Wiener index of iterated line graphs of trees homeomorphic to the claw $K_{1,3}$

M. Knor∗ P. Potočnik† R. Škrekovski‡

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Abstract

Let $G$ be a graph. Denote by $L^i(G)$ its $i$-iterated line graph and denote by $W(G)$ its Wiener index. Dobrynin, Entringer and Gutman stated the following problem: Does there exist a non-trivial tree $T$ and $i \geq 3$ such that $W(L^i(T)) = W(T)$? In a series of five papers we solve this problem. In a previous paper we proved that $W(L^i(T)) > W(T)$ for every tree $T$ that is not homeomorphic to a path, claw $K_{1,3}$ and to the graph of “letter H”, where $i \geq 3$. Here we prove that $W(L^i(T)) > W(T)$ for every tree $T$ homeomorphic to the claw, $T \neq K_{1,3}$ and $i \geq 4$.

1 Introduction

Let $G$ be a graph. For any two of its vertices, say $u$ and $v$, denote by $d_G(u, v)$ (or by $d(u, v)$ if no confusion is likely) the distance from $u$ to $v$ in $G$. The Wiener index of $G$, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of $G$. Wiener index was introduced by Wiener in [12]. It is related to boiling point, heat of evaporation, heat of formation, chromatographic retention times, surface tension, vapour pressure, partition coefficients, total electron energy of polymers, ultrasonic sound

∗Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68, Bratislava, Slovakia, knor@math.sk.
†Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 21, 1111 Ljubljana, Slovenia, primoz.potocnik@fmf.uni-lj.si.
‡Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 21, 1111 Ljubljana, Slovenia, skrekovski@gmail.com.
velocity, internal energy, etc., see [8]. For this reason Wiener index is widely studied by chemists. The interest of mathematicians was attracted in 1970’s. It was reintroduced as the distance and transmission, see [5] and [11], respectively. Recently, there are whole special issues of journals devoted to (mathematical properties) of Wiener index, see [6] and [7], as well as several surveys, see e.g. [3] and [4].

By definition, if \( G \) has a unique vertex, then \( W(G) = 0 \). In this case, we say that the graph \( G \) is trivial. We set \( W(G) = 0 \) also when the set of vertices (and hence also the set of edges) of \( G \) is empty.

The line graph of \( G \), \( L(G) \), has vertex set identical with the set of edges of \( G \). Two vertices of \( L(G) \) are adjacent if and only if the corresponding edges are adjacent in \( G \). Iterated line graphs are defined inductively as follows:

\[
L^i(G) = \begin{cases} 
G & \text{if } i = 0, \\
L(L^{i-1}(G)) & \text{if } i > 0.
\end{cases}
\]

In [1] we have the following statement.

**Theorem 1.1** Let \( T \) be a tree on \( n \) vertices. Then \( W(L(T)) = W(T) - \binom{n}{2} \).

Since \( \binom{n}{2} > 0 \) if \( n \geq 2 \), there is no nontrivial tree for which \( W(L(T)) = W(T) \). However, there are trees \( T \) satisfying \( W(L^2(T)) = W(T) \), see e.g. [2]. In [3], the following problem was posed:

**Problem 1.2** Is there any tree \( T \) satisfying the equality \( W(L^i(T)) = W(T) \) for some \( i \geq 3 \)?

As observed above, if \( T \) is a trivial tree then \( W(L^i(T)) = W(T) \) for every \( i \geq 1 \), although here the graph \( L^i(T) \) is empty.

Denote by \( H \) the tree on six vertices out of which two have degree 3 and four have degree 1. Since \( H \) can be drawn to resemble the letter \( H \), it is often called the H-graph. Graphs \( G_1 \) and \( G_2 \) are homeomorphic if and only if the graphs obtained from \( G_1 \) and \( G_2 \), respectively, by substituting the vertices of degree two together with the two incident edges with a single edge, are isomorphic. In [10] we proved the following:

**Theorem 1.3** Let \( T \) be a tree, not homeomorphic to a path, claw \( K_{1,3} \) and the graph \( H \). Then \( W(L^i(T)) > W(T) \) for all \( i \geq 3 \).

