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Complete solution of equation $W(L^3(T)) = W(T)$ for Wiener index of iterated line graphs of trees

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Abstract

Let $G$ be a graph. Denote by $L^i(G)$ its $i$-iterated line graph and denote by $W(G)$ its Wiener index. In [14] we show that there is an infinite class $T$ of trees $T$ satisfying $W(L^3(T)) = W(T)$, which disproves a conjecture of Dobrynin and Entringer. In this paper we prove that except of the trees of $T$, there is no non-trivial tree $T$ satisfying $W(L^3(T)) = W(T)$. Consequently, for a tree $T$ and $i \geq 3$, the equation $W(L^i(T)) = W(T)$ holds if and only if $T \in T$ and $i = 3$.

Keywords: Wiener index, tree, iterated line graph.

1 Introduction

Let $G$ be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For any two vertices $u, v$ let $d(u, v)$ be the distance from $u$ to $v$. The Wiener index of $G$, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken over unordered pairs of vertices of $G$. The Wiener index was introduced by Wiener in [22]. Since it is related to several properties of chemical

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molecules (see [13]), it is widely studied by chemists. The interest of mathematicians was attracted in 1970’s, when it was reintroduced as the transmission and the distance of a graph; see [21] and [9], respectively. Recently, several special issues of journals were devoted to (mathematical properties) of the Wiener index (see [11] and [12]). For surveys and some up-to-date papers related to the Wiener index of trees and line graphs see [5, 6], [8, 19, 20, 24] and [2, 3, 7, 10, 23], respectively.

By the definition, if \( G \) has a unique vertex, then \( W(G) = 0 \). In this case, we say that the graph \( G \) is trivial. We set \( W(G) = 0 \) also when the set of vertices of \( G \) is empty.

The line graph of \( G \), \( L(G) \), has vertex set identical with the set of edges of \( G \) and two vertices of \( L(G) \) are adjacent if and only if the corresponding edges are adjacent in \( G \). Iterated line graphs are defined inductively as follows:

\[
L^i(G) = \begin{cases} 
G & \text{if } i = 0, \\
L(L^{i-1}(G)) & \text{if } i > 0.
\end{cases}
\]

The Wiener index of the line graph of a tree \( T \) can easily be computed from \( W(T) \) by using the following result from [1]:

**Theorem 1.1** Let \( T \) be a tree on \( n \) vertices. Then \( W(L(T)) = W(T) - \binom{n}{2} \).

Since \( \binom{n}{2} > 0 \) if \( n \geq 2 \), there is no nontrivial tree for which \( W(L(T)) = W(T) \). However, there are trees \( T \) satisfying \( W(L^2(T)) = W(T) \), see e.g. [4]. In [5], the following problem was posed:

**Problem 1.2** Is there any tree \( T \) satisfying equality \( W(L^i(T)) = W(T) \) for some \( i \geq 3 \)?

As observed above, if \( T \) is a trivial tree, then \( W(L^i(T)) = W(T) \) for every \( i \geq 1 \), although here the graph \( L^i(T) \) is empty. The real question is, if there is a nontrivial tree \( T \) and \( i \geq 3 \) such that \( W(L^i(T)) = W(T) \).

In papers [15, 16, 17, 18] (see [18, Corollary 1.4]) we solved Problem 1.2 for \( i \geq 4 \):

**Theorem 1.3** Let \( T \) be a tree and \( i \geq 4 \). Then we have

\[
\begin{align*}
W(L^i(T)) &= W(T) & \text{if } T \text{ is trivial,} \\
W(L^i(T)) &< W(T) & \text{if } T \text{ is a nontrivial path or the claw } K_{1,3}, \\
W(L^i(T)) &> W(T) & \text{otherwise.}
\end{align*}
\]

In this paper we consider Problem 1.2 for the remaining case \( i = 3 \). Let \( H_0 \) be the tree on six vertices, out of which two have degree 3 and four have degree 1. In [16, Corollary 1.6], we proved:

**Theorem 1.4** Let \( T \) be a tree which is not homeomorphic to a path, claw \( K_{1,3} \) or \( H_0 \), and let \( i \geq 3 \). Then \( W(L^i(T)) > W(T) \).
(Recall that two graphs \( G_1 \) and \( G_2 \) are homeomorphic if and only if there is a third graph \( H \), such that both \( G_1 \) and \( G_2 \) can be obtained from \( H \) by means of edge subdivision.)

By Theorem 1.4, to solve Problem 1.2 for \( i = 3 \), it suffices to consider paths and trees homeomorphic to the claw \( K_{1,3} \) and \( H_0 \).

First, let us concentrate to paths. Denote by \( P_n \) a path on \( n \) vertices. If \( n \geq 2 \), then \( W(P_n) > W(P_{n-1}) \), since \( P_{n-1} \) is a subgraph embedded isometrically in \( P_n \). Since \( L(P_n) = P_{n-1} \) if \( n \geq 2 \), while \( L(P_1) \) is an empty graph, we have \( W(L^i(P_n)) \neq W(P_n) \) for every \( i \geq 1 \) if \( P_n \) is a nontrivial path.

Similarly, there is no solution of Problem 1.2 among trees homeomorphic to the claw \( K_{1,3} \); namely, in Section 3 we prove the following:

**Theorem 1.5** Let \( T \) be a tree homeomorphic to \( K_{1,3} \). Then \( W(L^3(T)) \neq W(T) \).

However, there is a non-trivial solution of Problem 1.2 among trees homeomorphic to \( H_0 \). Denote by \( H_{a,b,c,d,e} \) a specific tree homeomorphic \( H_0 \), defined as follows: In \( H_{a,b,c,d,e} \), the two vertices of degree 3 are joined by a path of length \( e+1 \), \( e \geq 0 \). Hence, this path has \( e \) vertices of degree 2. Further, at one vertex of degree 3 there start two pendant paths of lengths \( a \) and \( b \), where \( a, b \geq 1 \), and at the other vertex of degree 3 there start another two pendant paths of lengths \( c \) and \( d \), where \( c, d \geq 1 \). Thus \( H_{a,b,c,d,e} \) has \( a + b + c + d + e + 2 \) vertices (see Figure 1 for \( H_{3,3,4,2,2} \)). By symmetry, we may assume that \( a \geq b \), \( c \geq d \) and \( b \geq d \). That is, we assume that the shortest pendant path in \( H_{a,b,c,d,e} \) has length \( d \).

![Figure 1: The graph \( H_{a,b,c,d,e} \).](image)

In Section 3, we prove the following:

**Theorem 1.6** The equation \( W(L^3(H_{a,b,c,d,e})) = W(H_{a,b,c,d,e}) \) holds if and only if \( d = e = 1 \) and there are \( i, j \in \mathbb{Z} \), \( i \geq j \), such that

\[
\begin{align*}
a & = 128 + 3i^2 + 3j^2 - 3ij + i \\
b & = 128 + 3i^2 + 3j^2 - 3ij + j \\
c & = 128 + 3i^2 + 3j^2 - 3ij + i + j.
\end{align*}
\]
We remark that the “if” part of Theorem 1.6 was already proved in [14]. The smallest tree satisfying (1) is $H_{128, 128, 128, 1, 1}$ on 388 vertices obtained when $i = j = 0$.

