EFFICIENT DOMINATION IN CUBIC VERTEX-TRANSITIVE GRAPHS

MARTIN KNOR AND PRIMOŽ POTOČNIK

Abstract. An independent set of vertices S of a graph *dominates the graph efficiently* if every vertex of the graph is either in S or has precisely one neighbour in S. In this paper we prove that a connected cubic vertex-transitive graph on a power of 2 vertices has a set that dominates it efficiently if and only if it is not isomorphic to a Möbius ladder.

1. Introduction

Let X be a simple graph. A vertex u of X is dominated by a vertex v if either $u = v$ or u is adjacent to v. A set $S \subseteq V(X)$ is a *dominating set* of X provided that every vertex of $V(X)$ is dominated by a vertex in S, and is an *efficient dominating set* provided that every vertex of $V(X)$ is dominated by exactly one vertex in S. Note that an efficient dominating set is an independent set S of vertices of X, such that every vertex of X that is not contained in S has precisely one neighbour in S.

A graph is said to *admit efficient domination* if its vertex set contains an efficient dominating set. The notion of an efficient domination set has several interesting interpretations in other areas of discrete mathematics; for example, an efficient domination set is precisely a *perfect 1-code in a graph* (see [2]) as well as a *closed neighbourhood packing of a graph* (see [21]).

Determining whether a given graph admits efficient domination is an NP-complete problem (see [1]). In order to obtain any efficient characterisation of graphs admitting efficient domination, it is thus necessary to restrict to a suitably chosen class of graphs. Efficient domination has been studied in the context of several very special families of graphs, such as Cartesian and direct products of cycles [5, 8, 13], or circulants and other Cayley or general vertex-transitive graphs [6, 7, 9, 14]. The graphs in these special classes often exhibit a considerable level of symmetry, such as vertex-transitivity. (A graph is vertex-transitive if its automorphism group acts transitively on the set of the vertices.) It is thus natural to pose the following general problem:

Problem 1.1. *Characterise vertex-transitive graphs that admit efficient domination.*

In this paper we shall consider the above problem in the context of vertex-transitive graphs of the smallest interesting valency, namely valency 3. (Note that a regular graph of valence 2 admits efficient domination if and only if it is isomorphic to a disjoint union of cycles the lengths of which are all divisible by 3.)

Regular graphs of valence 3 are often called *cubic*. The study of cubic vertex-transitive graphs has a long and fruitful history, going back to Tutte's seminal paper [23], and later on, a heroic work of Coxeter, Frucht and Powers [4], who compiled an extensive hand-made

Key words and phrases. domination, cubic graph, vertex-transitive.

The authors were supported in part by a Slovak–Slovenian bilateral project.

census of cubic Cayley graphs. In the last decade, the major source of information on small cubic vertex-transitive graphs was a webpage [20], maintained by Gordon Royle, which contains an incomplete census of cubic vertex-transitive graphs on up to 258 vertices. This work was superseded only recently by Spiga, Verret and the second author of the present paper, who compiled a complete census of all cubic vertex-transitive graphs on up to 1280 vertices [15, 16].

By applying a brute force, depth-first search algorithm to the graphs presented in [15], we were able to decide which cubic vertex-transitive graphs on at most 76 vertices admit efficient domination (the complexity of such an algorithm prevented us from going much further than this number of vertices). The data is summarised in Table 1. (Note that orders that are not divisible by 4 are not listed, since cubic graphs of these orders clearly do not admit an efficient domination.) The data in Table 1 refers to connected graphs only. In fact, unless stated otherwise, all the graphs in this paper are assumed to be connected.

Table 1: Cubic vertex-transitive graphs of given order that admit efficient domination

$\|V\|$	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76
#	1	2	4	4	7	11	6	10	12	12	7	32	10	16	38	26	12	37	11
#D	1	1	2	3	2	6	2	9	3	6	1	23	2	8	4	25	2	19	1

First row: order of graphs;
Second row: number of cubic vertex-transitive graphs (of specified order);
Third row: number of cubic vertex-transitive graphs that admit efficient domination.

An obvious pattern that can be observed in Table 1 occurs at orders that are powers of 2, where all but one graph seem to admit efficient domination. The main result of this paper is a proof that this behaviour is not a speciality of small orders (see Theorem 1.2). The proof is inductive and uses the theory of lifting automorphisms along covering projections, as was presented in [10].

The *Möbius ladder* M_n is the cubic graph obtained from the cycle C_{2n} by adding a perfect matching connecting pairs of opposite vertices in C_{2n}. The edges of this perfect matching will be called *spokes*. Observe that M_n has $2n$ vertices and that the smallest Möbius ladders M_2 and M_3 are isomorphic to K_4 and $\mathrm{K}_{3,3}$, respectively.

Theorem 1.2. *Let m be an integer greater than or equal to 2 and let X be a connected simple cubic vertex-transitive graph with 2^m vertices. Then X does not admit efficient domination if and only if $m \geq 3$ and X is isomorphic to the Möbius ladder $\mathrm{M}_{2^{m-1}}$.*

The proof of the theorem is presented in Section 4. In Section 2 we prove some auxiliary results concerning certain special families of cubic vertex-transitive graphs, while Section 3 introduces the necessary theory of quotients and covers of graphs that is used essentially in the proof of Theorem 1.2.

2. Special families

In this section, we prove Theorem 1.2 for four specific families of graphs, which occur as special cases in the proof of Theorem 1.2. In particular, Lemma 2.2 proves one direction of Theorem 1.2.

For $n \geq 3$, let P_n denote the Cartesian product $\mathrm{C}_n \square \mathrm{K}_2$, called the *prism* on $2n$ vertices. Observe that P_4 is a cube.

Lemma 2.1. *A prism P_n admits efficient domination if and only if n is divisible by 4.*

Proof. Denote the vertices of P_n by $u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}$ in such a way that $E(\mathrm{P}_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i : i \in \mathbb{Z}_n\}$. Suppose that P_n has an efficient domination set S. Without loss of generality, we assume that $u_0 \in S$. Then u_1 and v_0 are dominated by u_0, implying that $v_0, v_1, u_1, u_2 \notin S$. But since v_1 has to be dominated by a vertex in S and two of its neighbours, namely v_0 and u_1, are already excluded from S, the third of its neighbours, namely v_2, is contained in S.

