Diameter and connectivity of 3-arc graphs

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Abstract

An arc of a graph is an oriented edge and a 3-arc is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of length two. The 3-arc graph of a given graph G, $X(G)$, is defined to have vertices the arcs of G. Two arcs uv, xy are adjacent in $X(G)$ if and only if (v, u, x, y) is a 3-arc of G. This notion was introduced in recent studies of arc-transitive graphs. In this paper we study diameter and connectivity of 3-arc graphs. In particular, we obtain sharp bounds for the diameter and connectivity of $X(G)$ in terms of the corresponding invariant of G.

Key words: 3-arc graph, diameter, connectivity, splitting construction, 3-arc graph construction
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1 Introduction

In this paper we study a new graph operator, namely the 3-arc graph construction which was first introduced [11, 16] in studying those arc-transitive graphs whose automorphism group contains a subgroup acting imprimitively on the vertex set. (A graph is arc-transitive if its automorphism group is transitive on the set of oriented edges.) This construction has been proved to be very useful in classifying or characterizing [11] certain families of arc-transitive graphs. For example, the cross-ratio graphs in [5] can be defined [15] equivalently as 3-arc graphs of $(\Gamma, 2)$-arc transitive complete graphs, where Γ is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, q)$, and the main result in [17] relies heavily on this construction as well. In two recent papers [7, 12] the 3-arc graph construction has also been used to construct some families of arc-transitive graphs. In this paper we will investigate this construction from a pure combinatorial point of view without involving arc-transitivity with focus on diameter and connectivity.

Let G be a graph. An *arc* of G is an ordered pair of adjacent vertices. For adjacent vertices u, v of G, we use uv to denote the arc from u to v, vu ($\neq uv$) the arc from v to u, and $\{u, v\}$ the edge between u and v. A *3-arc* of G is a 4-tuple (v, u, x, y) of vertices of G such that both

v, u, x and u, x, y are paths of length two. It is allowed to have $v = y$, and in this case the 3-arc (v, u, x, y) becomes the oriented cycle (v, u, x, v) of length three. A set Δ of 3-arcs of G is said to be *self-paired* if $(v, u, x, y) \in \Delta$ implies $(y, x, u, v) \in \Delta$.

Definition 1 Let G be a graph and Δ a self-paired set of 3-arcs of G. The *3-arc graph* [11, 16] of G with respect to Δ, $X(G, \Delta)$, is defined to have vertex set the set of arcs of G such that two vertices corresponding to two arcs uv, xy are adjacent if and only if $(v, u, x, y) \in \Delta$. The edge of $X(G, \Delta)$ between uv and xy will be denoted by $\{uv, xy\}$.

In the case when Δ is the set of all 3-arcs of G, the corresponding graph $X(G, \Delta)$ is called the *3-arc graph of* G, denoted by $X(G)$.

Since Δ is self-paired, $X(G, \Delta)$ is an undirected graph. In particular, $X(G)$ is an undirected graph with $2\,|E(G)|$ vertices and $\sum_{\{u,x\} \in E(G)} (\deg_G(u) - 1)(\deg_G(x) - 1)$ edges.

We can view X as a graph operator which outputs the 3-arc graph $X(G)$ for any given G. This operator is closely related to the well known line graph operator L. In fact, we can obtain $X(G)$ from the line graph $L(G)$ of G by the following operations. First, we split each vertex $\{u, v\}$ of $L(G)$ (that is, an edge of G) into two vertices, namely uv and vu. Then, for any two vertices $\{u, v\}, \{x, y\}$ of $L(G)$ which are distance two apart in $L(G)$, say, u and x are adjacent in G, we join uv and xy by an edge. The graph obtained this way is isomorphic to $X(G)$. On the other hand, define $P\{u, v\} = \{uv, vu\}$ for each vertex $\{u, v\}$ of $L(G)$, and let $\mathcal{P} = \{P\{u, v\} : \{u, v\} \in E(G)\}$. Then $\mathcal{P}$ is a partition of the vertex set of $X(G)$ into parts of size two, and the quotient graph of $X(G)$ with respect to $\mathcal{P}$ is isomorphic to the graph obtained from the square of $L(G)$ by deleting the edges of $L(G)$. (The square of a graph is defined to have the same vertex set in which two vertices are adjacent if and only if their distance in the original graph is one or two.) Obviously, there is a bijection between the edges of $X(G)$ and those of the 2-path graph $P_2(G)$, which is defined to have vertices the paths of length two in G such that two vertices are adjacent if and only if the union of the corresponding paths is a path or a cycle of length three, see [4]. Since $P_2(G)$ is a spanning subgraph of the second iterated line graph $L^2(G) = L(L(G))$ (see e.g. [8]), we have yet another relation between 3-arc graphs and line graphs.

There is an extensive literature on line graphs. See for example [6, 14] for surveys and [13, 9] for diameter and connectivity of iterated line graphs respectively. Some results on diameter of path graphs can be found in [2], while the connectivity of P_2-path graphs is studied e.g. in [10] and [1]. In contrast, we know little about the 3-arc graph operator X, despite its usefulness in algebraic graph theory. In this paper we will focus on diameter and connectivity of 3-arc graphs.