Since the case when \( T \) is a path is trivial, it remains to consider graphs homeomorphic to the claw \( K_{1,3} \) and those homeomorphic to \( H \). In this paper we concentrate on graphs homeomorphic to the claw \( K_{1,3} \). The remaining two cases, namely the trees homeomorphic to \( H \) for \( i \geq 3 \) and trees homeomorphic to \( K_{1,3} \) for \( i = 3 \), are dealt with in a forthcoming paper.
First, consider the case of the claw $K_{1,3}$ itself. Then $L^i(K_{1,3})$ is a cycle of length 3 for every $i \geq 1$. Since $W(K_{1,3}) = 9$ and the Wiener index of the cycle of length 3 is 3, we have $W(L^i(K_{1,3})) < W(K_{1,3})$ for every $i \geq 1$. For other trees homeomorphic to $K_{1,3}$, we prove the opposite inequality, provided that $i \geq 4$:

**Theorem 1.4** Let $T$ be a tree homeomorphic to $K_{1,3}$, such that $T \neq K_{1,3}$. Then $W(L^i(T)) > W(T)$ for every $i \geq 4$.

In [9] we proved the following statement:

**Theorem 1.5** Let $G$ be a connected graph. Then $f_G(i) = W(L^i(G))$ is a convex function in variable $i$.

Hence, to prove Theorem 1.4 it suffices to prove:

**Theorem 1.6** Let $T$ be a tree homeomorphic to $K_{1,3}$, such that $T \neq K_{1,3}$. Then $W(L^4(T)) > W(T)$.

## 2 Proofs

Let $a, b, c \geq 1$. Denote by $C_{a,b,c}$ a tree that has three paths of lengths $a$, $b$ and $c$, starting at a common vertex of degree 3. Obviously, $C_{a,b,c}$ is homeomorphic to $K_{1,3}$ and $C_{1,1,1} = K_{1,3}$. By symmetry, we may assume $a \geq b \geq c$, see Figure 1 for $C_{5,4,3}$.

![Figure 1: The graph $C_{5,4,3}$.](image)

Denote

$$\Delta C_{a,b,c} = W(L^4(C_{a,b,c})) - W(C_{a,b,c}).$$

Our aim is to prove $\Delta C_{a,b,c} > 0$ if $a \geq 2$. We start with the case $a \leq 3$. This case will serve as the base of induction in the proof of Theorem 1.6.

**Lemma 2.1** Let $3 \geq a \geq b \geq c \geq 1$ and $a \neq 1$. Then $\Delta C_{a,b,c} > 0$. 


Proof. Since \(3 \geq a \geq b \geq c \geq 1\) and \(a \neq 1\), there are 9 cases to consider. In Table 1 we present \(\Delta C_{a,b,c}\) for each of these cases. The results were found by a computer, though it is rather easy to find \(W(C_{a,b,c})\) by hand, and \(W(L^4(C_{a,b,c}))\) can be found by applying Proposition 2.3 to \(L^2(C_{a,b,c})\).

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\[
\begin{array}{|c|c|c|c|}
\hline
(a, b, c) & W(C_{a,b,c}) & W(L^4(C_{a,b,c})) & \Delta C_{a,b,c} \\
\hline
(3, 3, 3) & 138 & 642 & 504 \\
(3, 3, 2) & 102 & 533 & 431 \\
(3, 3, 1) & 75 & 257 & 182 \\
(3, 2, 2) & 72 & 435 & 363 \\
(3, 2, 1) & 50 & 192 & 142 \\
(3, 1, 1) & 32 & 65 & 33 \\
(2, 2, 2) & 48 & 348 & 300 \\
(2, 2, 1) & 31 & 138 & 107 \\
(2, 1, 1) & 18 & 38 & 20 \\
\hline
\end{array}
\]

Table 1: \(\Delta C_{a,b,c}\) for \(a \leq 3\).

In what follows we assume that \(a \geq 4\). Denote

\[
\delta_0(a, b, c) = W(C_{a,b,c}) - W(C_{a-1,b,c})
\]

\[
\delta_4(a, b, c) = W(L^4(C_{a,b,c})) - W(L^4(C_{a-1,b,c})).
\]

Then

\[
\Delta C_{a,b,c} - \Delta C_{a-1,b,c} = \delta_4(a, b, c) - \delta_0(a, b, c),
\]

so if we prove \(\delta_4(a, b, c) - \delta_0(a, b, c) \geq 0\), we obtain \(\Delta C_{a,b,c} \geq \Delta C_{a-1,b,c}\).

We distinguish 4 vertices in \(C_{a,b,c}\). Denote by \(y\) the vertex of degree 3, and denote by \(x_1, x_2\) and \(x_3\) the pendant vertices so that \(d(x_1, y) = a, d(x_2, y) = b\) and \(d(x_3, y) = c\), see Figure 1. As is the custom, by \(V(G)\) we denote the vertex set of \(G\).