By repeating this argument with v_2 in place of u_0, we see that $u_4 \in S$, and proceeding in this way, we see that $u_k \in S$ if and only if $k \equiv 0 \bmod 4$ and $v_k \in S$ if and only if $k \equiv 2 \bmod 4$.

On the other hand, since u_{n-1} and v_0 are already dominated by u_0, we see that $u_{n-2}, v_{n-1} \notin S$, implying that $v_{n-2} \in S$. In view of the previous paragraph, this implies that $n - 2 \equiv 2 \bmod 4$, or equivalently, that $4 \mid n$.

To prove the sufficiency, observe that if $4 \mid n$, then $S = \{u_0, u_4, \dots, u_{n-4}, v_2, v_6, \dots, v_{n-2}\}$ is an efficient dominating set. □

Lemma 2.2. *The Möbius ladder* M_n *admits efficient domination if and only if* $n \equiv 2 \pmod 4$*, or equivalently, if and only if* $|V(\mathrm{M}_n)| \equiv 4 \pmod 8$*.*

Proof. Denote the vertices and edges of the prism P_n as in the proof of Lemma 2.1. It is easy to see that M_n can be obtained from P_n by removing the edges $u_{n-1}u_0, v_{n-1}v_0$ and replacing them by edges $u_{n-1}v_0, v_{n-1}u_0$. In the same way as in in the proof of Lemma 2.1, one can show that for $k \le n-1$ we have $u_k \in S$ if and only if $k \equiv 0 \bmod 4$ and $v_k \in S$ if and only if $k \equiv 2 \bmod 4$. However, here it also follows that $u_{n-2} \in S$, showing that $n \equiv 2 \bmod 4$.

On the other hand, if indeed $n \equiv 2 \bmod 4$, then $S = \{u_0, u_4, \dots, u_{n-2}, v_2, v_6, \dots, v_{n-4}\}$ is an efficient domination set. □

The next two lemmas deal with graphs of a very specific structure. For a graph X and a set of vertices $B \subseteq V(X)$, let $X[B]$ denote the subgraph of X induced by B. Similarly, for two disjoint sets $B_1, B_2 \subseteq V(X)$, let $X[B_1, B_2]$ denote the bipartite graph with vertex set $B_1 \cup B_2$ and an edge between a vertex $u \in B_1$ and a vertex $v \in B_2$ whenever uv is an edge of X.

Lemma 2.3. *Let X be a connected cubic graph the vertex set of which admits a partition $\mathcal{B} = \{B_0, B_1, \dots, B_{4k-1}\}$ into $4k$ sets B_i of equal size, such that the following holds:*

(i) *for each $i \in \mathbb{Z}_{4k}$, the graph $X[B_i]$ is edgeless;*
(ii) *for each $i \in \mathbb{Z}_{4k}$, the bipartite graph $X[B_{2i}, B_{2i+1}]$ is a perfect matching;*
(iii) *for each $i \in \mathbb{Z}_{4k}$, the bipartite graph $X[B_{2i-1}, B_{2i}]$ is a disjoint union of 4-cycles.*

Then X admits efficient domination.

Proof. We construct an efficient dominating set $S = S_0 \cup S_1 \cup \dots \cup S_{k-1}$, such that every S_i contains only vertices from $B_{4i} \cup B_{4i+1}$ and such that S_i dominates all the vertices of $B_{4i-1} \cup B_{4i} \cup B_{4i+1} \cup B_{4i+2}$ (the addition in subscripts computed within $\mathbb{Z}_{4k}$).

Let C_0 be an auxiliary graph with vertex set $B_0 \cup B_1$ and with edges of the following three types: Every 4-cycle of $X[B_{4k-1}, B_0]$ contains two vertices of B_0 and these two vertices are joined by an edge in C_0. Analogously, every 4-cycle of $X[B_1, B_2]$ contains two vertices of B_1 and these two vertices are joined by an edge in C_0. Finally, C_0 contains also all the edges of perfect matching $X[B_0, B_1]$. Observe that C_0 consists of cycles whose lengths are multiples of 4. Though C_0 is a bipartite graph, every vertex of C_0 has one neighbour in B_0 and one in B_1. Let S_0 be an independent set in C_0 of maximum size. Then $|S_0| = \frac{1}{2}|V(C_0)|$. We show that S_0 is a dominating set in $X[B_{4k-1} \cup B_0 \cup B_1 \cup B_2]$.

Let F be any 4-cycle in $X[B_{4k-1}, B_0]$ or in $X[B_1, B_2]$. Since C_0 contains two vertices of F and these two vertices are adjacent in C_0, one vertex of F is in S_0. Denote this vertex by u and denote by v the other vertex of F in C_0. Further, denote by z the other neighbour of v in C_0, that is, $z \neq u$. In X, the vertex u dominates all the vertices of F except v. However, since $u \in S_0$, we see that $z \in S_0$, and v is dominated by z in X. Thus, S_0 dominates all the vertices of F, which implies that S_0 is a dominating set in $X[B_{4k-1} \cup B_0 \cup B_1 \cup B_2]$. Since X is cubic and $|S_0| = |B_j|$, where $j \in \{4k-1, 0, 1, 2\}$, the set S_0 is an efficient dominating set in $X[B_{4k-1} \cup B_0 \cup B_1 \cup B_2]$.

Now construct $S_1, S_2, \ldots, S_{k-1}$ analogously as S_0. Then $S = S_0 \cup S_1 \cup \cdots \cup S_{k-1}$ is an efficient dominating set in X. □

Lemma 2.4. *Let X be a connected cubic graph the vertex set of which can be partitioned into two sets, B_0 and B_1, of equal size, in such a way that the graph $X[B_i]$ is a perfect matching for each $i \in \{0, 1\}$ and the graph $X[B_0, B_1]$ is a disjoint union of cycles of length 4. Then the graph X admits efficient domination.*

Proof. We proceed similarly as in the proof of Lemma 2.3. Let C be a graph, possibly with parallel edges, obtained from $X[B_0]$ by adding $\frac{1}{2}|B_0|$ edges in such a way that for every 4-cycle F of $X[B_0, B_1]$ we add to C an edge e_F joining the two endvertices of $V(F) \cap B_0$. Then C consists of even cycles. Let S be an independent set in C of maximum size. We show that S is a dominating set in X.