Obviously, adding or deleting isolated vertices does not affect $X(G)$. Moreover, if G contains two connected components other than isolated vertices, then $X(G)$ is a disconnected graph; if G contains a degree-one vertex, say, u, which is adjacent to v, then uv is an isolated vertex of $X(G)$. *Therefore, we will consider only connected graphs G with minimum degree $\delta(G) \geq 2$.*

We use $\deg_G(u)$ to denote the degree of a vertex u in G, $d_G(u, v)$ the distance in G between u and v, and $(u, \ldots, v)$ a path connecting u and v. The reader is referred to [3] for terminology undefined in the paper.

2 Results

Unlike the line graph $L(G)$, the 3-arc graph $X(G)$ is not necessarily connected even for connected G. Our first result, Theorem 2 below, tells us precisely when $X(G)$ is connected. Define $G^{\div}$ to be the graph obtained from G by replacing each vertex u of degree two by a pair u', u'' of nonadjacent vertices, each joined to exactly one neighbour of u. Note that u', u'' are degree-one vertices of $G^{\div}$. Thus, $G^{\div}$ contains no degree-two vertex, and it has twice as many degree-one vertices as is the number of degree-two vertices in G. In particular, if $\delta(G) \geq 3$, then $G^{\div} = G$.

Theorem 2 *Let G be a connected graph with $\delta(G) \geq 2$. Then $X(G)$ is connected if and only if $G^{\div}$ is connected. In particular, if $\delta(G) \geq 3$, then $X(G)$ is connected.*

Next we consider the connectivity κ. $X(G)$ can be disconnected when $1 \leq \kappa(G) \leq 2$. In the case $\kappa(G) \geq 3$, we can bound the connectivity of $X(G)$ in terms of the connectivity of G.

Theorem 3 *Let G be a graph with connectivity $\kappa(G) \geq 3$. Then*

$$\kappa(X(G)) \geq (\kappa(G) - 1)^2.$$

Moreover, this bound is best possible.

In fact, for any maximally connected k-regular graph G (that is, $\kappa(G) = k$), where $k \geq 3$, $X(G)$ is a $(k-1)^2$-regular graph and thus cannot be more than $(k-1)^2$-connected. Hence $\kappa(X(G)) = (\kappa(G) - 1)^2$ and the bound in Theorem 3 is attained by G.

Denote by diam the diameter of a graph. We will prove the following results.

Theorem 4 *Let G be a connected graph with $\delta(G) \geq 3$. Then*

$$\mathrm{diam}(G) \leq \mathrm{diam}(X(G)) \leq \mathrm{diam}(G) + 2$$

with both bounds attainable. In addition, the lower bound holds as long as G has at least two vertices.

Theorem 5 *Let r and s be arbitrary integers such that $4 \leq r \leq s - 4$ and $s \geq 10$. Then there exists a graph $G_{r,s}$ such that $\mathrm{diam}(G_{r,s}) = r$ and $\mathrm{diam}(X(G_{r,s})) = s$.*

By Theorem 4 any graph $G_{r,s}$ satisfying the conditions of Theorem 5 must satisfy $\delta(G_{r,s}) = 2$, because otherwise we would have $r \leq s \leq r + 2$ which violates $r \leq s - 4$. Theorem 5 shows that $\mathrm{diam}(X(G))$ can be arbitrarily large when $\mathrm{diam}(G) \geq 4$ (and $\delta(G) = 2$). This is not the case if $\mathrm{diam}(G) \leq 3$ as indicated by the following result.

Theorem 6 *Let G be a connected graph such that $X(G)$ is connected. Then the following hold:*

(a) *if* $\mathrm{diam}(G) = 1$, *then* $\mathrm{diam}(X(G)) = 2$*;*

(b) *if* $\mathrm{diam}(G) = 2$, *then* $\mathrm{diam}(X(G)) \leq 7$*;*

(c) *if* $\mathrm{diam}(G) = 3$, *then* $\mathrm{diam}(X(G)) \leq 14$.

Let G be the graph obtained from a 6-cycle $(u_0, u_1, u_2, u_3, u_4, u_5, u_0)$ by adding two chords $\{u_0, u_2\}$ and $\{u_2, u_4\}$. Then $\mathrm{diam}(G) = 2$ and $\mathrm{diam}(X(G)) = 6$ (with the diameter achieved by $d_{X(G)}(u_0u_2, u_4u_2)$). This suggests that the bound (b) in Theorem 6 may be improved slightly. As regards to (c), we believe that it is far from being optimal.

We will prove Theorems 2 and 3 in Section 4, and Theorems 4, 5 and 6 in Section 5, after a preliminary result is given in Section 3.

3 Paths in 3-arc graphs

The *trace* of an edge $\{u_0v_0, u_1v_1\}$ of $X(G)$ is defined to be the edge $\{u_0, u_1\}$ of G. It is clear that, for two adjacent edges of $X(G)$, say $\{u_0v_0, u_1v_1\}$ and $\{u_1v_1, u_2v_2\}$, the traces $\{u_0, u_1\}$ and $\{u_1, u_2\}$ are either adjacent in G (if $u_0 \neq u_2$) or identical (if $u_0 = u_2$). In the former case we have $\deg_G(u_1) \geq 3$ as u_0, u_2 and v_1 are distinct neighbours of u_1, while in the latter case we have $\deg_G(u_1) \geq 2$ as $u_0 \neq v_1$. In general, if $P = (u_0v_0, u_1v_1, u_1v_1, \ldots, u_kv_k)$ is a path or walk in $X(G)$, then the traces of $\{u_0v_0, u_1v_1\}, \{u_1v_1, u_2v_2\}, \ldots, \{u_{k-1}v_{k-1}, u_kv_k\}$ form a walk $(u_0, u_1, \ldots, u_k)$ in G, which we call the *trace* of P.