Lemma 2.2 Let \(a, b, c \geq 1\). Then

\[
\delta_0(a, b, c) = \binom{a+b+1}{2} + \binom{a+c+1}{2} - \binom{a+1}{2}.
\]

Proof. Since \(C_{a-1,b,c}\) is a subgraph of \(C_{a,b,c}\) with \(V(C_{a,b,c}) - V(C_{a-1,b,c}) = \{x_1\}\), we have

\[
\delta_0(a, b, c) = W(C_{a,b,c}) - W(C_{a-1,b,c}) = \sum_u d(u, x_1),
\]

where the sum goes through all \(u \in V(C_{a,b,c}) \setminus \{x_1\}\). For vertices \(u\) of the \(x_1-x_2\) path, the sum of all \(d(u, x_1)\) is \(1+2+\cdots+(a+b) = \binom{a+b+1}{2}\). For vertices of the \(x_1-x_3\) path
which do not lay on \( x_1 - x_2 \) path, the sum of \( d(u, x_1) \) is \((a+1) + (a+2) + \cdots + (a+c) = \binom{a+c+1}{2} - \binom{a+1}{2}\), see Figure 1. Since the paths \( x_1 - x_2 \) and \( x_1 - x_3 \) contain all vertices of \( C_{a,b,c} \), we have 

\[
\delta_0(a, b, c) = \binom{a+b+1}{2} + \binom{a+c+1}{2} - \binom{a+1}{2}.
\]

For two subgraphs \( S_1 \) and \( S_2 \) of \( G \), by \( d(S_1, S_2) \) we denote the shortest distance in \( G \) between a vertex of \( S_1 \) and a vertex of \( S_2 \). If \( S_1 \) and \( S_2 \) share an edge then we set \( d(S_1, S_2) = -1 \).

Analogously as a vertex of \( L(G) \) corresponds to an edge of \( G \), a vertex of \( L^2(G) \) corresponds to a path of length two in \( G \). For \( x \in V(L^2(G)) \) we denote by \( B_2(x) \) the corresponding path in \( G \). Let \( x \) and \( y \) be two distinct vertices of \( L^2(G) \). It was proved in [9] that

\[
d_{L^2(G)}(x, y) = d_G(B_2(x), B_2(y)) + 2.
\]

Let \( u, v \in V(G) \), \( u \neq v \). Denote by \( \beta_i(u, v) \) the number of pairs \( x, y \in V(L^2(G)) \), with \( u \) being the center of \( B_2(x) \) and \( v \) being the center of \( B_2(y) \), such that \( d(B_2(x), B_2(y)) = d(u, v) - 2 + i \). Since \( d(u, v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u, v) \), we have \( \beta_i(u, v) = 0 \) for all \( i \notin \{0, 1, 2\} \). Denote by \( \deg(w) \) the degree of \( w \) in \( G \). In [9] we have the following statement:

**Proposition 2.3** Let \( G \) be a connected graph. Then

\[
W(L^2(G)) = \sum_{u \neq v} \left[ \binom{\deg(u)}{2} \binom{\deg(v)}{2} d(u, v) + \beta_1(u, v) + 2\beta_2(u, v) \right] + \sum_u \left[ 3 \binom{\deg(u)}{3} + 6 \binom{\deg(u)}{4} \right],
\]

(2)

where the first sum goes through unordered pairs \( u, v \in V(G) \) and the second one goes through \( u \in V(G) \).

We apply Proposition 2.3 to \( L^2(C_{a,b,c}) \) and \( L^2(C_{a-1,b,c}) \). This enables us to calculate \( \delta_4(a, b, c) \) using degrees and distances of the second iterated line graph.

Denote by \( w_1 \) the pendant vertex of \( L^2(C_{a,b,c}) \) corresponding to the path of length 2 terminating at \( x_1 \). Since \( a \geq 4 \), the unique neighbour of \( w_1 \) has degree 2. Denote by \( w \) this neighbour, see Figure 2. For every vertex \( u \in V(L^2(C_{a,b,c})) \setminus \{w, w_1\} \), denote by \( n(u) \) the number of neighbours of \( u \), whose distance to \( w \) is at least \( d(u, w) \). We have:

**Lemma 2.4** Let \( a \geq 4 \) and \( b, c \geq 1 \). Then

\[
\delta_4(a, b, c) = \sum_u \left[ \binom{\deg(u)}{2} d(u, w) + \binom{n(u)}{2} \right],
\]

where the sum goes through all vertices of \( V(L^2(C_{a,b,c})) \setminus \{w, w_1\} \).
Proof. Observe that $L^2(C_{a-1,b,c})$ is a subgraph of $L^2(C_{a,b,c})$ and $V(L^2(C_{a-1,b,c})) = \{w_1\}$. Since $\deg(w_1) = 1$, the vertex $w_1$ cannot be the center of a path of length 2, implying that $\beta_i(u, w_1) = 0$ for every $u$ and $i$. Since $\binom{\deg(u)}{2} = 0$, all summands of (2) containing $w_1$ contribute 0 to $W(L^2(C_{a,b,c}))$. The vertices of $L^2(C_{a-1,b,c})$, except $w$, have the same degree in $L^2(C_{a,b,c})$ as in $L^2(C_{a-1,b,c})$. The degree of $w$ is 1 in $L^2(C_{a-1,b,c})$, and it is 2 in $L^2(C_{a,b,c})$. Therefore $\sum_{u} [3\binom{\deg(u)}{3} + 6\binom{\deg(u)}{4}]$ has the same value in $L^2(C_{a,b,c})$ as in $L^2(C_{a-1,b,c})$, so these sums will cancel out. Thus, we have

$$\delta_4(a, b, c) = W(L^2(L^2(C_{a,b,c}))) - W(L^2(L^2(C_{a-1,b,c}))) = \sum_{u} \left[ \binom{\deg(u)}{2} \left( 2d(u, w) + \beta_1(u, w) + 2\beta_2(u, w) \right) \right],$$

where the sum goes through $u \in V(L^2(C_{a-1,b,c})) \setminus \{w\}$.

Let $u \in V(L^2(C_{a-1,b,c})) \setminus \{w\}$. Since $\deg(w_1) = 1$ and $\deg(w) = 2$ in $L^2(C_{a,b,c})$, the unique path of length 2 centered at $w$ contains an endvertex closer to $u$ than $w$. Hence, $\beta_2(u, w) = 0$. Consequently, $\beta_1(u, w)$ equals the number of paths of length 2 centered at $u$, both endvertices of which have distance to $w$ at least $d(u, w)$. Hence, $\beta_1(u, w) = \binom{n(u)}{2}$, which completes the proof.

Using Lemma 2.4 we prove the induction step.

**Lemma 2.5** Let $a \geq b \geq c \geq 1$ and $a \geq 4$. Then $\delta_4(a, b, c) \geq \delta_0(a, b, c)$.

**Proof.** We distinguish 8 more vertices in $L^2(C_{a,b,c})$. Denote by $w_2$ and $w_3$ pendant vertices corresponding to the paths of length 2 containing $x_2$ and $x_3$, respectively, see Figure 1 and 2. Denote by $z_1$, $z_2$ and $z_3$ the vertices corresponding to the paths of length 2, whose endvertex is $y$; and denote by $z_4$, $z_5$ and $z_6$ the vertices
corresponding to the paths of length 2 centered at \( y \). Of course, if \( b \leq 2 \) or \( c \leq 2 \), then some of these vertices are not defined.

For \( u \in V(L^2(C_{a-1,b,c})) \setminus \{ w \} \), denote

\[
\delta_4(a, b, c) = \sum_u h(u),
\]

where the sum goes through all vertices of \( V(L^2(C_{a,b,c})) \setminus \{ w, w_1 \} \). If \( u \in \{ w_2, w_3 \} \) then \( \deg(u) = 1 \) and \( n(u) = 0 \), so \( h(u) = 0 \). Thus, vertices of degree 1 contribute 0 to \( \sum_u h(u) \). Denote

\[
S_i = \sum_u h(u),
\]

where the sum is taken over all interior vertices \( u \) of the \( w_i - z_i \) path, \( u \neq w \) and \( 1 \leq i \leq 3 \). Then \( \delta_4(a, b, c) = \sum_{i=1}^3 S_i + \sum_{i=1}^6 h(z_i) \).