Let F be any 4-cycle in $X[B_0, B_1]$. Then one of the endvertices of e_F is in S. Denote this vertex by u and denote by v the other vertex of e_F. Further, denote by z a neighbour of v in C such that vz is an edge in $X[B_0]$. Then $u, z \in S$. Observe that if (u, v, u) is a cycle of length 2 in C, then $z = u$; otherwise $z \neq u$. In X, the vertex u dominates all the vertices of F except v, which is dominated by z. Thus, S dominates all the vertices of F which implies that S dominates X. Since X is cubic and $|S| = \frac{1}{2}|B_0|$, the set S is an efficient dominating set in X. □

3. Concerning graphs, covers and quotients

The main tool that will be used in the proof of Theorem 1.2 is the technique of normal quotients and regular covers. When talking about normal quotients, it is convenient to use a slightly more general definition of a graph, which allows the graphs to have loops, parallel edges and semiedges. In what follows, we briefly introduce this concept of a graph and refer the reader to [10, 12] for more detailed explanation.

A *graph* is an ordered 4-tuple $(D, V; \mathrm{beg}, \mathrm{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\mathrm{beg} : D \to V$ is a mapping which assigns to each dart x its *initial vertex* $\mathrm{beg}\, x$, and $\mathrm{inv} : D \to D$ is an involution which interchanges every dart x with its *inverse dart*, also denoted by x^{-1}.

The orbits of inv are called *edges*. The edge containing a dart x is called a *semiedge* if $\mathrm{inv}\, x = x$, a *loop* if $\mathrm{inv}\, x \neq x$ while $\mathrm{beg}\,(x^{-1}) = \mathrm{beg}\, x$, and is called a *link* otherwise. The *endvertices of an edge* are the initial vertices of the darts contained in the edge. Two links are *parallel* if they have the same endvertices.

A graph with no semiedges, no loops and no parallel links is called a *simple graph* and can be given uniquely in the usual manner, by its vertex-set and edge-set. Conversely, any simple graph, given in terms of its vertex-set V and edge-set E can be easily viewed as the graph $(D, V; \mathrm{beg}, \mathrm{inv})$, where $D = \{(u, v) \mid uv \in E\}$, $\mathrm{inv}(u, v) = (v, u)$ and $\mathrm{beg}(u, v) = u$ for any $(u, v) \in D$.

Let $X = (D, V; \mathrm{beg}, \mathrm{inv})$ and $X' = (D', V'; \mathrm{beg}', \mathrm{inv}')$ be two graphs. A *morphism of graphs*, $f \colon X \to X'$, is a function $f \colon V \cup D \to V' \cup D'$ such that $f(V) \subseteq V'$, $f(D) \subseteq D'$,

$f \circ \mathrm{beg} = \mathrm{beg}' \circ f$ and $f \circ \mathrm{inv} = \mathrm{inv}' \circ f$. A graph morphism is an *epimorphism* (*automorphism*) if it is a surjection (bijection, respectively). The group of automorphisms of a graph X is denoted by $\mathrm{Aut}\,(X)$. The graph X is called *vertex-transitive* (*dart-transitive*, respectively), provided that $\mathrm{Aut}\,(X)$ acts transitively on vertices (darts, respectively) of X. (Note that in the context of simple graphs, a *dart* is often called an *arc* of a graph; hence the term *arc-transitive* is also used as a synonym for dart-transtive.)

The *valency* of a vertex v is the number of darts having v as their initial vertex. A graph is *cubic* if all of its vertices have valency 3. The following lemma, the proof of which is trivial and is omitted, can serve as an illustration of the concepts defined above.

Lemma 3.1. *A connected cubic vertex-transitive graph is not simple if and only if it is isomorphic to one of the following graphs:*

1. *the dipole* D_3*, having two vertices and three parallel edges between them;*
2. *the graph* D_2'*, having two vertices, two parallel edges between them, and a semiedge attached to every vertex;*
3. *the graph* $\bar{\mathrm{C}}_{2n}$ *obtained from the cycle* C_{2n} *by attaching an edge parallel to every second edge of the cycle;*
4. *the graph* K_2° *obtained from the complete graph* K_2 *by attaching a loop to each of the two vertices;*
5. *the graph* K_2'' *obtained from* K_2 *by attaching a pair of semiedges to each of the two vertices;*
6. *the graph* C_n' *obtained from* C_n *by attaching a semiedge to every vertex of the cycle;*
7. *the graph* $\mathrm{K}_1^{\circ\prime}$ *obtained from* K_1 *by attaching a loop and a semiedge;*
8. *the graph* K_1''' *obtained from* K_1 *by attaching three semiedges.*

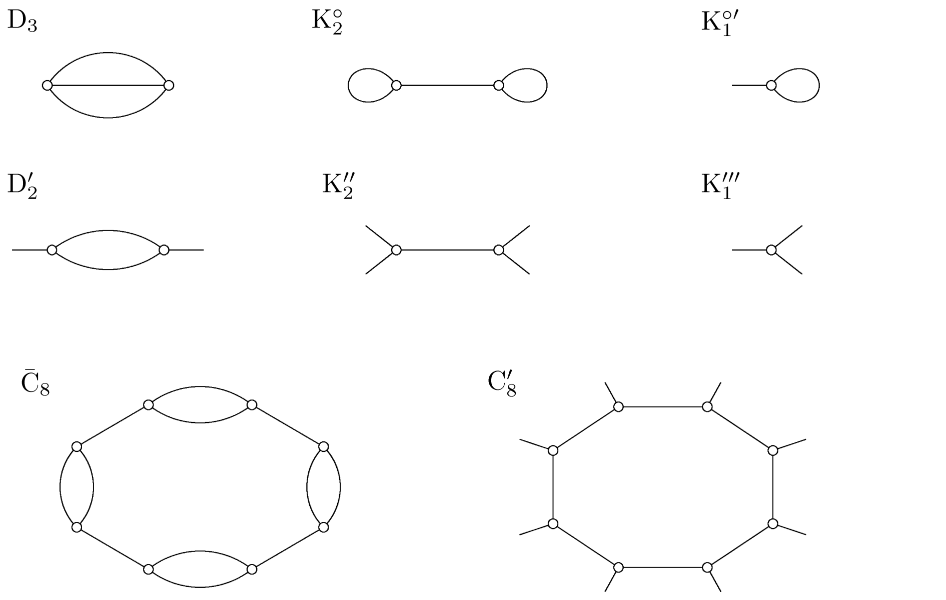

FIGURE 1. Quotients from Lemma 3.1.