The following lemma regarding the trace of a shortest path will be used in the next two sections. Denote by $G^{\times}$ the subgraph of a graph G induced by vertices of degree at least three.

Lemma 7 *Let G be a connected graph with $\delta(G) \geq 2$ and let $P = (u_0v_0, u_1v_1, \ldots, u_kv_k)$ be a shortest path in $X(G)$.*

(a) *If $k \geq 2$, then $(u_1, u_2, \ldots, u_{k-1})$ is either a path or a cycle in G.*

(b) *If $k \geq 4$, then $u_2, u_3, \ldots, u_{k-2}$ all have degrees at least three and $(u_2, u_3, \ldots, u_{k-2})$ is a shortest path in $G^{\times}$.*

Proof First we show that in the trace of P no edge can appear twice except possibly $\{u_0, u_1\} = \{u_1, u_2\}$ or $\{u_{k-2}, u_{k-1}\} = \{u_{k-1}, u_k\}$. By way of contradiction suppose that $\{u_i, u_{i+1}\} = \{u_j, u_{j+1}\}$ for some $i < j$ with $(i, j) \neq (0, 1), (k-2, k-1)$. We show that there exists a path in $X(G)$ between u_0v_0 and u_kv_k which is shorter than P. In fact, if $u_i = u_j$ and $u_{i+1} = u_{j+1}$, then $u_i \neq v_{j+1}$ and $v_i \neq u_{j+1}$, and hence P can be shortened to $(u_0v_0, \ldots, u_iv_i, u_{j+1}v_{j+1}, u_{j+2}v_{j+2}, \ldots, u_kv_k)$. So we assume $u_i = u_{j+1}$ and $u_{i+1} = u_j$ in the following. If $i + 1 < j$, then P can be shortened to $(u_0v_0, \ldots, u_iv_i, u_{i+1}v_{i+1}, u_{j+1}v_{j+1}, \ldots, u_kv_k)$. Hence we may further assume $i + 1 = j$ so that $u_i = u_{i+2}$. Since $(i, j) \neq (0, 1), (k-2, k-1)$, we have $2 \leq i+1 = j \leq k-2$. If $u_{i-1} = u_{i+1}$, then $u_{i-1} = u_j$ and $u_i = u_{j+1}$, but this case was already excluded. The case $u_j = u_{j+2}$ can be treated similarly. If $u_{i-1} = u_{j+2}$, then $\{u_{i-1}, u_i\} = \{u_{j+1}, u_{j+2}\}$, and since $u_i = u_{j+1}$ and $(i-1) + 1 < j+1$, this case was already solved. Hence we may assume that u_{i-1}, $u_{i+1}\,(=u_j)$ and u_{j+2} are pairwise distinct. However, this implies that P can be shortened to $(u_0v_0, \ldots, u_{i-1}v_{i-1}, u_iu_{i+1}, u_{j+2}v_{j+2}, \ldots, u_kv_k)$.

Now we prove (a). Suppose $u_i = u_j$ for some $1 \leq i < j \leq k-1$ and suppose that $\deg_G(u_i) \geq 3$. Then u_i has a neighbour x other than u_{i-1} and u_{j+1}, and so P can be shortened to $(u_0v_0, \ldots, u_{i-1}v_{i-1}, u_ix, u_{j+1}v_{j+1}, \ldots, u_kv_k)$, a contradiction. Hence we may assume $\deg_G(u_i) = 2$. As $1 \leq i < k-1$, the trace of P contains u_{i-1} and u_{i+1}. These two vertices must be distinct from v_i, so that $u_{i-1} = u_{i+1}$. Consequently, the edge $\{u_{i-1}, u_i\} = \{u_i, u_{i+1}\}$ appears twice on the trace and since $i < k-1$, by previous part of this proof we have $i = 1$. Analogously we can prove $j = k-1$, which finishes the proof of (a).

In fact, we proved more. We proved that all $u_2, u_3, \ldots, u_{k-2}$ have degrees at least 3. Hence, it remains to prove that $u_2, u_3, \ldots, u_{k-2}$ is a shortest path in $G^\times$. Let $(z_2, z_3, \ldots, z_{t-2})$ be any path connecting $z_2 = u_2$ and $z_{t-2} = u_{k-2}$ in $G^\times$. Denote $z_1 = u_1$ and $z_{t-1} = u_{k-1}$. Since the degrees of $z_2, z_3, \ldots, z_{t-2}$ are at least three, for every i there is a neighbour w_i of z_i distinct from z_{i-1} and z_{i+1}, $2 \le i \le t-2$. But then $Q = (u_0v_0, u_1v_1, z_2w_2, z_3w_3, \ldots, z_{t-2}w_{t-2}, u_{k-1}v_{k-1}, u_kv_k)$ is a path in $X(G)$. Hence we obtain (b) by taking for $(z_2, z_3, \ldots, z_{t-2})$ the shortest path connecting $z_2 = u_2$ and $z_{t-2} = u_{k-2}$ in $G^\times$. $\square$

4 Proof of Theorems 2 and 3

In the proof of Theorem 2 we use Lemma 7.

Proof of Theorem 2 Let G be a connected graph with $\delta(G) \ge 2$. Suppose first that $G^\div$ is connected. We prove that there is a path between any two distinct vertices u_1v_1 and u_2v_2 of $X(G)$.