We can summarize our results regarding Problem 1.2 in the following theorem:

**Theorem 1.7** Let $T$ be a tree and $i \geq 3$. Then we have

(i) $W(L^i(T)) = W(T)$ if $T$ is trivial or $i = 3$ and $T$ is $H_{a, b, c, 1, 1}$, where $a, b, c$ satisfy (1);

(ii) $W(L^i(T)) \neq W(T)$ if $i = 3$ and $T$ is homeomorphic to $K_{1, 3}$ or $H_0$ with the exception of trees mentioned in (i);

(iii) $W(L^i(T)) < W(T)$ if $T$ is a nontrivial path or the claw $K_{1, 3}$;

(iv) $W(L^i(T)) > W(T)$ otherwise.

It is obvious that trees mentioned in (ii) either satisfy $W(L^3(T)) < W(T)$ or $W(L^3(T)) > W(T)$. If $T \neq K_{1, 3}$, in some cases we prove $W(L^3(T)) > W(T)$, but in the others using congruences we can only show $W(L^3(T)) \neq W(T)$, see below.

In the next section we present a lemma, with the help of which we prove Theorems 1.5 and 1.6 in Sections 3 and 4, respectively.

## 2 Preliminaries

A degree of a vertex, say $v$, is denoted by $\deg(v)$, or when convenient, by $d_v$. Analogously as a vertex of $L(G)$ corresponds to an edge of $G$, a vertex of $L^2(G)$ corresponds to a path of length two in $G$. For $x \in V(L^2(G))$ we denote the corresponding path by $B_2(x)$. For two subgraphs $S_1$ and $S_2$ of $G$, the shortest distance in $G$ between a vertex of $S_1$ and a vertex of $S_2$ is denoted by $d(S_1, S_2)$. If $S_1$ and $S_2$ share an edge, then we set $d(S_1, S_2) = -1$.

Let $x$ and $y$ be two vertices of $L^2(G)$, such that $u$ is the center of $B_2(x)$, the vertex $v$ is the center of $B_2(y)$ and $u \neq v$. Then

$$d_{L^2(G)}(x, y) = d(B_2(x), B_2(y)) + 2.$$ 

Let $u, v \in V(G)$, $u \neq v$. Let $\beta_i(u, v)$ denote the number of pairs $x, y \in V(L^2(G))$, with $u$ being the center of $B_2(x)$ and $v$ being the center of $B_2(y)$, such that $d(B_2(x), B_2(y)) = d(u, v) - 2 + i$. Since $d(u, v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u, v)$, we have $\beta_i(u, v) = 0$ for all $i \notin \{0, 1, 2\}$. Moreover, $\sum_{i=0}^{2} \beta_i(u, v) = \binom{d_u}{2} \binom{d_v}{2}$.

Let

$$h(u, v) = \left( \binom{d_u}{2} \binom{d_v}{2} - 1 \right) d(u, v) + \beta_1(u, v) + 2\beta_2(u, v).$$

In [14, Lemma 2.2] we have the following statement:
Lemma 2.1 Let $G$ be a connected graph. Then

$$W(L^2(G)) - W(G) = \sum_{u \neq v} h(u, v) + \sum_u \left[ 3 \left( \frac{d_u}{3} \right) + 6 \left( \frac{d_u}{4} \right) \right],$$

where the first sum is taken over unordered pairs of vertices $u, v \in V(G)$, such that either $d_u \neq 2$ or $d_v \neq 2$, and the second one is taken over $u \in V(G)$.

Observe that $W(P_n) = \left( (n-1) + \ldots + 1 \right) + \left( (n-2) + \ldots + 1 \right) + \ldots + 1 = \binom{n+1}{3}$. Using this fact, one can show that $W(L^3(H_{a,b,c,d,e}))$ is a polynomial of third degree in $a, b, c, d$ and $e$, and so is also $W(L^3(H_{a,b,c,d,e}))$ (the situation with the claw being similar). However, if we calculate $W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e})$ with the help of Lemma 2.1, we obtain a polynomial the degree of which is at most 2, since the pairs of vertices $u, v$ with $d_u = d_v = 2$ do not contribute to $W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e})$ (for detailed calculation see the proofs below).

3 Proof of Theorem 1.5

Proof of Theorem 1.5. Let $C_{a,b,c}$ be a tree homeomorphic to the claw $K_{1,3}$ in which the paths connecting the vertices of degree 1 with the vertex of degree 3 have lengths $a$, $b$ and $c$, where $a \geq b \geq c \geq 1$. The tree $C_{a,b,c}$ has exactly $a + b + c + 1$ vertices, see Figure 2 for $C_{4,3,2}$.

We prove Theorem 1.5 by counting the distances in $L(C_{a,b,c})$ instead of in $C_{a,b,c}$ and $L^3(C_{a,b,c})$. In $L(C_{a,b,c})$ we distinguish 6 vertices $x_1, x_2, x_3, y_1, y_2$ and $y_3$. The vertices $x_1, x_2$ and $x_3$ correspond to the pendant edges of $C_{a,b,c}$, while the vertices $y_1, y_2$ and $y_3$ correspond to the edges incident with the vertex of degree 3 in $C_{a,b,c}$, see Figure 2 for $L(C_{4,3,2})$. Observe that if $c = 1 \ (b = 1 \text{ or } a = 1)$, then $x_3 = y_3$ ($x_2 = y_2$ or $x_1 = y_1$), and in such a case, $\deg(x_3) = 2 \ (\deg(x_2) = 2 \text{ or } \deg(x_1) = 2$, respectively).

In what follows, the graph $L(C_{a,b,c})$ is denoted by $LC$. Further, for $i \in \{1, 2, 3\}$, let $V_i$ be the set of vertices of $V(LC')$ of degree $i$. For $x \in V_1$ and $y \in V_3$, define

$$S^1(x) = \sum_u h(u, x) \quad \text{where } u \in V(LC) \setminus V_1,$$

$$M^1 = \sum_{u \neq v} h(u, v) \quad \text{where } u, v \in V_1,$$

$$S^3(y) = \sum_u h(u, y) \quad \text{where } u \in V_2,$$

$$M^3 = \sum_{u \neq v} h(u, v) \quad \text{where } u, v \in V_3,$$

$$D = \sum_u \left[ 3 \left( \frac{u}{3} \right) + 6 \left( \frac{u}{4} \right) \right] \quad \text{where } u \in V_3.$$

Observe that $\sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3} S^3(y) + M^3$ sums $h(u, v)$ for all pairs $\{u, v\}$ of vertices such that either $\deg(u) \neq 2$ or $\deg(v) \neq 2$. 

6
Denote \( P = W(L^3(C_{a,b,c})) - W(C_{a,b,c}) \). Since \( C_{a,b,c} \) has \( a + b + c + 1 \) vertices, we have \( W(C_{a,b,c}) = W(LC) + \binom{a+b+c+1}{2} \), by Theorem 1.1. Thus, by Lemma 2.1, we have
\[
P = W(L^2(LC)) - W(LC) - \binom{a+b+c+1}{2} \\
= \sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3} S^3(y) + M^3 + D - \binom{a+b+c+1}{2}.
\] (3)

This naturally splits the problem into four cases according to the size of \( V_1 \). In each of these cases we evaluate \( S^1 \)'s, \( M^1 \), \( S^3 \)'s, \( M^3 \) and \( D \), and we solve the equation \( P = 0 \). To avoid fractions, in some cases we solve the equation \( 2P = 0 \) instead of \( P = 0 \).