We shall now describe the concept of a normal quotient (a concept that has roots in the work of Peter Lorimer [11] and was later developed into a powerful tool to study simple arc-transitive graphs by Cheryl E. Praeger [17, 18]). Note that the extension of this method to the more general graphs (as defined in this section) captures more information and can thus be used in some instances where the original method does not yield the desired results.

Let $N \le \mathrm{Aut}\,(X)$ and let D_N and V_N denote the sets of N-orbits on darts and vertices of X, respectively. Further, for a dart x of X and its N-orbit $[x] \in D_N$ let $\mathrm{beg}_N\,[x] = [\mathrm{beg}\ x]$ be the N-orbit of the vertex $\mathrm{beg}\ x$, and let $\mathrm{inv}_N\,[x] = [\mathrm{inv}\ x]$ be the N-orbit of the dart $\mathrm{inv}\ x$. This defines the *quotient graph* $X_N = (D_N, V_N; \mathrm{beg}_N, \mathrm{inv}_N)$ together with the obvious epimorphism $\wp_N \colon X \to X_N$, mapping $x \in V \cup D$ onto its N-orbit $[x]$, called the *quotient projection relative to* N.

If the group N acts on V semiregularly (that is, if the stabiliser N_v of any vertex v of X is trivial), then the quotient projection $\wp_N \colon X \to X_N$ is also a local bijection on darts and $\wp_N$ is called a *regular covering projection* (or also an N-covering projection if we want to specify the group N). In this case, the graph X can be reconstructed from X_N in terms of the voltage assignments on X_N; let us explain this in more detail.

Let $Y = (D_Y, V_Y, \mathrm{beg}_Y, \mathrm{inv}_Y)$ be an arbitrary connected graph, let N be a group and let $\zeta \colon D_Y \to N$ be a mapping (called a *voltage assignment*) satisfying the condition $\zeta(x) = \zeta(\mathrm{inv}_Y\, x)^{-1}$ for every $x \in D_Y$. Then $\mathrm{Cov}(Y, \zeta)$ is a graph with $D_Y \times N$ and $V_Y \times N$ as the sets of darts and vertices, respectively, and the functions beg and inv defined by $\mathrm{beg}(x, a) = (\mathrm{beg}_Y\ x, a)$ and $\mathrm{inv}(x, a) = (\mathrm{inv}_Y\ x, a\zeta(x))$. Note that there is a natural covering projection $\wp_\zeta \colon \mathrm{Cov}(Y, \zeta) \to Y$ mapping (x, a) onto x for any vertex or dart (x, a) of $\mathrm{Cov}(Y, \zeta)$. The following is a well-known fact in the theory of graph coverings (see [10, 12, 22], for example).

Lemma 3.2. *Let $\wp_N \colon X \to X_N$ be a regular covering projection and let T be a spanning tree in the graph X_N. Then the graph X is isomorphic to the graph $\mathrm{Cov}(X_N, \zeta)$ for some voltage assignment $\zeta \colon D(X_N) \to N$ which maps the darts of the tree T onto the trivial element of the group N. The isomorphism $f \colon X \to \mathrm{Cov}(X_N, \zeta)$ can be chosen in such a way that $\wp_N = f \circ \wp_\zeta$.*

Observe that the covering graph $\mathrm{Cov}(Y, \zeta)$ is connected if and only if the set $\{\zeta(x) : x \in D_Y\}$ generates the group N. Since ζ can be assumed to be trivial on the darts of a spanning tree, this implies that N can be generated by β elements, where β is the number of cotree edges of Y (this number is also known as the *Betti number of* Y). This fact is particularly useful when the group N is elementary abelian (that is, isomorphic to $\mathbb{Z}_p^\alpha$ for some prime p and some integer α). Namely, in this case we can conclude that $\alpha \le \beta$, whenever $\mathrm{Cov}(Y, \zeta)$ is connected.

Regular covering projections behave particularly nicely towards the group of automorphisms: Suppose that G is a subgroup of $\mathrm{Aut}\,(X)$ and that N is a normal subgroup of G (that is, $N \trianglelefteq G \le \mathrm{Aut}\,(X)$). Further, suppose that $\wp_N \colon X \to X_N$ is the corresponding quotient projection. If G acts transitively on the set of vertices (darts), then G/N acts transitively (but not necessarily faithfully) on the vertices (darts, respectively) as a group of automorphisms of X_N. If, in addition, N acts semiregularly on the vertices of X, then the quotient group G/N acts faithfully on the set $V_N \cup D_N$. In this case we say that the group G/N (and each of its elements) *lifts along* $\wp_N$. In particular, a group $H \le \mathrm{Aut}\,(X_N)$ *lifts along* $\wp_N$ if there exists some $G \le \mathrm{Aut}\,(X)$ containing N as a normal subgroup such that $G/N = H$.

There exists a very nice combinatorial condition for an automorphism of a graph Y to lift along a derived covering projection $\wp_\zeta \colon \mathrm{Cov}(Y, \zeta) \to Y$. For a directed cycle $C = (x_1, x_2, \ldots, x_n)$, traversing the darts $x_1, x_2 \ldots, x_n$ (in that order), define the voltage $\zeta(C)$ to be the product $\zeta(C) = \zeta(x_1)\zeta(x_2)\ldots\zeta(x_n)$. If this product happens to be the identity of the voltage group, then we say that C has a trivial voltage. The following criterion for an automorphism of a graph to have a lift was proved in [22].

Lemma 3.3. *Let Y be a connected graph with dart-set D and let $\zeta\colon D \to N$ be a voltage assignment. Then a group $H \leq \mathrm{Aut}\,(Y)$ lifts along the covering projection $\wp_\zeta\colon \mathrm{Cov}(Y,\zeta) \to Y$ if and only if each $g \in H$ preserves the set of cycles with trivial voltage.*

Lemma 3.3 will be used to determine all vertex-transitive $\mathbb{Z}_2$-covers of a Möbius ladder. But first we need to determine the automorphism group of M_n.