Consider the case $u_1 = u_2$ first. In this case, if $\deg_G(u_1) \ge 3$, then there is a neighbour $x \neq v_1, v_2$ of u_1. Let y be a neighbour of x other than u_1. Then (u_1v_1, xy, u_2v_2) is a path of length two in $X(G)$ connecting u_1v_1 and u_2v_2, and we are done. So we may suppose $\deg_G(u_1) = 2$. Let u_1' and u_1'' be the two vertices of $G^\div$ obtained by splitting u_1. Since $G^\div$ is connected, there is a path from u_1' to u_1'' in $G^\div$. All internal vertices on this path must have degree at least three in G. Hence there exists a cycle C in G containing u_1 such that all its vertices except u_1 have degree at least three in G. Let $W_0 = (u_1, v_2, \ldots, v_1, u_1)$ be the walk in G starting at u_1, then traversing all edges of C and terminating at u_1. That is, we prescribe the direction in which W_0 traverses C.

Now suppose $u_1 \neq u_2$. Since $G^\div$ is connected, there is a path W_0 in G starting at u_1 and terminating at u_2, such that all internal vertices of W_0 have degree at least three. Moreover, if $\deg_G(u_1) = 2$, we may assume $W_0 = (u_1, w_1, \ldots, u_2)$, where w_1 is the unique neighbour of u_1 other than v_1; if $\deg_G(u_2) = 2$, we may assume $W_0 = (u_1, \ldots, w_2, u_2)$, where w_2 is the unique neighbour of u_2 other than v_2.

In both possibilities above, the internal vertices of W_0 have degree at least three. Let $W_0 = (u_1, w_1, \ldots, w_2, u_2)$. From the choice of W_0, the case $w_1 = v_1$ occurs only when $\deg_G(u_1) \ge 3$, and in this case we extend W_0 by adding the prefix (u_1, x_1, u_1), where $x_1 \neq v_1\,(=w_1)$ is a neighbour of u_1. Analogously, the case $w_2 = v_2$ occurs only when $\deg_G(u_2) \ge 3$, and in this case we extend W_0 by adding the suffix (u_2, x_2, u_2), where $x_2 \neq v_2\,(=w_2)$ is a neighbour of u_2. Let W be the walk obtained this way in these two cases, and define $W = W_0$ otherwise. In the following we construct a path P in $X(G)$ connecting u_1v_1 and u_2v_2 with trace W.

If W differs from W_0 at the beginning, then P starts with $(u_1v_1, x_1y_1, u_1z_1, \ldots)$, where y_1 is a neighbour of x_1 different from u_1, and z_1 is a neighbour of u_1 different from x_1 and $v_1\,(=w_1)$. If W differs from W_0 at the end, then P terminates with $(\ldots, u_2z_2, x_2y_2, u_2v_2)$, where y_2 is a neighbour of x_2 different from u_2, and z_2 is a neighbour of u_2 different from x_2 and $v_2\,(=w_2)$. Denote $W_0 = (a_0, a_1, \ldots, a_k)$, where $a_0 = u_1$ and $a_k = u_2$. In all cases it suffices to construct the part P_0 of P whose trace is W_0. Note that the end-vertices of P_0 are already defined, namely,

$P_0 = (a_0b_0, \ldots, a_kb_k)$, where $a_0b_0 = u_1z_1$ if $a_1 = v_1$ and $a_0b_0 = u_1v_1$ otherwise, and $a_kb_k = u_2z_2$ if $a_{k-1} = v_2$ and $a_kb_k = u_2v_2$ otherwise. Since $\deg_G(a_i) \geq 3$, $0 < i < k$, there exists a neighbour b_i of a_i in G other than a_{i-1} and a_{i+1}. Let $P_0 = (a_0b_0, a_1b_1, a_2b_2, \ldots, a_{k-1}b_{k-1}, a_kb_k)$. Then P_0 is a path in $X(G)$ with trace W_0. Adding the prefix or suffix to P_0 whenever applicable, we obtain the desired path P connecting u_1v_1 and u_2v_2. Up to now we have proved that if $G^{\div}$ is connected then so is $X(G)$.

Now suppose that $G^{\div}$ is a disconnected graph. Then, since G is connected, it contains a vertex u of degree two such that u' and u'' are in different connected components of $G^{\div}$. Denote by v_1 and v_2, respectively, the two neighbours of u in G. Suppose that there is a path in $X(G)$ connecting uv_1 with uv_2, and denote by $P = (uv_1, x_1y_1, x_2y_2, \ldots, x_{k-1}y_{k-1}, uv_2)$ a shortest one. Observe that $x_1 = v_2$ and $x_{k-1} = v_1$. In the next we consider the trace of P. Since uv_1 and uv_2 are not adjacent in $X(G)$, $k \geq 2$. By Lemma 7, all $x_2, x_3, \ldots, x_{k-2}$ have degrees at least three. If one of x_1 and x_{k-1} has degree two in G then G has adjacent vertices of degree two and consequently $X(G)$ is disconnected. Hence, we may assume that $x_1, x_2, \ldots, x_{k-1}$ is a path connecting v_2 with v_1 in $G^{\times}$, so that $u'', x_1, x_2, \ldots, x_{k-1}, u'$ is a path in $G^{\div}$, a contradiction. □

Possibly due to the relation explained in the introduction, the paths constructed in the proof of Theorem 3 are very similar to those constructed for 2-iterated line graphs [9] and 2-path graphs [10].

Proof of Theorem 3 We will use the following version of Menger's theorem: A graph G is k-connected if and only if it has more than k vertices and for each pair of nonadjacent vertices there exist k internally-vertex-disjoint paths connecting them.