Regarding the values of \( a, b \) and \( c \), we distinguish 4 cases:

**Case 1.** \( a \geq 4 \) and \( b, c \geq 3 \).

If \( u \) is a vertex of degree 2, then \( n(u) = 1 \) and \( \binom{\deg(u)}{2} = 1 \). Hence \( h(u) = d(u, w) \). Thus,

\[
\begin{align*}
S_1 &= 1 + 2 + \cdots + (a-4) = \binom{a-3}{2} \\
S_2 &= a + (a+1) + \cdots + (a+b-4) = \binom{a+b-3}{2} - \binom{a}{2} \\
S_3 &= a + (a+1) + \cdots + (a+c-4) = \binom{a+c-3}{2} - \binom{a}{2}.
\end{align*}
\]

If \( u \in \{ z_1, z_2, z_3 \} \), then \( \deg(u) = 3 \) and \( n(u) = 2 \). Thus \( h(u) = 3d(u, w) + 1 \). If \( u \in \{ z_4, z_5 \} \), then \( \deg(u) = 4 \) and \( n(u) = 3 \), so \( h(u) = 6d(u, w) + 3 \). Finally, if \( u = z_6 \), then \( \deg(u) = 4 \) and \( n(u) = 2 \), so \( h(u) = 6d(u, w) + 1 \). This gives

\[
\begin{align*}
h(z_1) &= 3(a-3) + 1 \\
h(z_2) &= h(z_3) = 3(a-1) + 1 \\
h(z_4) &= h(z_5) = 6(a-2) + 3 \\
h(z_6) &= 6(a-1) + 1.
\end{align*}
\]

Since \( \delta_4(a, b, c) = \sum_{i=1}^3 S_i + \sum_{i=1}^6 h(z_i) \), we have

\[
\delta_4(a, b, c) = \binom{a-3}{2} + \binom{a+b-3}{2} + \binom{a+c-3}{2} - 2\binom{a}{2} + 3a-8 + 2(3a-2) + 2(6a-9) + (6a-5).
\]

Denote \( P = \delta_4(a, b, c) - \delta_0(a, b, c) \). By Lemma 2.2 we have \( \delta_0(a, b, c) = \binom{a+b+1}{2} + \binom{a+c+1}{2} - \binom{a+1}{2} \). Expanding the terms we get

\[
P = 17a - 4b - 4c - 17.
\]

Since \( a \geq b \) and \( a \geq c \), we have \( P \geq 9a - 17 \). Finally, since \( a \geq 4 \), we have \( P = \delta_4(a, b, c) - \delta_0(a, b, c) \geq 0 \).
Case 2. $a \geq 4$, $b \geq 3$ and $c \leq 2$.

We calculate first $\delta_4(a, b, 1)$. In $L^2(C_{a,b,1})$ we have $S_3 = 0$; note that $z_3$ is not defined here and that $\deg(z_5) = \deg(z_6) = 3$ (see Figure 2). Analogously as in Case 1 we get:

\[
\begin{align*}
S_1 &= \binom{a-3}{2} \\
S_2 &= \binom{a+b-3}{2} - \binom{a}{2} \\
S_3 &= 0
\end{align*}
\]

\[
\begin{align*}
h(z_1) &= 3(a-3) + 1 \\
h(z_2) &= 3(a-1) + 1 \\
h(z_4) &= 6(a-2) + 3 \\
h(z_5) &= 3(a-2) + 1 \\
h(z_6) &= 3(a-1)
\end{align*}
\]

since $n(z_3) = 2$ and $n(z_6) = 1$. Thus,

$$
\delta_4(a, b, 1) = \binom{a-3}{2} + \binom{a+b-3}{2} - \binom{a}{2} + (3a-8) + (3a-5) + (3a-3).
$$

Denote $P = \delta_4(a, b, 1) - \delta_0(a, b, 2)$. By Lemma 2.2 we have $\delta_0(a, b, 2) = \binom{a+b+1}{2} - \binom{a+1}{2}$. Expanding the terms we get

$$
P = 9a - 4b - 18.
$$

Since $a \geq b$, we have $P \geq 5a - 18$, and as $a \geq 4$, we have $P \geq 0$. Since $\delta_4(a, b, 2) \geq \delta_4(a, b, 1)$ and $\delta_0(a, b, 2) \geq \delta_0(a, b, 1)$, we conclude $\delta_4(a, b, i) - \delta_0(a, b, i) \geq P \geq 0$ for $i \in \{1, 2\}$.

Case 3. $a \geq 4$, $b = 2$ and $c \leq 2$.