![Figure 2: The graphs \( C_{a,b,c} \) and \( L(C_{a,b,c}) = LC \).](image)

**Case 1.** \( a, b, c \geq 2 \), that is, \( |V_1| = 3 \).

We start with evaluating \( S^1(x) \), where \( x \in V_1 \). Since \( \deg(x) = 1 \), we have \( \beta_j(u, x) = 0 \), \( 0 \leq j \leq 2 \). Hence, \( h(u, x) = -d(u, x) \), see (2). The sum of distances from \( x_1 \) to all interior vertices of \( x_1 - x_2 \) path is \( 1 + 2 + \ldots + (a+b-2) = \binom{a+b-1}{2} \) (see Figure 2). The sum of distances from \( x_1 \) to all interior vertices of \( x_1 - x_3 \) path, not included in the previous calculation, is \( a + (a+1) + \ldots + (a+c-2) = \binom{a+c-1}{2} - \binom{a}{2} \).

In this way we get \( S^1(x_1) \) and analogously we calculate \( S^1(x_2) \) and \( S^1(x_3) \):
\[
S^1(x_1) = -\binom{a+b-1}{2} - \binom{a+c-1}{2} + \binom{a}{2},
S^1(x_2) = -\binom{a+b-1}{2} - \binom{b+c-1}{2} + \binom{b}{2},
S^1(x_3) = -\binom{a+c-1}{2} - \binom{b+c-1}{2} + \binom{c}{2}.
\]

Now \( h(x_1, x_2) = -(a+b-1) \). Using the symmetry we obtain
\[
M^1 = -(a+b-1) - (a+c-1) - (b+c-1).
\]

In \( S^3(y) \) we sum \( h(u, y) \), where \( \deg(u) = 2 \) and \( \deg(y) = 3 \). Hence, \( \binom{d_u}{2} \binom{d_y}{2} - 1 = 2 \). Since \( \beta_0(u, y) = 2, \beta_1(u, y) = 1 \) and \( \beta_2(u, y) = 0 \), we have \( h(u, y) = 2d(u, y) + 1 \). Thus, the sum of \( h(u, y_1) \)'s for interior vertices \( u \) of \( y_1 - x_1 \) path is \( 2(1 + 2 + \ldots + (a-2)) + (a-2) = 2\binom{a-1}{2} + (a-2) \) (see Figure 2). Analogously, the sum of \( h(u, y_1) \)'s
for interior vertices of \( y_2 - x_2 \) path is \( 2(2+3+\ldots+(b-1)) + (b-2) = 2\binom{b}{2} - 2 + (b-2) = 2\binom{b}{2} + (b-4) \). In this way we get

\[
\begin{align*}
S^3(y_1) &= 2\binom{a-1}{2} + (a-2) + 2\binom{b}{2} + (b-4) + 2\binom{c}{2} + (c-4), \\
S^3(y_2) &= 2\binom{a}{2} + (a-4) + 2\binom{b-1}{2} + (b-2) + 2\binom{c}{2} + (c-4), \\
S^3(y_3) &= 2\binom{a}{2} + (a-4) + 2\binom{b}{2} + (b-4) + 2\binom{c-1}{2} + (c-2).
\end{align*}
\]

Consider \( h(y_1, y_2) \). Here \( \binom{d_u}{2} - 1 = 8, \beta_0(y_1, y_2) = 4, \beta_1(y_1, y_2) = 5 \) and \( \beta_2(y_1, y_2) = 0 \) (see Figure 2). This means that \( h(y_1, y_2) = 8 + 5 = 13 \), and analogously also \( h(y_1, y_3) = 13 \) and \( h(y_2, y_3) = 13 \). Hence

\[
M^3 = 13 + 13 + 13.
\]

Finally, since \( LC \) has exactly three vertices of degree 3 and no vertex of higher degree, we have

\[
D = \sum_u \left[ 3\binom{d_u}{3} + 6\binom{d_u}{4} \right] = 3\binom{3}{3} = 9.
\]

By (3), expanding the terms (using a computer package, for instance), we get

\[
\begin{align*}
P &= (a^2+b^2+c^2) - 3(ab+ac+bc) + (a+b+c) + 21 \\
&= (a+b+c)^2 - 5(ab+ac+bc) + (a+b+c) + 21.
\end{align*}
\]

Now substitute \( x = a+b+c \) and consider the equation \( P = 0 \) over \( \mathbb{Z}_5 \). We get

\[
x^2 + x + 1 = 0,
\]

which has no solution in \( \mathbb{Z}_5 \). Consequently, \( P = 0 \) has no integer solution and \( W(L^3(C_{a,b,c})) - W(C_{a,b,c}) \neq 0 \) in this case.

**Case 2.** \( a, b \geq 2, c = 1 \), that is, \( |V_1| = 2 \).

In this case the vertex \( x_3 = y_3 \) has degree 2, so we do not need to find \( S^1(x_3) \) and \( S^3(y_3) \), see (3), but we must include the distances to \( x_3 \) in \( S^1(x_1), S^1(x_2), S^3(y_1) \) and \( S^3(y_2) \). Analogously as in the previous case we have

\[
\begin{align*}
S^1(x_1) &= -(\binom{a+b-1}{2}) - a, \\
S^1(x_2) &= -(\binom{a+b-1}{2}) - b, \\
M^1 &= -(a+b-1), \\
S^3(y_1) &= 2\binom{a-1}{2} + (a-2) + 2\binom{b}{2} + (b-4) + 2 + 1, \\
S^3(y_2) &= 2\binom{a}{2} + (a-4) + 2\binom{b-1}{2} + (b-2) + 2 + 1, \\
M^3 &= 13, \\
D &= 2 \cdot 3\binom{3}{3} = 6.
\end{align*}
\]
By (3), expanding the terms we get
\[ 2P = (a^2 + b^2) - 6ab - 5(a+b) + 30 = (a+b)^2 - 8ab - 5(a+b) + 30. \] (4)

Now consider the equation \( 2P = 0 \) over \( \mathbb{Z}_5 \). We get \( (a'+b')^2 + 2a'b' = 0 \). It is a matter of routine to check that the only solution in \( \mathbb{Z}_5 \) is \( a' = b' = 0 \). Hence, in (4) we have \( 25 \mid (a+b)^2, 25 \mid 8ab \) and \( 25 \mid 5(a+b) \). Since \( 25 \nmid 30 \), (4) has no integer solution. Thus, \( P = 0 \) has no solution also in this case.

**Case 3.** \( a \geq 2, b = c = 1 \), that is, \( |V_1| = 1 \).

The vertices \( x_2 = y_2 \) and \( x_3 = y_3 \) have degree 2, so we do not need to find \( S^1(x_2), S^1(x_3), S^3(y_2) \) and \( S^3(y_3) \). We have
\[
S^1(x_1) = -\binom{a}{2} - a - a, \\
M^1 = 0, \\
S^3(y_1) = 2\binom{a-1}{2} + (a-2) + 2 + 1 + 2 + 1, \\
M^3 = 0, \\
D = 3\binom{3}{3} = 3.
\]

By (3), expanding the terms we get
\[ P = -6a + 6 < 0 \]
as \( a \geq 2 \). Thus, \( P = 0 \) has no solution in this case.