Lemma 3.4. *If $n \geq 4$, then* $\mathrm{Aut}\,(\mathrm{M}_n) \cong D_{2n}$.

Proof. Observe that if $n \geq 4$, then every spoke of M_n lies on two 4-cycles while a "non-spoke" edge lies on just one 4-cycle. Since D_{2n} is obviously a subgroup of $\mathrm{Aut}\,(\mathrm{M}_n)$, the set of spokes forms an orbit under $\mathrm{Aut}\,(\mathrm{M}_n)$. Consequently, the automorphism groups of M_n and the cycle C_{2n} obtained from M_n by removing the spokes are the same. Hence $\mathrm{Aut}\,(\mathrm{M}_n) \cong \mathrm{Aut}\,(\mathrm{C}_{2n}) \cong D_{2n}$, as claimed. □

Lemma 3.5. *Let n be an even integer greater than 2 and let $\wp\colon X \to \mathrm{M}_n$ be a $\mathbb{Z}_2$-covering projection along which a vertex-transitive subgroup of* $\mathrm{Aut}\,(\mathrm{M}_n)$ *lifts. Then X is isomorphic to the prism* P_{2n}.

Proof. Denote the vertices of M_n by $u_0, u_1, \ldots, u_{2n-1}$ in such a way that the edge set of M_n is $\{u_iu_{i+1};\ 0 \leq i \leq 2n-1\} \cup \{u_iu_{n+i};\ 0 \leq i \leq n-1\}$, the addition in subscripts computed in $\mathbb{Z}_{2n}$. In view of Lemma 3.2, we may assume that $X = \mathrm{Cov}(Y;\zeta)$ for some voltage assignment $\zeta\colon D(\mathrm{M}_n) \to \mathbb{Z}_2$. Moreover, ζ may be chosen in such a way that $\zeta(x) = 0$ for every dart on the spanning path $u_0, u_1, \ldots, u_{2n-1}$ of M_n. (In Figure 2, the edges of this spanning tree are depicted by thick lines.)

By Lemma 3.4, the automorphism group of M_n is isomorphic to D_{2n} in its natural action on the vertex set of M_n. Hence $\mathrm{Aut}\,(\mathrm{M}_n)$ has just two minimal vertex-transitive subgroups, namely $\langle\rho\rangle \cong C_{2n}$, where ρ is a rotation mapping $u_i \to u_{i+1}$, and $\langle\tau,\rho^2\rangle \cong D_n$, where τ is a reflection mapping $u_i \to u_{2n-1-i}$. In view of Lemma 3.3 one of these two subgroups preserves the set of cycles with voltage 0.

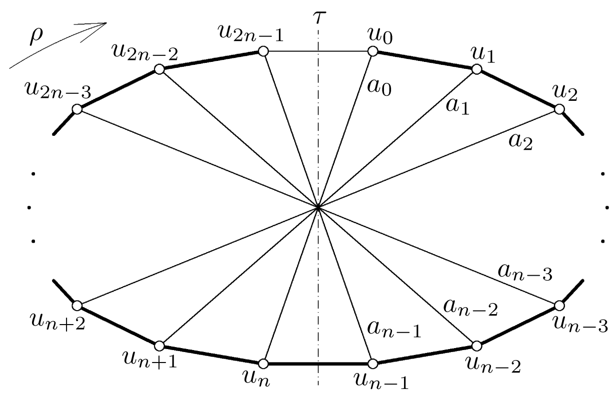

FIGURE 2. Möbius ladder M_n.

In what follows, we denote by C_i the $(n+1)$-cycle $(u_i, u_{i+1}, \ldots, u_{i+n})$, $0 \leq i \leq 2n-1$. Further, for a walk W in M_n, we denote by $\zeta(W)$ the sum of the voltages of the darts that W traverses in positive sense, and we denote by a_i the voltage of the dart (u_i, u_{n+i}); i.e. $a_i = \zeta(u_i, u_{n+i})$, $0 \leq i \leq n-1$. We distinguish two cases:

Case 1. *The automorphism ρ lifts.* Since the voltages on the path $u_0, u_1, \ldots, u_{2n-1}$ are all 0, we have $\zeta(C_i) = a_i$ for $0 \le i \le n-1$. Observe that $\rho(C_0) = C_1$. Hence, if $a_0 = 0$, then $\zeta(C_0) = 0$ and consequently $\zeta(C_1) = 0$ by Lemma 3.3. Since $a_1 = \zeta(C_1)$, we obtain $a_0 = a_1$. On the other hand, if $a_0 = 1$, then $\zeta(C_0) = 1$ and consequently $\zeta(C_1)$ cannot be 0 by Lemma 3.3. Thus $\zeta(C_1) = 1 = a_1$, and hence $a_0 = a_1$, as above. Since $\rho(C_i) = C_{i+1}$ for $i \in \{0, \ldots, n-2\}$, we see in a similar way as above that $a_0 = a_1 = \cdots = a_{n-1}$. However, $\rho(C_{n-1}) = C_n$. Hence $a_0 = a_{n-1} = \zeta(C_{n-1}) = \zeta(C_n) = a_0 + \zeta(u_{2n-1}u_0)$, which gives $\zeta(u_{2n-1}u_0) = 0$. Observe that if all the voltages in M_n are trivial, then the covering graph is disconnected, which contradicts our assumptions. Therefore $a_0 = a_1 = \cdots = a_{n-1} = 1$ and the lift of M_n is the prism P_{2n}.