Denote $k = \kappa(G)$. Let x_1y_1 and x_2y_2 be distinct and nonadjacent vertices of $X(G)$. We prove $\kappa(X(G)) \geq (k-1)^2$ by constructing $(k-1)^2$ internally-vertex-disjoint paths connecting x_1y_1 and x_2y_2 in $X(G)$.

Case 1: Consider the case $x_1 = x_2$ first. Since $\delta(G) \geq k$, x_1 has $k-2$ neighbours which are distinct from y_1 and y_2. Denote these neighbours by $y_3, y_4, \ldots, y_k$. Further, for $3 \leq i \leq k$, y_i has $k-1$ neighbours, say, $z_{i,1}, z_{i,2}, \ldots, z_{i,k-1}$, which are distinct from x_1. Define $P_{i,j} = (x_1y_1, y_iz_{i,j}, x_1y_2)$, $3 \leq i \leq k$, $1 \leq j \leq k-1$. These are $(k-2)(k-1)$ internally-vertex-disjoint paths in $X(G)$ connecting x_1y_1 and x_1y_2. Since $k = \kappa(G)$, $G - \{x_1, y_3, y_4, \ldots, y_k\}$ is connected. Let $P = (a_1, a_2, \ldots, a_{t-1})$ be a path in $G - \{x_1, y_3, y_4, \ldots, y_k\}$ connecting $a_1 = y_1$ and $a_{t-1} = y_2$. Since $\delta(G) \geq k$, we may choose $k-2$ neighbours $u_3, u_4, \ldots, u_k$ of a_1 other than x_1 and a_2, and $k-2$ neighbours $v_3, v_4, \ldots, v_k$ of a_{t-1} other than x_1 and a_{t-2}. Define $P_i = (x_1y_1, y_2v_i, x_1y_i, y_1u_i, x_1y_2)$, $3 \leq i \leq k$. These are internally-vertex-disjoint paths, and none of them contains any internal vertex of any $P_{i,j}$. Now we have found $(k-1)(k-2) + (k-2) = (k-1)^2 - 1$ internally-vertex-disjoint paths connecting x_1y_1 and x_1y_2, so it remains to construct the last one. If $\deg_G(x_1) > k$ then x_1 had a neighbour y_0 distinct from $y_1, y_2, y_3, \ldots, y_k$ and we can find another $(k-1)$ paths of type $P_{i,j}$. Hence, suppose that $\deg_G(x_1) = k$. Set $a_0 = x_1 = x_2 = a_t$, and for $1 \leq i \leq t-1$ choose a neighbour b_i of a_i distinct from a_{i-1} and a_{i+1}. Since $\deg_G(x_1) = k$, $a_i \notin \{y_3, y_4, \ldots, y_k\}$, $b_1 \neq a_0$ and $b_{t-1} \neq a_t$, we have $b_i \neq x_1$. Choose a neighbour c_i of b_i distinct from a_i, $1 \leq i \leq t-1$. In the case $b_i = y_j$ for some $3 \leq j \leq k$, we simply set $c_i = x_1$. Consider the walk $W = (a_0a_{t-1}, a_1a_2, b_1c_1, a_1a_0, a_2a_3, b_2c_2, a_2a_1, \ldots, a_{t-1}a_t, b_{t-1}c_{t-1}, a_{t-1}a_{t-2}, a_ta_1)$ (noting that $a_0a_{t-1} = x_1y_2$ and $a_ta_1 = x_1y_1$). This walk is internally-vertex-disjoint with $P_{i,j}$'s and P_l's constructed above. It may happen that $b_ic_i = b_jc_j$ for some $i \neq j$, and so W may not be a path. However, by deleting redundant subwalks from W when necessary we can obtain a path connecting x_1y_1 and x_1y_2 as required.

Case 2: Now we consider the case $x_1 \neq x_2$.