We find $\delta_4(a, 2, 1)$. In $L^2(C_{a,2,1})$ we have $S_2 = S_3 = 0$. Again, the vertex $z_3$ is not defined here, $\deg(z_2) = 2$ and $\deg(z_5) = \deg(z_6) = 3$ (see Figure 2). Analogously as in the previous cases we get:

\[
\begin{align*}
S_1 &= \binom{a-3}{2} \\
h(z_1) &= 3(a-3) + 1 \\
h(z_2) &= (a-1) \\
h(z_4) &= 6(a-2) + 3 \\
h(z_5) &= 3(a-2) + 1 \\
h(z_6) &= 3(a-1)
\end{align*}
\]

since $n(z_2) = 1$, $n(z_5) = 2$ and $n(z_6) = 1$. Thus,

$$
\delta_4(a, 2, 1) = \binom{a-3}{2} + (3a-8) + (a-1) + (6a-9) + (3a-5) + (3a-3).
$$

Denote $P = \delta_4(a, 2, 1) - \delta_0(a, 2, 2)$. By Lemma 2.2 we have $\delta_0(a, 2, 2) = 2\binom{a+3}{2} - \binom{a+1}{2}$. Expanding the terms we get

$$
P = 8a - 26.
$$

Since $a \geq 4$, we have $P \geq 0$. Since $\delta_4(a, 2, 2) \geq \delta_4(a, 2, 1)$ and $\delta_0(a, 2, 2) \geq \delta_0(a, 2, 1)$, we conclude $\delta_4(a, 2, i) - \delta_0(a, 2, i) \geq P \geq 0$ for $i \in \{1, 2\}$. 


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Case 4. $a \geq 4$ and $b = c = 1$.

In $L^2(C_{a,1,1})$ we have $S_2 = S_3 = 0$. Note that the vertices $z_2$ and $z_3$ are not defined, while $\deg(z_4) = \deg(z_5) = 3$ and $\deg(z_6) = 2$ (see Figure 2). Analogously as in the previous cases we get:

$$
S_2 = h(z_4) = h(z_5) = 3(a-2) + 1
$$

$$
h(z_1) = 3(a-3) + 1
$$

since $n(z_4) = n(z_5) = 2$ and $n(z_6) = 0$. Thus,

$$
\delta_4(a,1,1) = \binom{a-3}{2} + (3a-8) + 2(3a-5) + (a-1).
$$

Denote $P = \delta_4(a,1,1) - \delta_0(a,1,1)$. By Lemma 2.2 we have $\delta_0(a,1,1) = 2\left(\binom{a+2}{2} - \binom{a+1}{2}\right)$. Expanding the terms we get

$$
P = 4a - 15.
$$

Since $a \geq 4$, we have $P \geq 0$, and hence $\delta_4(a,1,1) - \delta_0(a,1,1) \geq P \geq 0$. 

Proof of Theorem 1.6. Let $T$ be the tree $C_{a,b,c}$ with $a \geq b \geq c \geq 1$, such that $a \neq 1$. If $a \leq 3$, then $\Delta C_{a,b,c} = W(L^4(C_{a,b,c})) - W(C_{a,b,c}) > 0$, by Lemma 2.1.

Now suppose that $a \geq 4$. Consider lexicographical ordering of triples $(a', b', c')$ with $a' \geq b' \geq c' \geq 1$ and $a' \geq 2$. Further, assume that $\Delta C_{a',b',c'} > 0$ for every triple $(a', b', c')$, such that $a' \geq b' \geq c' \geq 1$ and $a' \geq 2$, which is in the lexicographical ordering smaller than $(a, b, c)$.

Let $(a^*, b^*, c^*)$ be ordering of triple $(a-1, b, c)$, such that the multisets $\{a^*, b^*, c^*\}$ and $\{a-1, b, c\}$ are the same and $a^* \geq b^* \geq c^*$. Then $C_{a-1,b,c}$ and $C_{a^*,b^*,c^*}$ are isomorphic graphs. Moreover, since $a \geq 4$, we have $a^* \geq 3$. By (1) and Lemma 2.5 we see that

$$
\Delta C_{a,b,c} - \Delta C_{a^*,b^*,c^*} = \Delta C_{a,b,c} - \Delta C_{a-1,b,c} = \delta_4(a, b, c) - \delta_0(a, b, c) \geq 0.
$$

Since $(a^*, b^*, c^*)$ is in the lexicographical ordering smaller than $(a, b, c)$ and $a^* \geq 2$, by the induction hypothesis we have $\Delta C_{a^*,b^*,c^*} > 0$. Hence, $\Delta C_{a,b,c} = W(L^4(C_{a,b,c})) - W(C_{a,b,c}) > 0$. 

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