**Case 4.** \( a = b = c = 1 \), that is, \( |V_1| = 0 \).

In this case \( C_{a,b,c} = K_{1,3} \) and \( L'(K_{1,3}) \) is a cycle of length 3 for every \( i \geq 1 \). Since \( W(G) = 3 \) if \( G \) is a cycle of length 3, while \( W(K_{1,3}) = 9 \), we have \( W(L^3(C_{1,1,1})) - W(C_{1,1,1}) \neq 0 \) also in this case. \( \square \)

### 4 Proof of Theorem 1.6

**Proof of Theorem 1.6.** We proceed analogously as in the proof of Theorem 1.5. That is, we prove Theorem 1.6 by counting the distances in \( L(H_{a,b,c,d,e}) \) instead of those in \( H_{a,b,c,d,e} \) and \( L^3(H_{a,b,c,d,e}) \). In \( L(H_{a,b,c,d,e}) \) we distinguish 10 vertices \( x_1, x_2, \ldots, x_4 \) and \( y_1, y_2, \ldots, y_6 \). The vertices \( x_1, \ldots, x_4 \) correspond to pendant edges of \( H_{a,b,c,d,e} \), while the vertices \( y_1, \ldots, y_6 \) correspond to edges incident with vertices of degree 3 in \( H_{a,b,c,d,e} \) (see Figure 3). Observe that if \( e = 0 \), then \( y_5 = y_6 \) and \( \deg(y_5) = 4 \). If \( d = 1 \) (\( c = 1, b = 1 \) or \( a = 1 \), then \( x_4 = y_4 \) (\( x_3 = y_3, x_2 = y_2 \) or \( x_1 = y_1 \)), and in such a case \( \deg(x_4) = 2 \) (\( \deg(x_3) = 2, \deg(x_2) = 2 \) or \( \deg(x_1) = 2 \), respectively).
In what follows, the graph $L(H_{a,b,c,d,e})$ is denoted by $LH$. Further, for $i \in \{1, 2, 3, 4\}$, let $V_i$ be the set of vertices of $V(LH)$ of degree $i$. For $x \in V_1$ and $y \in V_3 \cup V_4$, define
\[
S^1(x) = \sum_u h(u, x) \quad \text{where } u \in V(LH) \setminus V_1,
M^1 = \sum_{u \neq v} h(u, v) \quad \text{where } u, v \in V_1,
S^3(y) = \sum_u h(u, y) \quad \text{where } u \in V_2,
M^3 = \sum_{u \neq v} h(u, v) \quad \text{where } u, v \in V_3 \cup V_4,
D = \sum_u \left[ 3 \binom{u}{3} + 6 \binom{u}{4} \right] \quad \text{where } u \in V_3 \cup V_4.
\]
Observe that once again, $\sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3 \cup V_4} S^3(y) + M^3$ sums $h(u, v)$ for all pairs $\{u, v\}$ of vertices such that either $\text{deg}(u) \neq 2$ or $\text{deg}(v) \neq 2$.

Denote $P = W(L^2(LH)) - W(H_{a,b,c,d,e})$. Since $H_{a,b,c,d,e}$ has $a+b+c+d+e+2$ vertices, we have $W(H_{a,b,c,d,e}) = W(LH) + \binom{a+b+c+d+e+2}{2}$, by Theorem 1.1. Thus, by Lemma 2.1, we have
\[
P = W(L^2(LH)) - W(LH) - \binom{a+b+c+d+e+2}{2} = \sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3 \cup V_4} S^3(y) + M^3 + D - \binom{a+b+c+d+e+2}{2}. \tag{5}
\]

If $e = 0$, then we have one vertex of degree 4 in $LH$, while if $e \geq 1$, then the greatest degree of a vertex in $LH$ is 3. By symmetry, we distinguish eleven cases. In the first five cases we have $e \geq 1$ and in the next five we have $e = 0$. In each of these first ten cases (the last case will be solved in a different way) we evaluate $S^1$s, $M^1$, $S^3$s, $M^3$ and $D$ and we solve the equation $P = 0$. To avoid fractions, in some cases we solve the equation $2P = 0$.

![Figure 3: The graph $LH = L(H_{a,b,c,d,e})$ for $e \geq 1$ and $e = 0.$](image)

**Case 1.** $a, b, c, d \geq 2$, $e \geq 1$. 


We start with evaluating $S^1(x)$, where $x \in V_1$. Since \( \deg(x) = 1 \), we have $\beta_j(u, x) = 0$, $0 \leq j \leq 2$. Hence, $h(u, x) = -d(u, x)$. The sum of distances from $x_1$ to all interior vertices of $x_1 - x_2$ path is $1 + 2 + \ldots + (a+b-2) = \left(\frac{a+b-1}{2}\right)$ (see Figure 3). The sum of distances from $x_1$ to all interior vertices of $x_1 - x_3$ path, not included in the previous calculation, is $\left(\frac{a+b+c}{2}\right) - \left(\frac{1}{2}\right)$. Finally, the sum of distances from $x_1$ to all interior vertices of $x_1 - x_4$ path, not included previously, is $\left(\frac{a+b+d}{2}\right) - \left(\frac{a+c+1}{2}\right)$.

In this way we get $S^1(x_1)$ and analogously we calculate $S^1(x_2)$, $S^1(x_3)$ and $S^1(x_4)$:

\[
S^1(x_1) = -\left(\frac{a+b-1}{2}\right) - \left(\frac{a+e+c}{2}\right) + \left(\frac{a}{2}\right) - \left(\frac{a+b+d}{2}\right) + \left(\frac{a+c+1}{2}\right),
\]
\[
S^1(x_2) = -\left(\frac{a+b-1}{2}\right) - \left(\frac{b+c+e}{2}\right) + \left(\frac{b}{2}\right) - \left(\frac{b+b+d}{2}\right) + \left(\frac{b+c+1}{2}\right),
\]
\[
S^1(x_3) = -\left(\frac{a+e+c}{2}\right) - \left(\frac{b+c+e}{2}\right) + \left(\frac{c+e+1}{2}\right) - \left(\frac{c+d-1}{2}\right) + \left(\frac{c}{2}\right),
\]
\[
S^1(x_4) = -\left(\frac{a+b+d}{2}\right) - \left(\frac{b+c+e}{2}\right) + \left(\frac{e+d+1}{2}\right) - \left(\frac{e+d-1}{2}\right) + \left(\frac{e}{2}\right).
\]

Now $h(x_1, x_2) = -(a+b-1)$ and $h(x_1, x_3) = -(a+e+c)$. Using the symmetry we obtain

\[
M^1 = -(a+b-1) - (a+e+c) - (a+b+d) - (b+e+c) - (b+b+d) - (c+d-1).
\]

In $S^3(y)$ we sum $h(u, y)$, where \( \deg(u) = 2 \) and \( \deg(y) = 3 \). Hence, \( \left(\frac{d_a}{2}\right) \left(\frac{d_b}{2}\right) - 1 = 2 \). Since $\beta_0(u, y) = 2$, $\beta_1(u, y) = 1$ and $\beta_2(u, y) = 0$, we have $h(u, y) = 2d(u, y) + 1$. Thus, the sum of $h(u, y_1)$ for interior vertices $u$ of $y_1 - x_1$ path is $2(1 + 2 + \ldots + (a-2)) + (a-2) = 2\left(\frac{a-1}{2}\right) + (a-2)$ (see Figure 3). Analogously, the sum of $h(u, y_1)$ for interior vertices of $y_2 - x_2$ path is $2(2 + 3 + \ldots + (b-1)) + (b-2) = 2\left(\frac{b}{2}\right) + (b-4)$; the sum of $h(u, y_1)$ for interior vertices of $y_3 - y_6$ path is $2(2 + 3 + \ldots + (e-1)) + (e-1) = 2\left(\frac{e+1}{2}\right) + (e-3)$; and the sum of $h(u, y_1)$ for interior vertices of $y_3 - x_3$ path is $2((e+3) + (e+4) + \ldots + (e+e)) + (c-2) = 2\left(\frac{e+c+1}{2}\right) - 2\left(\frac{e+3}{2}\right) + (c-2)$. In this way we get

