power of the n-path, where n is a large integer. Then, $|V(G)| = n, |E(G)| = m = 2n-3$ and $\operatorname{diam}(G) = \lfloor \frac{n}{2} \rfloor$. Hence, the graph G has diameter which grows linearly with the number of vertices n. However, the clustering coefficient of G is at least $\frac{1}{2}$, as every vertex has clustering coefficient at least $\frac{1}{2}$. Therefore, $(A3)$ holds for G. Obviously, G satisfies also $(A1)$.

The graph $L(G)$ consists of $n' = |V(L(G))| = 2n-3$ vertices and $\operatorname{diam}(L(G)) = \operatorname{diam}(G)$. Hence, $\operatorname{diam}(L(G)) = \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n'+3}{4} \rfloor$, so that $(A2)$ does not hold for $L(G)$. $\square$

As regards $(A1)$, even if a graph satisfies all $(A1)$, $(A2)$ and $(A3)$, its line graph does not necessarily satisfy $(A1)$:

Proposition 8. *There exist small worlds, whose line graphs are not small worlds. In particular, these line graphs do not satisfy* $(A1)$.

Proof. Let G be obtained from $K_{1,k}$ by attaching a 3-cycle at every leaf. Thus, G has $n = 3k+1$ vertices, $m = 4k$ edges and diameter 4. Obviously, G satisfies $(A1)$ and $(A2)$.

Regarding the degree distribution in G observe that $2k$ vertices are of degree 2, k others are of degree 3 and one (the central vertex) is of degree k. Moreover, each vertex of degree 2 has clustering coefficient 1, each vertex of degree 3 has clustering coefficient $\frac{1}{3}$, and the vertex of degree k has clustering coefficient 0. Thus,

$$\mathrm{CC}(G) = \frac{2k \cdot 1 + k \cdot \frac{1}{3} + 1 \cdot 0}{3k+1} = \frac{7k}{9k+3} = \frac{7}{9} - o(1).$$

As $\mathrm{CC}(G)$ is large enough, we conclude that G is a small world. However, its line graph $L(G)$ does not satisfy $(A1)$ since it has $m = 4k$ vertices and $\binom{k}{2} + 5k$ edges, which implies that $|E(L(G))| > \frac{1}{32}|V(L(G))|^2$. $\square$

4 Line graphs as small worlds

Now we pose assumptions on G under which $L(G)$ is a small world.

Theorem 9. *Let G be a graph in which all vertices are of degree $O(\sqrt{\lg m})$ except a constant number of them, and these exceptional vertices are of degree at most $O(\sqrt{m \lg m})$. Suppose that* $\operatorname{diam}(G) \in O(\lg m)$ *and that G has at most $c\,m$ bad edges, where $c < 1$ is some prescribed constant. Then, $L(G)$ is a small world with clustering coefficient at least $\frac{(1-c)}{3}$.*

Proof. Since edges in $L(G)$ correspond to pairs of adjacent edges in G, we easily infer that

$$|E(L(G))| = \sum_{u \in V(G)} \binom{\deg(u)}{2} \leq c_1\, n \lg m + c_2\, m \lg m \in O(m \lg m),$$

where c_1 and c_2 are constants. Hence, $L(G)$ satisfies $(A1)$. By Theorem 1,

$$\mathrm{diam}(L(G)) \leq \mathrm{diam}(G) + 1 \in O(\lg m),$$

so $L(G)$ satisfies $(A2)$ as well. Finally, by Lemma 3, $\mathrm{CC}(L(G)) \geq \frac{1}{3}(1-c)$. Since $c < 1$, $L(G)$ also satisfies $(A3)$. □

If G does not have vertices of "big" degree, we can prove the following statement.

Theorem 10. *Let G be a graph whose all vertices are of degree $O(\lg m)$. Suppose that $\mathrm{diam}(G) \in O(\lg m)$ and G has at most $c\,m$ bad edges, where $c < 1$ is some prescribed constant. Then, $L(G)$ is a small world with clustering coefficient at least $\frac{(1-c)}{3}$.*

Proof. If e is an edge whose endvertices have degrees a and b in G, then the vertex v_e corresponding to e has degree $a+b-2$ in $L(G)$. Hence, all vertices of $L(G)$ have degrees in $O(\lg m)$. Consequently, $|E(L(G))| \in O(m \lg m)$, so that $L(G)$ satisfies $(A1)$. The properties $(A2)$ and $(A3)$ follow analogously as in the proof of Theorem 9. □

For regular graphs we obtain the following consequence of Theorem 10:

Corollary 11. *If a regular graph satisfies $(A1)$ and $(A2)$, then its line graph is a small world.*

Proof. Denote by k the degree of vertices in G. Observe that there are $\frac{1}{2}kn$ edges in G. Since G satisfies $(A1)$, we have $\frac{1}{2}kn \in O(n \lg n)$ and consequently $k \in O(\lg n)$. The graph G is connected and so $m \geq n-1$. Hence, $k \in O(\lg m)$.

Since G satisfies $(A2)$, we have $\mathrm{diam}(G) \in O(\lg n)$, and hence also $\mathrm{diam}(G) \in O(\lg m)$. Finally, G is not a cycle since it satisfies $(A2)$. Hence, G has no vertices of degree at most 2, so that all edges of G are good. This means that G satisfies all the assumptions of Theorem 10, and so $L(G)$ is a small world. □

Although Theorems 9 and 10 cover many cases (see the next section), they yield just sufficient conditions on G, under which $L(G)$ is a small world. Therefore, we pose here the following general problem.