Case 2. *The automorphisms τ and ρ^2 lift.* Analogously as above, since $\zeta(C_i) = a_i$ and $\rho^2(C_i) = C_{i+2}$, $0 \le i \le n-1$, we obtain $a_0 = a_2 = \cdots = a_{n-2}$ and $a_1 = a_3 = \cdots = a_{n-1}$ (observe that n is even). Moreover, since $\tau(C_0) = C_{n-1}$, we see that $a_0 = \zeta(C_0) = \zeta(C_{n-1}) = a_{n-1}$. Hence we see that $a_0 = a_1 = \cdots = a_{n-1}$. Since $\rho^2(C_{n-1}) = C_{n+1}$, we deduce that $a_{n-1} = \zeta(C_{n-1}) = \zeta(C_{n+1}) = a_1 + \zeta(u_{2n-1}u_0)$, which gives $\zeta(u_{2n-1}u_0) = 0$. Since not all the voltages in M_n are trivial, we see that $a_0 = a_1 = \cdots = a_{n-1} = 1$, and the lift of M_n is the prism P_{2n}, as required. □

In the context of "generalized graphs" $(V, D; \mathrm{beg}, \mathrm{inv})$, one needs to be careful when defining domination. The appropriate extension of domination from simple graphs is as follows:

Definition 3.6. *For a non-negative integer k we say that a vertex v of a graph $X = (V, D; \mathrm{beg}, \mathrm{inv})$ is* k-dominated *by a set $S \subset V$ provided that*

$$k = |\{x \in D \mid \mathrm{beg}(x) = v \text{ and } \mathrm{beg}(x^{-1}) \in S\}|.$$

If v is k-dominated by S for some $k \ge 1$, then we say that S dominates *v. A set $S \subseteq V$ dominates the graph X* efficiently *if every $v \in S$ is 0-dominated by S and every $v \in V \setminus S$ is 1-dominated by S. If X is dominated efficiently by some $S \subseteq V$, then we say that X* admits efficient domination*.*

Note that in view of the above definition S dominates every $v \in S$ such that $v = \mathrm{beg}(x)$ for some semiedge or some loop x. Moreover, if vertices u and v are adjacent by a pair of parallel edges and one is contained in S, then the other is k-dominated by S for some $k \ge 2$. This shows that a non-simple graph in which every vertex is an endpoint of a semi-edge, a loop or a pair of parallel edges, does not admit an efficient domination. In particular, no vertex-transitive non-simple graph admits an efficient domination.

The following lemma, which will be used substantially in the proof of Theorem 1.2, follows directly from the fact that covering projections are local bijections, that is, that for each vertex $\tilde{v}$ of a graph $\tilde{X}$ the set of darts $\tilde{x}$ with $\mathrm{beg}(\tilde{x}) = \tilde{v}$ projects by a covering projection $\wp\colon \tilde{X} \to X$ bijectively onto the set of darts x for which $\mathrm{beg}(x) = \wp(\tilde{v})$.

Lemma 3.7. *Let $\wp\colon \tilde{X} \to X$ be a covering projection. If a set $S \subseteq V(X)$ dominates X efficiently, then the preimage $\wp^{-1}(S) = \{\tilde{v} \in V(\tilde{X}) \mid \wp(\tilde{v}) \in S\}$ dominates the graph $\tilde{X}$ efficiently.*

4. Proof of Theorem 1.2

In the proof of Theorem 1.2, we will use some basic notions and results from group theory. For example, recall that a finite group is called a p-group (where p is a fixed prime) provided that the order of the group is of the form p^m for some integer $m \ge 0$; see [19, Chapter 4] for basic facts about p-groups. Further, we shall need the famous Lagrange theorem, stating that whenever a prime p divides the order of a group G, there exists an element $g \in G$ of order p. Finally, we shall use the well-known Burnside's $p^\alpha q^\beta$ theorem (see [3]), stating that every group whose order is divisible by at most two primes is soluble (see [19, Chapter 5] for basic facts on soluble groups). We shall also need the following folklore result (see also [4, pages 3–5]):

Lemma 4.1. *Let X be a connected cubic graph, let v be a vertex of X and let G be a group of automorphisms of X acting transitively on the vertex set of X but intransitively on the arcs of X. Then the vertex-stabiliser G_v is a (possibly trivial) 2-group.*

Proof. For a vertex u of X let $G_u^{X(u)}$ denote the permutation group induced by the action of G_u on the neighbourhood $X(u)$ of u in X. Observe that since G acts transitively on $V(X)$ but intransitively on the arcs of X, the permutation group $G_u^{X(u)}$ is intransitive, and thus, as an abstract group, either trivial or isomorphic to the group of order 2. Now suppose that G_v is not a 2-group and let p be an odd prime dividing the order of G_v. In view of Lagrange's theorem, there exists $g \in G_v$ of order p. Among all the vertices of X that are not fixed by g, let w be one which is closest to v and let $[v = v_0, v_1, \ldots, v_{m-1}, v_m = w]$ be a shortest path from v to w. By the choice of w, it follows that g fixes v_{m-1}, and thus $g \in G_{v_{m-1}}$. Now let U be the orbit of w under the action of the group $\langle g\rangle$. Since w is not fixed by g, we see that $|U| \geq 2$. On the other hand, by the well-known orbit-stabiliser theorem, $|U|$ divides the order of the group $\langle g\rangle$, implying that $|U| = p$. On the other hand, U is clearly a subset of $X(v_{m-1})$, and in fact, a proper subset (since G_u is intransitive on $X(u)$ for every $u \in V(X)$). Since X is a cubic graph, this implies that $|U| \leq 2$, which contradicts the fact that p is an odd prime. This contradiction shows that G_v is indeed a 2-group. □

We now have all the ingredients for the proof of Theorem 1.2. Suppose that Theorem 1.2 is false and let X be a minimal counter-example; that is, let X be a smallest connected simple cubic vertex-transitive graph on a power of 2 vertices, not isomorphic to a Möbius ladder on 8 or more vertices, which does not admit an efficient domination. Let m be the positive integer such that $|V(X)| = 2^m$, let $G = \mathrm{Aut}\,(X)$ and let v be a vertex of X.

The only connected simple vertex-transitive graphs on 4 or 8 vertices are the complete graph K_4, the cube Q_3 and the Möbius ladder M_4 (see [20]). Since the first two admit a perfect domination while the last one is a Möbius ladder, we may assume that $m \geq 4$.

Since X is vertex-transitive, it follows that $|G| = 2^m|G_v|$. If G acts transitively on the darts of X, then the famous theorem of Tutte [23] states that $|G_v| = 3 \cdot 2^r$ for some non-negative integer r not exceeding 4. On the other hand, if G acts intransitively on the darts of X, then Lemma 4.1 implies that G_v is a 2-group. In both cases the order of G is divisible by at most two primes, implying that G is soluble.

Let N be a minimal normal subgroup of G. Since G is soluble, N is elementary abelian (see, for example, [19, Theorem 5.24]). If N were a 3-group, then the length of each of its orbits would be either 1 or divisible by 3. Since $|V(X)|$ is not divisible by 3, this would imply that N fixes at least one vertex of X. However, being normal in a transitive permutation group G, N would then fix every vertex of N, which is clearly a contradiction.