Subcase 2.1: Suppose first that x_1 and x_2 are not adjacent in G. Since G is k-connected, there are k internally-vertex-disjoint paths connecting x_1 with x_2 in G. Denote these paths by $R_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,t_i})$, $0 \leq i \leq k-1$, where we set $a_{i,0} = x_1$ and $a_{i,t_i} = x_2$. Since $k \geq 3$, we may assume that R_{k-1} does not pass through y_1 and y_2. Since $\delta(G) \geq k$, for $0 \leq i \leq k-1$ and $1 \leq j \leq t_i-1$ we may choose $k-2$ neighbours $b_{i,j,1}, b_{i,j,2}, \ldots, b_{i,j,k-2}$ of $a_{i,j}$ different from $a_{i,j-1}$ and $a_{i,j+1}$. Define $P'_{i,j} = (a_{i,1}b_{i,1,j}, a_{i,2}b_{i,2,j}, \ldots, a_{i,t_i-1}b_{i,t_i-1,j})$, $0 \leq i \leq k-1$, $1 \leq j \leq k-2$, which are $k(k-2)$ vertex-disjoint paths in $X(G)$. If $y_1 \neq a_{i,1}$, then we extend $P'_{i,j}$ $(1 \leq j \leq k-2)$ at the beginning by adding x_1y_1. Similarly, if $y_2 \neq a_{i,t_i-1}$, then we extend $P'_{i,j}$ $(1 \leq j \leq k-2)$ at the end by adding x_2y_2. There is at most one i with $y_1 = a_{i,1}$ (which is less than $k-1$ since R_{k-1} does not contain y_1), and for this i we extend $P'_{i,j}$ $(1 \leq j \leq k-2)$ at the beginning by adding $(x_1y_1, a_{i+j,1}a_{i+j,2}, x_1a_{n_{i,j},1})$ where the addition in subscript is modulo $k-1$, $n_{i,j} \equiv i+j+1 \pmod{k-1}$ if $1 \leq j < k-2$ and $k > 3$, $n_{i,j} \equiv i+1 \pmod{k-1}$ if $j = k-2$ and $k > 3$, and $n_{i,j} = k-1$ if $k = 3$. Observe that these prefixes are, with the exception of x_1y_1, vertex-disjoint. Similarly, there is at most one $i < k-1$ such that $y_2 = a_{i,t_i-1}$, and for this i we extend $P'_{i,j}$ $(1 \leq j \leq k-2)$ at the end by adding $(x_2a_{n_{i,j},t_n-1}, a_{i+j,t_{i+j}-1}a_{i+j,t_{i+j}-2}, x_2y_2)$ where the subscripts have the same meaning as above. Denote the extended form of $P'_{i,j}$ by $P_{i,j}$. Then $P_{i,j}$'s are $(k-1)^2-1$ internally-vertex-disjoint paths connecting x_1y_1 and x_2y_2. It remains to construct the last path, which starts with $(x_1y_1, a_{k-1,1}a_{k-1,2})$ and terminates with $(a_{k-1,t_{k-1}-1}a_{k-1,t_{k-1}-2}, x_2y_2)$. To abbreviate the notation set $q = k-1$. Choose a neighbour $c_j \neq a_{q,j}$ of $b_{q,j,1}$, $1 \leq j \leq t_1-1$. Assuming that the path R_{k-1} has no redundant parts, i.e., it is as short as possible, we get $b_{q,j,1} \neq x_1, x_2$. However, it may happen that $b_{q,j,1} = a_{m,n}$ for some m and n. In this case we choose $c_j = a_{m,n-1}$ if $n \leq t_m/2$ and $c_j = a_{m,n+1}$ otherwise. The walk $W = (x_1y_1, a_{q,1}a_{q,2}, b_{q,1,1}c_1, a_{q,1}x_1, a_{q,2}a_{q,3}, b_{q,2,1}c_2, a_{q,2}a_{q,1}, \ldots, a_{q,t_q-1}x_2, b_{q,t_q-1,1}c_{t_q-1}, a_{q,t_q-1}a_{q,t_q-2}, x_2y_2)$ is internally-vertex-disjoint with all $P_{i,j}$'s. Therefore, we can obtain from W a path between x_1y_1 and x_2y_2 which is internally-vertex-disjoint with all $P_{i,j}$'s. Altogether we have constructed $(k-1)^2$ internally-vertex-disjoint paths in $X(G)$ between x_1y_1 and x_2y_2.

Subcase 2.2: Now we deal with the case where x_1 and x_2 are adjacent in G. Since G is k-connected, there are $k-1$ internally-vertex-disjoint paths of length at least two connecting x_1 and x_2. Denote these paths by $R_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,t_i})$, $0 \leq i \leq k-2$, where $a_{i,0} = x_1$ and $a_{i,t_i} = x_2$. For $0 \leq i \leq k-2$ and $1 \leq j \leq t_i - 1$, let $b_{i,j,1}, b_{i,j,2}, \ldots, b_{i,j,k-2}$ be $k-2$ neighbours of $a_{i,j}$ distinct from $a_{i,j-1}$ and $a_{i,j+1}$. Since x_1, x_2 are adjacent in G and x_1y_1, x_2y_2 are not adjacent in $X(G)$, we have $\{x_1, y_1\} \cap \{x_2, y_2\} \neq \emptyset$, and hence by symmetry we need to consider the following two possibilities only.

The first possibility is that $y_1 = x_2$ and $y_2 = x_1$. In this case, for $0 \leq i \leq k-2$ and $1 \leq j \leq k-2$, define $P_{i,j} = (x_1y_1, a_{i,1}b_{i,1,j}, a_{i,2}b_{i,2,j}, \ldots, a_{i,t_i-1}b_{i,t_i-1,j}, x_2y_2)$ and $Q_i = (x_1y_1, a_{i,1}a_{i,2}, x_1a_{i+1,1}, x_2a_{i+1,t_{i+1}-1}, a_{i,t_i-1}a_{i,t_i-2}, x_2y_2)$, where subscripts are taken modulo $k-1$. Obviously, these are $(k-1)(k-2)+(k-1) = (k-1)^2$ internally-vertex-disjoint paths in $X(G)$ connecting x_1y_1 and x_2y_2.

In the second possibility, we may assume $y_1 = x_2$ and $y_2 \neq x_1$. In the case when y_2 appears on some path R_i, we may assume without loss of generality that $y_2 = a_{0,t_0-1}$. Consider the paths $P'_{i,j} = (x_1y_1, a_{i,1}b_{i,1,j}, a_{i,2}b_{i,2,j}, \ldots, a_{i,t_i-1}b_{i,t_i-1,j})$, $0 \leq i \leq k-2$, $1 \leq j \leq k-2$. We extend $P'_{i,j}$ $(1 \le i \le k-2,\ 1 \le j \le k-2)$ at the end by adding x_2y_2. Then we extend $P'_{0,j}$ $(1 \le j \le k-2)$ at the end by adding $(x_2a_{j,t_j-1}, a_{j+1,t_{j+1}-1}a_{j+1,t_{j+1}-2}, x_2y_2)$ if $j < k-2$ and $(x_2x_1, a_{1,t_1-1}a_{1,t_1-2}, x_2y_2)$ if $j = k-2$. (Note that only the latter case applies when $k = 3$.) Denote by $P_{i,j}$ the extension of $P'_{i,j}$ obtained this way. Define $Q_i = (x_1y_1, a_{i,1}a_{i,2}, x_1a_{i+1,1}, x_2y_2)$, $0 \le i \le k-2$, where subscripts are taken modulo $k-1$. Then $P_{i,j}$'s and Q_i's are $(k-1)(k-2)+(k-1) = (k-1)^2$ internally-vertex-disjoint paths in $X(G)$ connecting x_1y_1 and x_2y_2.