\[
S^3(y_1) = 2\left(\frac{a-1}{2}\right) + (a-2) + 2\left(\frac{b}{2}\right) + (b-4) + 2\left(\frac{e+1}{2}\right) + (e-3) + 2\left(\frac{e+c+1}{2}\right) - 2\left(\frac{e+3}{2}\right) + (c-2) + 2\left(\frac{e+d+1}{2}\right) - 2\left(\frac{e+d-1}{2}\right) + (d-2),
\]
\[
S^3(y_2) = 2\left(\frac{a}{2}\right) + (a-4) + 2\left(\frac{b-1}{2}\right) + (b-2) + 2\left(\frac{e+1}{2}\right) + (e-3) + 2\left(\frac{e+c+1}{2}\right) - 2\left(\frac{e+3}{2}\right) + (c-2) + 2\left(\frac{e+d+1}{2}\right) - 2\left(\frac{e+d-1}{2}\right) + (d-2),
\]
\[
S^3(y_3) = 2\left(\frac{a+e+1}{2}\right) - 2\left(\frac{e+3}{2}\right) + (a-2) + 2\left(\frac{b+e+1}{2}\right) - 2\left(\frac{e+3}{2}\right) + (b-2) + 2\left(\frac{e+1}{2}\right) + (e-3) + 2\left(\frac{e-1}{2}\right) + (c-2) + 2\left(\frac{d}{2}\right) + (d-4),
\]
\[
S^3(y_4) = 2\left(\frac{b+e+1}{2}\right) - 2\left(\frac{b+c+1}{2}\right) + (a-2) + 2\left(\frac{b+e+1}{2}\right) - 2\left(\frac{b+e+1}{2}\right) + (b-2) + 2\left(\frac{e+1}{2}\right) + (e-3) + 2\left(\frac{e}{2}\right) + (c-4) + 2\left(\frac{d-1}{2}\right) + (d-2),
\]
\[
S^3(y_5) = 2\left(\frac{a}{2}\right) + (a-4) + 2\left(\frac{b}{2}\right) + (b-4) + 2\left(\frac{e}{2}\right) + (e-1) + 2\left(\frac{e+c}{2}\right) - 2\left(\frac{e+2}{2}\right) + (c-2) + 2\left(\frac{e+d}{2}\right) - 2\left(\frac{e+2}{2}\right) + (d-2),
\]
\[
S^3(y_6) = 2\left(\frac{a+e}{2}\right) - 2\left(\frac{e+2}{2}\right) + (a-2) + 2\left(\frac{b+e}{2}\right) - 2\left(\frac{e+2}{2}\right) + (b-2) + 2\left(\frac{e}{2}\right) + (e-1) + 2\left(\frac{e}{2}\right) + (c-4) + 2\left(\frac{d}{2}\right) + (d-4).
\]
Consider $h(y_i, y_j)$, where $1 \leq i < j \leq 6$. Here $(d_{y_i})^2(d_{y_j})^2 - 1 = 8$ and $\beta_0(y_i, y_j) = 4$. If $y_i$ and $y_j$ lie in a common triangle, then $\beta_1(y_i, y_j) = 5$ and $\beta_2(y_i, y_j) = 0$, while if $y_i$ and $y_j$ do not lie in a common triangle, then $\beta_1(y_i, y_j) = 4$ and $\beta_2(y_i, y_j) = 1$. This means that $h(y_1, y_2) = 13$, $h(y_1, y_3) = 8e+2+6 = 8e+22$, $h(y_1, y_6) = 8(e+1)+6 = 8e+14$ and $h(y_5, y_6) = 8e+6$. Hence

$$M^3 = 13 + (8e+22) + (8e+22) + 13 + (8e+14) + (8e+22) + 13 + (8e+14) + 13 + (8e+14) + 13 + (8e+6).$$

Finally,

$$D = \sum u \left[ 3 \left( \frac{a}{4} \right) + 6 \left( \frac{d}{4} \right) \right] = 6 \left[ \frac{3}{4} \right] = 18.$$ 

By (5), expanding the terms (using a computer package, for instance), we get

$$2P = 7(a^2+b^2+c^2+d^2+e^2) - 6(ab+ac+ad+bc+bd+cd) + 4(ab+be+ce+de) + 5(a+b+c+d) + 65e + 234 = 7(a+b+c+d+e)^2 - 20(ab+ac+ad+bc+bd+cd) - 10(ab+be+ce+de) + 5(a+b+c+d) + 65e + 234.$$ 

Now substitute $x = (a+b+c+d+e)$ and consider the equation $2P = 0$ over $\mathbb{Z}_5$. We get

$$2x^2 + 4 = 0,$$

which has no solution in $\mathbb{Z}_5$. Consequently, $P = 0$ has no integer solution and $W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}) \neq 0$ in this case.

**Case 2.** $a, b, c \geq 2$, $d = 1$, $e \geq 1$.

In this case the vertex $x_4 = y_1$ has degree 2, so we do not need to find $S^1(x_4)$ and $S^3(y_4)$, but we must include the distances to $x_4$ in $S^1(x_1)$, $S^1(x_2)$, $S^1(x_3)$, $S^3(y_1)$, $S^3(y_2)$, $S^3(y_3)$, $S^3(y_5)$ and $S^3(y_6)$. Analogously as in the previous case we have

$$S^1(x_1) = -(a+b-1) -(a+e+1),$$

$$S^1(x_2) = -(a+b-1) -(b+e+1),$$

$$S^1(x_3) = -(a+b-1) -(b+e+1),$$

$$S^1(x_4) = -(a+b-1) -(b+e+1),$$

$$S^1(x_5) = -(a+b-1) -(b+e+1),$$

$$S^1(x_6) = -(a+b-1) -(b+e+1),$$

$$M^1 = -(a+b-1) -(a+e+c) -(b+e+c),$$

$$S^3(y_1) = 2(a-1) + (a-2) + (b-2) + (e+1) + (e-3) + 2(b+e+1) - 2(e+3) + (c-2) + 2(e+2) + 1,$$

$$S^3(y_2) = 2(a-1) + (a-2) + (b-2) + (e+1) + (e-3) + 2(b+e+1) - 2(e+3) + (c-2) + 2(e+2) + 1,$$

$$S^3(y_3) = 2(a-1) + (a-2) + (b-2) + (e+1) + (e-3) + 2(b+e+1) - 2(e+3) + (c-2) + 2(e+2) + 1,$$

$$S^3(y_4) = 2(a-1) + (a-2) + (b-2) + (e+1) + (e-3) + 2(b+e+1) - 2(e+3) + (c-2) + 2(e+2) + 1,$$

$$S^3(y_5) = 2(a-1) + (a-2) + (b-2) + (e+1) + (e-3) + 2(b+e+1) - 2(e+3) + (c-2) + 2(e+2) + 1,$$