We may thus assume that N is an elementary abelian 2-group. Moreover, if G acts intransitively on the darts of X, then G is a 2-group and must therefore have a nontrivial centre. In this case we can thus choose N to be isomorphic to $\mathbb{Z}_2$. (We shall use this fact later in the proof.)

Let X_N be the quotient graph of X with respect to N and let $\wp\colon X \to X_N$ be the corresponding quotient projection. We shall distinguish two cases, depending on whether $\wp$ is a covering projection or not.

Case 1. Suppose first that $\wp$ is a covering projection, or equivalently, that N acts semiregularly on $V(X)$. Then X_N is a cubic connected (not necessarily simple) vertex-transitive graph, and a vertex-transitive group of automorphisms of X_N lifts along $\wp$.

If X_N is simple, then in view of the fact that X is a minimal counter-example to Theorem 1.2, it follows that X_N either admits an efficient domination or it is isomorphic to the Möbius ladder M_r for some $r \geq 3$. In the former case, X admits efficient domination by Lemma 3.7. In the latter case (ie. if X_N is isomorphic to the Möbius ladder), the graph X_N is nor dart-transitive and in particular G acts intransitively on the darts of X. Recall that in this case we may assume that $N \cong \mathbb{Z}_2$, which allows us to use Lemma 3.5 to conclude that X is a prism, which, in view of Lemma 2.1, also admits efficient domination. In both cases, we obtain a contradiction with the assumption that X is a counter-example to Theorem 1.2. This shows that X_N is not simple, and is therefore isomorphic to a graph from Lemma 3.1.

Now recall the comment (regarding the Betti numbers) that follows Lemma 3.2. If X_N is isomorphic to D_3 (resp. K_1'''), then the Betti number of X_N is 2 (resp. 3), and N is a subgroup of $\mathbb{Z}_2^2$ (resp. $\mathbb{Z}_2^3$). This shows that the order of X_N is at most 8, contradicting our assumption that $m \geq 4$.

We may thus assume that X_N is not isomorphic to K_1''' or D_3. Note however that none of the other graphs from Lemma 3.1 is dart-transitive, implying that G does not act transitively on the darts of X. Recall that this implies that $N \cong \mathbb{Z}_2$. Therefore, if X_N has at most 2 vertices, then X has at most 4 vertices, contradicting our assumptions.

This leaves us with the possibility that X_N is isomorphic either to $\bar{\mathrm{C}}_{2^k}$ or C'_{2^k} for some $k \geq 3$. In view of Lemma 3.2, we see that $X \cong \mathrm{Cov}(X_N, \zeta)$ where $\zeta\colon D(X_N) \to \mathbb{Z}_2$ is a voltage assignment that can be chosen so as to be trivial on any prescribed spanning tree of X_N.

If $X_N \cong \mathrm{C}'_{2^k}$, then we may assume that $\zeta(x) = 0$ for all the darts along the cycle except possibly for one pair of mutually inverse darts (call them x_0 and x_0^{-1}). Moreover, since X is a simple graph, all the semiedges of X_N must receive a non-trivial voltage. It is now clear that the resulting covering graph $\mathrm{Cov}(X_N, \zeta)$ is isomorphic either to the prism P_{2^k} (if $\zeta(x_0) = 0$) or to the Möbius ladder M_{2^k} (if $\zeta(x_0) = 1$); the latter clearly contradicting our assumptions on X. Hence X is a prism whose order is divisible by 4, and therefore, by Lemma 2.1, admits an efficient domination.

On the other hand, if $X_N \cong \bar{\mathrm{C}}_{2^k}$, then we may assume that $\zeta(x) = 0$ for all the darts on a path of length $2^k - 1$ containing all the edges of $\bar{\mathrm{C}}_{2^k}$ that have no parallel counterparts. Further, since X is a simple graph, any two parallel edges must receive distinct voltages. Since the voltage group N has only two elements, this shows that the voltage assignment ζ is uniquely determined and gives rise to the simple graph consisting of 2^k vertex-fibres, call them $F_0 = \{u_0, w_0\}$, $F_1 = \{u_1, w_1\}, \ldots, F_{2^k-1} = \{u_{2^k-1}, w_{2^k-1}\}$, where between two consecutive fibres F_i and F_{i+1} we have a perfect matching (if i is even) or a complete bipartite graph $\mathrm{K}_{2,2}$ (if i is odd). Such a graph X satisfies the conditions of Lemma 2.3, and therefore admits a perfect matching. This contradiction completes Case 1.

Case 2. Suppose now that $\wp\colon X \to X_N$ is not a covering projection. Then N does not act semiregularly on $V(X)$ and X_N is a vertex-transitive graph of valence 1 or 2. Furthermore, since N is a 2-group, it cannot act transitively on the set of $3 \cdot 2^{m-1}$ edges of X. This shows that X_N has at least two edges, implying that the valence of X_N is 2 (rather than 1). If X_N consisted of one vertex only, then N would be vertex-transitive. But since N is abelian, this would imply that N is regular, which contradicts our assumption that N is not semiregular. This shows that X_N has at least two vertices.

Before proceeding, let us first prove the following: If X_N contains a link e between two vertices u and v such that the preimage $F = \wp^{-1}(e)$ induces a disjoint union of cycles of length k for some $k \geq 4$, then $k = 4$. Indeed: since N acts transitively on the edges of F, the vertex-stabiliser $N_{\tilde{v}}$ of a vertex $\tilde{v} \in \wp^{-1}(v)$ acts transitively on the set of its F-neighbours in $\wp^{-1}(v)$. On the other hand, since N is abelian, $N_{\tilde{v}}$ fixes every vertex in the fibre $\wp^{-1}(v)$, showing that the two F-neighbours of $\tilde{v}$ have the same neighbourhood, and thus lie on a 4-cycle consisting of edges in F. The graph induced by F is thus a disjoint union of 4-cycles, as claimed. We shall refer to this conclusion as Implication F.

If X_N has only two vertices, say u and w, then it is isomorphic either to the graph with a single link between u and w and a semiedge attached to each of u and w or to the graph with two links between u and w.