That the bound $\kappa(X(G)) \ge (k-1)^2$ is best possible was explained right after the statement of Theorem 3. $\square$

5 Proof of Theorems 4, 5 and 6

Given vertex-disjoint graphs $G_1, G_2, \ldots, G_k$, define $G_1 \vee G_2 \vee \cdots \vee G_k$ to be the graph obtained from the union $G_1 \cup G_2 \cup \cdots \cup G_k$ by adding all possible edges joining a vertex of G_i with a vertex of G_{i+1}, $1 \le i \le k-1$. Let K_n denote the complete graph on n vertices.

Proof of Theorem 4 Let us prove the upper bound first. Suppose $\delta(G) \ge 3$ and let x_1y_1 and x_2y_2 be vertices of $X(G)$ with $d_{X(G)}(x_1y_1, x_2y_2) = \mathrm{diam}(X(G))$. Let z_1 be a neighbour of x_1 different from y_1, and z_2 a neighbour of x_2 different from y_2. Let $(a_1, a_2, \ldots, a_{k-1})$ be a shortest path in G between $a_1 = z_1$ and $a_{k-1} = z_2$. Set $a_0 = x_1$ and $a_k = x_2$. Since $\delta(G) \ge 3$, for each $1 \le i \le k-1$ there exists a vertex b_i adjacent to a_i and different from a_{i-1} and a_{i+1}. Since $a_1 \ne y_1$ and $a_{k-1} \ne y_2$, $(x_1y_1, a_1b_1, a_2b_2, \ldots, a_{k-1}b_{k-1}, x_2y_2)$ is a path in $X(G)$. Therefore, $\mathrm{diam}(X(G)) = d_{X(G)}(x_1y_1, x_2y_2) \le d_G(a_1, a_{k-1}) + 2 \le \mathrm{diam}(G) + 2$.

To prove the lower bound we require only that G is nontrivial, since otherwise $X(G)$ is an empty graph. Let x_1 and x_2 be vertices of G such that $d_G(x_1, x_2) = \mathrm{diam}(G)$. Let y_1 be a neighbour of x_1 and y_2 a neighbour of x_2. Assume that $X(G)$ is connected and denote by P a shortest path in $X(G)$ between x_1y_1 and x_2y_2. Then the trace of P is a walk starting at x_1 and terminating at x_2, and the length of this walk cannot be shorter than the distance between x_1 and x_2 in G. Hence, $\mathrm{diam}(G) = d_G(x_1, x_2) \le d_{X(G)}(x_1y_1, x_2y_2) \le \mathrm{diam}(X(G))$.

Let $G_2 = K_3 \vee K_1 \vee K_3$, and for $k \ge 3$ let $G_k = K_3 \vee K_1 \vee K_2 \vee \cdots \vee K_2 \vee K_1 \vee K_3$, where there are $k-3$ copies of K_2 in G_k. Then $\mathrm{diam}(G_k) = k$ and $\mathrm{diam}(X(G_k)) = k+2$, and hence the upper bound is attained by G_k. (The diameter of $X(G_k)$ is achieved by $d_{X(G_k)}(x_1y_1, x_2y_2)$, where x_1 and x_2 are from different copies of K_3 and y_1 and y_2 are from copies of K_1.) Let $H_k = K_3 \vee K_2 \vee \cdots \vee K_2 \vee K_3$, where there are $k-1 \ge 1$ copies of K_2. Then $\mathrm{diam}(H_k) = \mathrm{diam}(X(H_k)) = k$, and so the lower bound is attainable as well. $\square$

Proof of Theorem 5 Let $P_s = (a_0, a_1, \ldots, a_{s-4})$ be a path of length $s-4$. We add several vertices and edges to P_s:

(1) First we add two vertices b_0 and b_{s-4}, join b_0 to a_0 and a_2, and join b_{s-4} to a_{s-6} and a_{s-4};

(2) then we add vertices c_1, c_2, $\ldots$, c_{s-5} and join c_i to a_{i-1} and a_{i+1}, $1 \le j \le s-5$.

Denote by H_s the resulting graph.

(3) We then add to H_s some vertices $d_{i,j}$, $0 \le i < j \le s-4$, and join $d_{i,j}$ to a_i and a_j, in the following manner: We first add $d_{i,j}$ (and the corresponding edges) with $j - i = 3$. Then

we add $d_{i,j}$ (and the corresponding edges) with $j - i = 4$, and so on until we obtain a graph of diameter r.