$$S^3(y_6) = 2(a-1) + (a-2) + (b-2) + (e+1) + (e-3) + 2(b+e+1) - 2(e+3) + (c-2) + 2(e+2) + 1.$$
\[ S^3(y_3) = 2\binom{a+e+1}{2} - 2\binom{e+3}{2} + (a-2) + 2\binom{b+e+1}{2} - 2\binom{e+3}{2} + (b-2) \\
+ 2\binom{e+1}{2} + (e-3) + 2\binom{c-1}{2} + (c-2) + 2 + 1, \]
\[ S^3(y_5) = 2\binom{a}{2} + (a-4) + 2\binom{b}{2} + (b-4) + 2\binom{e}{2} + (e-1) \\
+ 2\binom{e+2}{2} - 2\binom{e+2}{2} + (c-2) + 2(e+1) + 1, \]
\[ S^3(y_6) = 2\binom{a+e}{2} - 2\binom{e+2}{2} + (a-2) + 2\binom{b+e}{2} - 2\binom{e+2}{2} + (b-2) \\
+ 2\binom{e}{2} + (e-1) + 2\binom{c}{2} + (c-4) + 2 + 1, \]
\[ M^3 = \left( 13 + (8e+22) + 13 + (8e+14) \right) + \left( (8e+22) + 13 + (8e+14) \right) \\
+ \left( (8e+14) + 13 \right) + \left( (8e+6) \right), \]
\[ D = 5 \cdot 3^3(3) = 15. \]

By (5), expanding the terms we get
\[ P = 3(a^2+b^2+c^2+e^2) - 3(ab+ac+bc) + (ae+be) + 2ce - 2(a+b) - c + 28e + 97. \]

Since \((a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a^2+b^2+c^2) - 2(ab+ac+bc) \geq 0\), we have
\[ 3(a^2+b^2+c^2) - 3(ab+ac+bc) \geq 0. \]
Hence, if \(e \geq 2\), then
\[ P \geq 3e^2 + (e-2)(a+b) + c(2e-1) + 28e + 97 > 0. \]

This means that if \(P = 0\) then \(e = 1\). For \(e = 1\) we obtain
\[ P = 3(a^2+b^2+c^2) - 3(ab+ac+bc) - a - b + c + 128. \]

Substituting \(a = 128 + x\), \(b = 128 + y\) and \(c = 128 + z\) we get
\[ P = 3(x^2+y^2+z^2) - 3(xy+xz+yz) - x - y + z. \]

Now we solve the equation \(P = 0\). This gives
\[ 3(x^2+y^2+z^2) - 3(xy+xz+yz) = x + y - z = 3t \]
or equivalently
\[ \frac{3}{2} \left( (x-y)^2 + (y-z)^2 + (z-x)^2 \right) = x + y - z = 3t, \]
where \(t\) is nonnegative integer. Since \(x\), \(y\) and \(z\) were defined using \(a\), \(b\) and \(c\), the differences \((z-y)\) and \((z-x)\) are integer numbers. Set \(i = (z-y)\) and \(j = (z-x)\).

Then \((x-y) = (z-y) - (z-x) = i - j\), so that
\[ 2t = (x-y)^2 + (y-z)^2 + (z-x)^2 = (i-j)^2 + (-i)^2 + j^2 = 2i^2 + 2j^2 - 2ij \]

13
and consequently $3t = 3i^2 + 3j^2 - 3ij = x + y - z$. This gives

$$x = 3t + (z-y) = 3i^2 + 3j^2 - 3ij + i,$$
$$y = 3t + (z-x) = 3i^2 + 3j^2 - 3ij + j,$$
$$z = x + y - 3t = 3i^2 + 3j^2 - 3ij + i + j,$$

which is equivalent to (1).

In [14] we proved that for every triple $a, b, c$ satisfying (1) and $e = 1$ it holds $P = 0$ (that is, $W(L^3(H_{a,b,c,1,1})) = W(H_{a,b,c,1,1})$). Thus, $P = 0$ in this case if and only if $e = 1$ and $a, b, c$ satisfy (1).

**Case 3.** $a, c \geq 2, b = d = 1, e \geq 1$.

The vertices $x_2 = y_2$ and $x_4 = y_4$ have degree 2, so we do not need to find $S^1(x_2)$, $S^1(x_4)$, $S^3(y_2)$ and $S^3(y_4)$. We have

$$S^1(x_1) = -a - \frac{a+e+c}{2} - (a+e+1),$$
$$S^1(x_3) = -(a+e+c) - (e+c+1) - c,$$
$$M^1 = -(a+e+c),$$
$$S^3(y_1) = 2\frac{a-1}{2} + (a-2) + 2 + 1 + 2\frac{e+1}{2} + (e-3) + 2\frac{e+e+1}{2} - 2\frac{e+3}{2} + (c-2) + 2(e+2) + 1,$$
$$S^3(y_3) = 2\frac{a+e+1}{2} - 2\frac{e+3}{2} + (a-2) + 2(e+2) + 1 + 2\frac{e+1}{2} + (e-3) + 2\frac{e-1}{2} + (c-2) + 2 + 1,$$
$$S^3(y_5) = 2\frac{a}{2} + (a-4) + 2 + 1 + 2\frac{e}{2} + (e-1) + 2\frac{e+2}{2} + (c-2) + 2(e+1) + 1,$$
$$S^3(y_6) = 2\frac{a+e}{2} - 2\frac{e+2}{2} + (a-2) + 2(e+1) + 1 + 2\frac{e}{2} + (e-1) + 2\frac{e}{2} + (c-4) + 2 + 1,$$
$$M^3 = \left((8e+22) + 13 + (8e+14)\right) + \left((8e+14) + 13\right) + \left((8e+6)\right),$$
$$D = 4 \cdot 3\frac{a}{2} = 12.$$

By (5), expanding the terms we get

$$2P = 5(a^2 + c^2 + e^2) - 6ac + 2(ae + ce) - 11(a + c) + 45e + 148.$$

Since $4(a-c)^2 = 4a^2 + 4e^2 - 8ac \geq 0$ and $(a+c-6)^2 = a^2 + c^2 + 2ac - 12(a+c) + 36 \geq 0$, we get

$$2P \geq 2 \left(a^2 + c^2 + 5e^2 + 2ac + 2(ae + ce) - 11(a + c) + 45e + 148\right) \geq 5e^2 + 2(ae + ce) + (a + c) + 45e + 112 > 0.$$

Thus, the equation $P = 0$ has no solution in this case.
Case 4. $a, b \geq 2$, $c = d = 1$, $e \geq 1$.

The vertices $x_3 = y_3$ and $x_4 = y_4$ have degree 2, so we do not need to find $S^1(x_3), S^1(x_4), S^3(y_3)$ and $S^3(y_4)$. We have

$$S^1(x_1) = -\binom{a+b-1}{2} - \binom{a+e+2}{2} + \binom{a}{2} - (a+e+1),$$
$$S^1(x_2) = -\binom{a+b-1}{2} - \binom{b+e+2}{2} + \binom{b}{2} - (b+e+1),$$
$$M^1 = -(a+b-1),$$
$$S^3(y_1) = 2\binom{a-1}{2} + (a-2) + 2\binom{b}{2} + (b-4) + 2\binom{e+1}{2} + (e-3) + 2(e+2) + 1 + 2(e+2) + 1,$$
$$S^3(y_2) = 2\binom{a}{2} + (a-4) + 2\binom{b-1}{2} + (b-2) + 2\binom{e+1}{2} + (e-3) + 2(e+2) + 1 + 2(e+2) + 1,$$
$$S^3(y_5) = 2\binom{a}{2} + (a-4) + 2\binom{b}{2} + (b-4) + 2\binom{e}{2} + (e-1) + 2(e+1) + 1 + 2(e+1) + 1,$$
$$S^3(y_6) = 2\binom{a+e}{2} - 2\binom{e+2}{2} + (a-2) + 2\binom{b+e}{2} - 2\binom{e+2}{2} + (b-2) + 2\binom{e}{2} + (e-1) + 2 + 1 + 2 + 1,$$
$$M^3 = \left(13 + 13 + (8e+14)\right) + \left(13 + (8e+14)\right) + \left((8e+6)\right),$$
$$D = 4 \cdot 3^{(3)} = 12.$$