Problem 1. *Characterize graphs, whose line graphs are small worlds.*

5 Line graphs of some common networks

Theorem 10 (or 9) can be applied to graphs which have "many" vertices and "small" diameter and degrees. Such graphs can be found among models of parallel computers. In this section we briefly mention some of these structures, as well as some other classes of graphs, satisfying the assumptions of Theorem 10.

Complete t-ary trees $T_{t,d}$. These are rooted trees where every non-leaf vertex has precisely t sons and every leaf has distance d from the root.

Notice that $T_{t,d}$ has $n = 1+t+t^2+\ldots+t^d = (t^{d+1}-1)/(t-1)$ vertices and the number of edges is one less, i.e. $m = n-1$. The diameter is $2d$. Hence, $\mathrm{diam}(T_{t,d}) \in O(\lg m)$ and also the maximum degree $t+1 \in O(\lg m)$. Since $t \geq 2$, the tree $T_{t,d}$ has no bad edges. Consequently, Theorem 10 implies that $L(T_{t,d})$ is a small world.

Toroidal graphs C_a^d. These graphs are defined to be Cartesian products of fixed length cycles, i.e., $C_a^d = C_a \Box C_a \Box \cdots \Box C_a$.

The numbers of vertices and edges of C_a^d are equal to $n = a^d$ and $m = \frac{a^d \cdot 2d}{2} = a^d \cdot d$. Hence, $m/n = \frac{a^d \cdot d}{a^d} \in O(\lg n)$. The diameter of C_a^d satisfies $\mathrm{diam}(C_a^d) = d\lfloor \frac{a}{2} \rfloor$ and hence, $\mathrm{diam}(C_a^d) \in O(\lg n)$, since we consider a to be a constant. Therefore, C_a^d satisfies (A1) and (A2). Since C_a^d is a regular graph of degree $2d$, by Corollary 11 we obtain that $L(C_a^d)$ is a small world.

Grid powers P_a^d. These graphs are Cartesian products of fixed length paths, $P_a^d = P_a \Box P_a \Box \cdots \Box P_a$, where a is a fixed number. They generalize hypercubes as $Q_d = P_a^d$ for $a = 2$. The graph P_a^d has $n = a^d$ vertices, diameter $d(a-1)$, its maximum degree is $2d$ and its minimum degree is d. By Theorem 10, we conclude that $L(P_a^d)$ is a small world.

Butterfly B_d. Vertices are pairs (w, i) where $0 \le i \le d$ and w is a d-bit binary number. Two vertices (w, i) and (w', i') are adjacent if and only if $i' = i + 1$ and either $w = w'$ or w and w' differ in precisely the i'th bit, see [11, pg 440].

The graph B_d has $n = 2^d(d+1)$ vertices whose degree is either 2 or 4. If $d \ge 2$ then it has no bad edges. Its diameter is in $O(\lg n)$, see [11, pg 442]. By Theorem 10, $L(B_d)$ is a small world.

We remark that the line graph of the wrapped butterfly is a small world as well.

Two-dimensional mesh of trees M_d. This graph is constructed from a $2^d \times 2^d$ grid of isolated vertices by adding $2 \cdot 2^d$ complete binary trees (one for each row and one for each column) each with 2^d leaves which are identified with all the vertices of one row (one column) of the grid, see [11, pg 280].

The number of vertices of M_d is $n = 3 \cdot 2^{2d} - 2^{d+1}$. It is easy to see that $\mathrm{diam}(M_d) = 4d \in O(\lg(n))$, the maximum degree is 3 and M_d has no bad edges. By Theorem 10, $L(M_d)$ is a small world.

We remark that higher dimensional meshes of trees satisfy the assumptions of Theorem 10 as well.

Cube-connected-cycles CCC_d. This graph is constructed from the d-dimensional hypercube, where $d \ge 3$, by replacing every vertex of the hypercube with a cycle of length d. Observe that CCC_d is 3-regular graph with $n = d2^d$ vertices. The diameter of CCC_d is in $\Theta(\lg n)$, see [11, pg 451]. Hence, by Corollary 11, $L(\mathrm{CCC}_d)$ is a small world.

Shuffle-exchange graph S_d. The vertices of this graph are binary numbers of length d. Two vertices, say u and v, are adjacent if and only if either u and v differ precisely in the last digit or u is a left or right cyclic shift of v.

Obviously, S_d has $n = 2^d$ vertices. The maximum degree of S_d is 3 and although it has vertices of degree 1, it does not have bad edges. The diameter of S_d is in $\Theta(\lg n)$, see [11, pg 474]. Thus, by Theorem 10, $L(S_d)$ is a small world.

Analogously, the underlying graph of de Bruijn digraph satisfies all the assumptions of Theorem 10.

6 Conclusion and future work

Line graph operator is an excellent tool for creating small world networks as it splits the neighborhood of every vertex into two cliques and it increases the diameter at most by 1. In this paper we study under which conditions the line graph operator creates a (Watts-Strogatz) model of small world network, that is, a network with properties (A1), (A2) and (A3). In the future we expect to study other properties of complex networks, such as the scale-freeness, correlation coefficient and for specific networks, like those discussed in section 5, even the routing.

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