Denote the resultant graph by $G_{r,s}$. First of all, we have to show that adding vertices $d_{i,j}$ successively in step (3) can indeed create a graph of diameter r. In fact, we have $\mathrm{diam}(H_s) = d_{H_s}(a_0, a_{s-4}) = s - 4$, and at each step of adding a single vertex $d_{i,j}$ together with the corresponding edges $\{d_{i,j}, a_i\}$ and $\{d_{i,j}, a_j\}$, the diameter can decrease by at most one, since we connect vertices at distance 3 by a path of length 2. Moreover, if we add all possible vertices $d_{i,j}$ with $j - i \geq 3$ together with the corresponding edges, then we get a graph of diameter 4. (As $s \geq 10$, we have $d_{G_{r,s}}(b_0, b_{s-4}) \geq 4$.) Since $4 \leq r \leq s - 4$, there exists a time at which we obtain a graph $G_{r,s}$ of diameter r.

Now we prove $\mathrm{diam}(X(G_{r,s})) = s$. Observe that all vertices of P_s have degree at least three in $G_{r,s}$, while all other vertices have degree two in $G_{r,s}$. From this one can see that $G^{\div}_{r,s}$ is connected. Hence $X(G_{r,s})$ is connected by Theorem 2.

Let P be a shortest path connecting two vertices of $X(G_{r,s})$, and let $W = (u_0, u_1, \ldots, u_t)$ be the trace of P. Then by Lemma 7, $u_2, u_3, \ldots, u_{t-2}$ all have degree at least three in $G_{r,s}$ and $(u_2, u_3, \ldots, u_{t-2})$ is a shortest path in $G^{\times}_{r,s}$. In view of the observation in the previous paragraph this implies that $(u_2, u_3, \ldots, u_{t-2})$ is a subpath of P_s, and hence P has length at most $2 + (s-4) + 2 = s$. Since P is an arbitrary shortest path in $X(G_{r,s})$, it follows that $\mathrm{diam}(X(G_{r,s})) \leq s$.

To prove the reverse inequality, consider the distance between a_0a_1 and $a_{s-4}a_{s-5}$ in $X(G_{r,s})$. Since in $X(G_{r,s})$ the vertex a_0a_1 is adjacent only to vertices xy such that $\deg_{G_{r,s}}(x) = 2$, the trace of any path in $X(G_{r,s})$ joining a_0a_1 with $a_{s-4}a_{s-5}$ must start with $(a_0, x, a_0, \ldots)$. Analogously, since $a_{s-4}a_{s-5}$ is adjacent only to vertices zw such that $\deg_{G_{r,s}}(z) = 2$, the trace of such a path must terminate with $(\ldots, a_{s-4}, z, a_{s-4})$. Thus, the trace of any path joining a_0a_1 with $a_{s-4}a_{s-5}$ is of the form $(a_0, x, a_0, \ldots, a_{s-4}, z, a_{s-4})$. By Lemma 7 all vertices of its subpath $(a_0, \ldots, a_{s-4})$ must have degree at least three, so they form a walk in P_s. Consequently the trace of any shortest path in $X(G_{r,s})$ between a_0a_1 and $a_{s-4}a_{s-5}$ must have length at least $2+(s-4)+2 = s$, so that $d_{X(G_{r,s})}(a_0a_1, a_{s-4}a_{s-5}) \geq s$. Hence, $\mathrm{diam}(X(G_{r,s})) \geq s$. $\square$

Proof of Theorem 6 If $diam(G) = 1$, then G is a complete graph. Moreover, it has at least four vertices as $X(G)$ is connected. It can be easily verified that $\mathrm{diam}(X(G)) = 2$.

Now suppose $\mathrm{diam}(G) = 2$ and $\mathrm{diam}(X(G)) \geq 8$. Then there exist u_0v_0 and u_8v_8 whose distance in $X(G)$ is eight. Let $P = (u_0v_0, u_1v_1, \ldots, u_8v_8)$ be a shortest path joining u_0v_0 and u_8v_8 in $X(G)$. By Lemma 7, $(u_2, u_3, \ldots, u_6)$ is a shortest path in $G^{\times}$. As u_2 and u_5 are not adjacent in $G^{\times}$, they are not adjacent in G. Since $\mathrm{diam}(G) = 2$, we have $d_G(u_2, u_5) = 2$ and so there exists a vertex x_1 of degree two in G which is adjacent to both u_2 and u_5. Similarly, there exists a vertex x_2 of degree two which is adjacent to both u_3 and u_6. Since x_1 and x_2 do not have any common neighbour, $d_G(x_1, x_2) \geq 3$, which contradicts our assumption $\mathrm{diam}(G) = 2$.

Finally, suppose $\mathrm{diam}(G) = 3$ and $\mathrm{diam}(X(G)) \geq 15$. Similarly to the above, there exists a shortest path P in $X(G)$ with length 15 and trace $(u_0, u_1, \ldots, u_{15})$, say, such that $(u_2, u_3, \ldots, u_{13})$ is a shortest path in $G^{\times}$. Since $d_G(u_2, u_{13}) \leq 3$, there exists a vertex x_1 of degree two which allows a "shortcut" between u_2 and u_{13} in G. Then x_1 is joined by an edge to u_2 or u_{13}. Without loss of generality assume that x_1 is adjacent to u_2. Then the other edge incident to x_1 connects x_1 with u_{13} or with a neighbour of u_{13}. Similarly, since $d_G(u_5, u_9) \leq 3$, there exists a vertex

x_2 of degree two which is adjacent to u_5 or a neighbour of u_5 and also to u_9 or a neighbour of u_9. In any case, no neighbour of x_1 is adjacent to a neighbour of x_2. Hence $d_G(x_1, x_2) \geq 4$, a contradiction. □

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