By (5), expanding the terms we get

$$2P = 5(a^2+b^2+e^2) - 6ab - 13(a+b) + 47e + 148.$$  

Since $4(a-b)^2 = 4a^2+4b^2-8ab \geq 0$ and $(a+b-7)^2 = a^2+b^2+2ab-14(a+b)+49 \geq 0$, we get

$$2P \geq a^2 + b^2 + 5e^2 + 2ab - 13(a+b) + 47e + 148$$
$$\geq 5e^2 + (a+b) + 47e + 99 > 0.$$  

Thus, the equation $P = 0$ has no solution in this case.

Case 5. $a \geq 2$, $b = c = d = 1$, $e \geq 1$.

The vertices $x_2 = y_2$, $x_3 = y_3$ and $x_4 = y_4$ have degree 2, so we have

$$S^1(x_1) = -a - \binom{a+e+2}{2} - (a+e+1),$$
$$M^1 = 0,$$
$$S^3(y_1) = 2\binom{a-1}{2} + (a-2) + 2 + 1 + 2\binom{e+1}{2} + (e-3) + 2(e+2) + 1 + 2(e+2) + 1,$$
$$S^3(y_3) = 2\binom{a}{2} + (a-4) + 2 + 1 + 2\binom{a}{2} + (e-1) + 2(e+1) + 1 + 2(e+1) + 1,$$
$$S^3(y_6) = 2\binom{a+e}{2} - 2\binom{e+2}{2} + (a-2) + 2(e+1) + 1 + 2\binom{a}{2} + (e-1) + 2 + 1 + 2 + 1,$$
$$M^3 = \left(13 + (8e+14)\right) + \left((8e+6)\right),$$
$$D = 3 \cdot 3^{(3)} = 9.$$
By (5), expanding the terms we get

\[ P = 2a^2 + 2e^2 - 10a + 17e + 48. \]

Since \((a-5)^2 = a^2 - 10a + 25 \geq 0\), we get

\[ P \geq a^2 + 2e^2 + 17e + 23 > 0. \]

Thus, the equation \( P = 0 \) has no solution in this case.

**Case 6.** \( a, b, c, d \geq 2, \ e = 0 \).

In this case, and also in the next four, we have \( y_5 = y_6 \) and the degree of \( y_5 \) is 4 (see Figure 3). This does not affect \( S^1(x_i), M^1 \) and \( S^3(y_j) \), where \( 1 \leq i, j \leq 4 \). Hence, analogously as above we get

\[
\begin{align*}
S^1(x_1) & = -\left(\frac{a+b+1}{2}\right) - (a+c) + (a) - (\frac{a+d}{2}) + (\frac{a+1}{2}), \\
S^1(x_2) & = -\left(\frac{a+b+1}{2}\right) - (b+c) + (b) - (\frac{b+d}{2}) + (\frac{b+1}{2}), \\
S^1(x_3) & = -\left(\frac{a+c}{2}\right) - (b+c) + (c+1) - (\frac{c+d-1}{2}) + (\frac{c}{2}), \\
S^1(x_4) & = -\left(\frac{a+d}{2}\right) - (b+d) + (\frac{d+1}{2}) - (\frac{c+d-1}{2}) + (\frac{d}{2}), \\
M^1 & = -(a+b-1) - (a+c) - (a+d) - (b+c) - (b+d) - (c+d-1), \\
S^2(y_1) & = 2(a-1) + (a-2) + 2(b) + (b-4) + 2(c-1) + (c-8) + 2(d) + (d-8), \\
S^2(y_2) & = 2(a) + (a-4) + 2(b+1) + (b-2) + 2(c+1) + (c-8) + 2(d+1) + (d-8), \\
S^2(y_3) & = 2(a+1) + (a-8) + 2(b+1) + (b-8) + 2(c-1) + (c-2) + 2(d) + (d-4), \\
S^2(y_4) & = 2(a+1) + (a-8) + 2(b+1) + (b-8) + 2(c) + (c-4) + 2(d) + (d-2), \\
S^3(y_5) & = 5(a) - 5 + 3(a-2) + 5(b) - 5 + 3(b-2) + 5(c) - 5 + 3(c-2) + 5(d) - 5 + 3(d-2).
\end{align*}
\]

Now we discuss the terms containing \( h(u, y_5) \). In \( S^3(y_5) \) we sum \( h(u, y_5) \), where \( \deg(u) = 2 \) and \( \deg(y_5) = 4 \). Hence \( \binom{4}{2} \binom{4+3}{2} - 1 = 5 \). Since \( \beta_0(u, y_5) = 3 \), \( \beta_1(u, y_5) = 3 \), and \( \beta_2(u, y_5) = 0 \), we have \( h(u, y_5) = 5d(u, y_5) + 3 \). Thus, the sum of \( h(u, y_5) \) for interior vertices \( u \) of \( y_1 - x_1 \) path is \( 5(2+3+\ldots+(a-1)) + 3(a-2) = 5(a) - 5 + 3(a-2) \) (see Figure 3). In this way we get

\[
\begin{align*}
S^3(y_5) & = 5(a) - 5 + 3(a-2) + 5(b) - 5 + 3(b-2) + 5(c) - 5 + 3(c-2) + 5(d) - 5 + 3(d-2).
\end{align*}
\]

Now consider \( h(y_i, y_5), 1 \leq i \leq 4 \). Here \( \binom{4}{2} \binom{2+3}{2} - 1 = 17 \) and \( \beta_0(y_i, y_5) = 2 \cdot 3 = 6 \). Since \( y_i \) and \( y_5 \) always lie in a common triangle, we have \( \beta_1(y_i, y_5) = 11 \) and \( \beta_2(y_i, y_5) = 1 \) (see Figure 3). Thus, \( h(y_i, y_5) = 17 \cdot 1 + 11 + 2 \cdot 1 = 30 \). As regards \( h(y_i, y_j), 1 \leq i < j \leq 4 \), analogously as above we get \( h(y_1, y_2) = 13 \) and \( h(y_1, y_3) = 8e + 22 = 22 \). Hence

\[
M^3 = (13+22+22+30) + (22+22+30) + (13+30) + 30.
\]
Finally,
\[ D = \sum_a \left[ 3\binom{a}{2} + 6\binom{a}{4} \right] = 4\left[ 3\binom{3}{2} + 3\binom{3}{4} + 6\binom{4}{4} \right] = 12 + 18. \]

By (5), expanding the terms we get
\[ P = 4(a^2 + b^2 + c^2 + d^2) - 3(ab + ac + ad + bc + bd + cd) + 3(a + b + c + d) + 137 \]
\[ = 4(a + b + c + d)^2 - 11(ab + ac + ad + bc + bd + cd) + 3(a + b + c + d) + 137. \]

Substitute \( x = (a + b + c + d) \) and consider the equation \( P = 0 \) over \( \mathbb{Z}_{11} \). We get
\[ 4x^2 + 3x + 5 = 0, \]
which has no solution in \( \mathbb{Z}_{11} \). Consequently, \( P = 0 \) has no integer solution and \( W(L^3(H_{a,b,c,d,0})) - W(H_{a,b,c,d,0}) \neq 0 \) in this case.