In the latter case, the $\wp$-preimage (call it F) of one of the two links between u and w is a perfect matching between the fibres $\wp^{-1}(u)$ and $\wp^{-1}(w)$. Since F is also an edge-orbit of N, this implies that every element g of the stabiliser $N_{\tilde{u}}$ of $\wp^{-1}(u)$ is also contained in the stabiliser $N_{\tilde{w}}$ of its F-neighbour $\tilde{w} \in \wp^{-1}(w)$. However, since N is abelian $g \in N_{\tilde{u}}$ fixes every vertex in the N-orbit $\tilde{u}^N = \wp^{-1}(u)$ and by the above, also every vertex in the N-orbit $\tilde{w}^N = \wp^{-1}(w)$. Hence $N_{\tilde{u}}$ is trivial, contrary to our assumption that N is not semiregular on the vertex-set of X.

If X_N has a single link between u and w and a semiedge attached to each of u and w, then either the preimage of that link is a perfect matching, or it induces in X a disjoint union of cycles of equal length. The former possibility leads to contradiction in the same way as in the previous paragraph. On the other hand, if the preimage of the link induces a disjoint union of cycles of equal length, then by Implication F above, these cycles have length 4. Further, the preimages of two semiedges form a perfect matching in each of the two vertex-fibres, showing that the graph X satisfies the assumptions of Lemma 2.4. The graph X thus admits an efficient domination, contrary to our assumptions.

This leaves us with the possibility that X_N has at least 3 vertices. Since its valence is 2, this implies that $X_N \cong \mathrm{C}_{4k}$ for some integer $k = 2^r$, $r \geq 0$. In particular, the vertex-set of X can be partitioned into $4k$ N-orbits B_i, $i \in \mathbb{Z}_{4k}$ such that an edge in B_i is adjacent only to vertices in B_{i-1} and B_{i+1}. Since the sets B_i are N-orbits, the graphs $X[B_i, B_{i+1}]$ induced by two consecutive sets B_i and B_{i+1} are regular. Since the valence of X is 3, this shows that for each $i \in \mathbb{Z}_{4k}$ one of the graphs $X[B_i, B_{i-1}]$ and $X[B_i, B_{i+1}]$ is regular of valence 1 and the other is regular of valence 2. In view of Implication F above, whenever $X[B_i, B_{i+1}]$ is of valence 2, then it is in fact a disjoint union of 4-cycles, showing that X satisfies the assumptions of Lemma 2.3. Hence X admits efficient domination. This contradiction finishes the proof of Theorem 1.2.

Acknowledgements. The first author acknowledges partial support by Slovak research grants VEGA 1/0871/11 and APVV-0223-10. Both authors acknowledge the support of bilateral Slovak-Slovenian grant.

References

[1] D. W. Bange, A. E. Barkauskas, P. J. Slater, Efficient Dominating Sets in Graphs, in R. D. Ringeisen,F. S. Roberts (Eds.) *Applications of discrete mathematics*, SIAM, Philadelphia (1988), 189–199.

[2] N. Biggs, Perfect codes in graphs *J. Combinatorial Theory Ser. B* **15** (1973), 289–296.

[3] W. Burnside, On groups of order $p^\alpha q^\beta$, *Proc. London Math. Soc.* (1904), 388–392.

[4] H. S. M. Coxeter, R. Frucht, D. L. Powers, *Zero-symmetric graphs: Trivalent Graphical Regular Representations of Groups*, Academic Press, New York (1981).

[5] I. J. Dejter, Perfect domination in regular grid graphs, *Australas. J. Combin.* **42** (2008), 99–114.

[6] I. J. Dejter, O. Serra, Efficient dominating sets in Cayley graphs, *Discrete Appl. Math.* **129** (2003), 319–328.

[7] J. Huang, J.-M. Xu, The bondage numbers and efficient dominations of vertex-transitive graphs, *Discrete Math.* **308** (2008), 571–582.
[8] S. Klavžar, S. Špacapan, J. Žerovnik, An almost complete description of perfect codes in direct products of cycles, Adv. in Appl. Math. **37** (2006), 2–18.
[9] J. Lee, Independent perfect domination sets in Cayley graphs, *J. Graph Theory* **37** (2001), 213–219.
[10] A. Malnič, R. Nedela and M. Škoviera, Lifting Graph Automorphisms by Voltage Assignments, *Europ. J. Combin.*, **21** (2000), 927–947.
[11] P. Lorimer, Vertex-transitive graphs: Symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55–68.
[12] A. Malnič, D. Marušič and P. Potočnik, Elementary abelian covers of graphs, to appear in *Journal of Algebraic Combinatorics*.
[13] M. Mollard, On perfect codes in Cartesian products of graphs, *European J. Combin.* **32** (2011), 398–403.
[14] N. Obradović, J. Peters, G. Ružić, Efficient domination in circulant graphs with two chord lengths *Inform. Process. Lett.* **102** (2007), 253–258.
[15] P. Potočnik, P. Spiga, G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, arXiv:1201.5317v1.
[16] P. Potočnik, P. Spiga, G. Verret, A census of small connected cubic vertex-transitive graphs, available online at `http://www.matapp.unimib.it/ spiga/`, accessed on April 12th, 2012.
[17] C. E. Praeger, Imprimitive symmetric graphs, *Ars Combin.* **19** (1985), 149–163.
[18] C. E. Praeger, An O'Nan-Scott Theorem for finite quasiprimitive permutation groups, and an application to 2-arc transitive graphs, *J. London Math. Soc.(2)* **47** (1993), 227–239.
[19] J. J. Rotman, *An Introduction to the Theory of Finite Simple Groups, Fourth Edition*, Graduate Texts in Mathematics **148**, Springer–Verlag, New York (1995).
[20] G. Royle, *Cubic Transitive Graphs*, available online at `http://mapleta.maths.uwa.edu.au/~gordon/remote/cubtrans/index.html`, accessed on November 16th, 2011.
[21] R. Rubalcaba, M. Walsh, Minimum fractional dominating functions and maximum fractional packing functions, *Discrete Math.* **309** (2009), 3280–3291.
[22] M. Škoviera, A contribution to the theory of voltage graphs, *Discrete Math.* **61** (1986), 281–292.
[23] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Phil. Soc* **43** (1947), 459–474.

Martin Knor, Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia
E-mail address: `knor@math.sk`

Primož Potočnik, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
E-mail address: `primoz.potocnik@fmf.uni-lj.si`