**Case 7.** \( a, b, c \geq 2, d = 1, e = 0. \)

In this case the vertex \( x_4 = y_4 \) has degree 2, so we do not need to find \( S^1(x_4) \) and \( S^3(y_4) \). Analogously as in the previous case we have
\[ S^1(x_1) = -(\binom{a+b-1}{2}) - (\binom{a+c}{2}) + (\binom{a}{2}) - (a+1), \]
\[ S^1(x_2) = -(\binom{a+b-1}{2}) - (b + c) + (\binom{b}{2}) - (b+1), \]
\[ S^1(x_3) = -(\binom{a+c}{2}) - (\binom{b+c}{2}) + (\binom{c+1}{2}) - c, \]
\[ M^1 = -(a+b-1) - (a+c) - (b+c), \]
\[ S^3(y_1) = 2(\binom{a-1}{2}) + (a-2) + 2\binom{b}{2} + (b-4) + 2(\binom{c+1}{2}) + (c-8) + 4 + 1, \]
\[ S^3(y_2) = 2(\binom{a}{2}) + (a-4) + 2\binom{b-1}{2} + (b-2) + 2(\binom{c+1}{2}) + (c-8) + 4 + 1, \]
\[ S^3(y_3) = 2(\binom{a}{2}) + (a-8) + 2\binom{b+1}{2} + (b-8) + 2(\binom{c-1}{2}) + (c-2) + 2 + 1, \]
\[ S^3(y_4) = 5(\binom{a}{2}) - 5 + 3(a-2) + 5\binom{b}{2} - 5 + 3(b-2) + 5\binom{c}{2} - 5 + 3(c-2) + 5 + 3, \]
\[ M^3 = (13+22+30) + (22+30) + 30, \]
\[ D = 3\left( \binom{3}{3} \right) + \left( \binom{4}{3} + 6\binom{4}{4} \right) = 9 + 18. \]

By (5), expanding the terms we get
\[ 2P = 7(a^2 + b^2 + c^2) - 6(ab + ac + bc) - 3(a+b) - c + 232. \]

Since \( 3(a-b)^2 + 3(b-c)^2 + 3(c-a)^2 = 6(a^2 + b^2 + c^2) - 6(ab + ac + bc) \geq 0 \) and also \( (a-2)^2 + (b-2)^2 + (c-1)^2 = (a^2 + b^2 + c^2) - 4(a+b) - 2c + 9 \geq 0 \), we get
\[ 2P \geq (a^2 + b^2 + c^2) - 3(a+b) - c + 232 \]
\[ \geq a + b + c + 223 > 0. \]
Thus, the equation $P = 0$ has no solution in this case.

**Case 8.** $a, c \geq 2$, $b = d = 1$, $e = 0$.

The vertices $x_2 = y_2$ and $x_4 = y_4$ have degree 2, so we have

$$S^1(x_1) = -a - \binom{a+c}{2} - (a+1),$$
$$S^1(x_3) = -(a+c) - (c+1) - c,$$
$$M^1 = -(a+c),$$
$$S^3(y_1) = 2\binom{a-1}{2} + (a-2) + 2 + 1 + 2\binom{c+1}{2} + (c-8) + 4 + 1,$$
$$S^3(y_2) = 2\binom{a+1}{2} + (a-8) + 4 + 1 + 2\binom{c-1}{2} + (c-2) + 2 + 1,$$
$$S^3(y_3) = 5\binom{a}{2} - 5 + 3(a-2) + 5 + 3 + 5\binom{c}{2} - 5 + 3(c-2) + 5 + 3,$$
$$M^3 = -(a+b-1),$$
$$D = 2\left(3\binom{3}{3}\right) + \left(3\binom{4}{3} + 6\binom{4}{4}\right) = 6 + 18.$$

By (5), expanding the terms we get

$$P = 3(a^2 + c^2) - 3ac - 5(a+c) + 92.$$  

Since $2(a-c)^2 = 2(a^2+c^2) - 4ac \geq 0$ and $(a-3)^2 + (c-3)^2 = (a^2+c^2) - 6(a+c) + 18 \geq 0$, we get

$$P \geq (a^2+c^2) + ac - 5(a+c) + 92 \geq ac + (a+c) + 74 > 0.$$  

Thus, the equation $P = 0$ has no solution in this case.

**Case 9.** $a, b \geq 2$, $c = d = 1$, $e = 0$.

The vertices $x_3 = y_3$ and $x_4 = y_4$ have degree 2, so we have

$$S^1(x_1) = -(a+b-1) - (a+1) - (a+1) - a,$$
$$S^1(x_2) = -(a+b-1) - (b+1) - (b+1) - b,$$
$$M^1 = -(a+b-1),$$
$$S^3(y_2) = 2\binom{a+1}{2} + (a-4) + 2\binom{b-1}{2} + (b-2) + 4 + 1 + 4 + 1,$$
$$S^3(y_3) = 5\binom{a}{2} - 5 + 3(a-2) + 5\binom{b}{2} - 5 + 3(b-2) + 5 + 3 + 5 + 3,$$
$$M^3 = (13+30) + 30,$$
$$D = 2\left(3\binom{3}{3}\right) + \left(3\binom{4}{3} + 6\binom{4}{4}\right) = 6 + 18.$$

By (5), expanding the terms we get

$$P = 3(a^2+b^2) - 3ab - 6(a+b) + 92.$$  

18
Since \(2(a-b)^2 = 2(a^2 + b^2) - 4ab \geq 0\) and \((a-3)^2 + (b-3)^2 = (a^2 + b^2) - 6(a+b) + 18 \geq 0\), we get

\[
P \geq (a^2 + b^2) + ab - 6(a+b) + 92 \\
\geq ab + 74 > 0.
\]

Thus, the equation \(P = 0\) has no solution in this case.

**Case 10.** \(a \geq 2, b = c = d = 1, e = 0\).

The vertices \(x_2 = y_2, x_3 = y_3\) and \(x_4 = y_4\) have degree 2, so we have

\[
S^1(x_1) = -(a+1) - (a+1) - (a+1) - a, \\
M^1 = 0, \\
S^3(y_1) = 2(a^2) + (a-2) + 2 + 1 + 4 + 1 + 4 + 1, \\
S^3(y_5) = 5(a^2) - 5 + 3(a-2) + 5 + 3 + 5 + 3 + 5 + 3, \\
M^3 = 30, \\
D = 3(\binom{3}{3}) + \left(3\binom{4}{3} + 6\binom{4}{4}\right) = 3 + 18.
\]

By (5), expanding the terms we get

\[2P = 5a^2 - 19a + 130.\]

Since \(5(a-2)^2 = 5a^2 - 20a + 20\), we get

\[2P \geq a + 110.\]

Thus, the equation \(P = 0\) has no solution in this case.

**Case 11.** \(a = b = c = d = 1, e \geq 0\).

In [15, Theorem 1.5] we proved that \(W(L^i(T)) > W(T)\) for every \(i \geq 3\) and for every tree \(T\) which is different from a path and the claw \(K_{1,3}\) and in which no leaf is adjacent to a vertex of degree 2. By this statement, for \(H = H_{1,1,1,1,e}\) we have \(W(L^3(H)) > W(H)\), which completes the proof.

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