SMALL RADIALLY MOORE GRAPHS

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Abstract. In this note we find radially Moore graphs of radius 3 with degrees 3, 4, 5, 6 and 7.

1. Introduction

A *digraph* is a pair $D = (V(D), E(D))$, where $V(D)$ is a finite set of elements, which we call *vertices*, while $E(D)$ is a set of ordered pairs of $V(D)$, which we call *arcs*. Let u be a vertex of D. The number of arcs terminating in u is the *in-degree* of u, $id_D(u)$, and the number of arcs starting in u is the *out-degree* of u, $od_D(u)$. If there is a constant t such that $id_D(v) = od_D(v) = t$ for every vertex $v \in V(D)$, then the digraph D is *regular of degree* t. Further, let u be a vertex of D. Then

out-eccentricity of u is	$e^+_D(u) = \max\{d_D(u,v) \mid v \in V(D)\}$;
in-eccentricity of u is	$e^-_D(u) = \max\{d_D(v,u) \mid v \in V(D)\}$;
eccentricity of u is	$e_D(u) = \max\{e^+_D(u), e^-_D(u)\}$;

where $d_D(x, y)$ is a distance from x to y in D. The *radius*, $rad(D)$, is the minimum eccentricity in D, while the *diameter*, $diam(D)$, is the maximum eccentricity.

Clearly, a regular digraph of degree t and with diameter s contains at most

$$MD_{s,t} = 1 + t + t^2 + \ldots + t^s$$

vertices. If it has exactly $MD_{s,t}$ vertices, then it is called a *Moore digraph*. Unfortunately, Moore digraphs exist only for $s = 1$ or $t = 1$, namely a complete graph K_{t+1} and a directed cycle C_{s+1}, respectively. Therefore we have to relax some of the conditions. There are many papers in which the authors fix the degree and the diameter and study the problem "how close" to the Moore bound $MD_{s,t}$ we can get. In [1] a different approach was suggested. Namely, to relax the strong condition of having diameter equal to s. This attempt was based on the fact that any regular digraph with degree t and radius s cannot have more then $MD_{s,t}$ vertices. If it has $MD_{s,t}$ vertices and its diameter equals $s + 1$, such digraph is a *radially Moore digraph*.

In [1] radially Moore digraphs were constructed for all pairs of positive integers s and t. Hence, the problem of existence of radially Moore digraphs was solved positively. Unfortunately, the situation for graphs is quite different.

A *graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is a finite set of *vertices*, while $E(G)$ is a set of unordered pairs of $V(G)$, which we call *edges*. The number of edges incident with a vertex u is the *degree* of u, $d_G(u)$. If there is a constant t such that $d_G(v) = t$ for every vertex $v \in V(G)$, then the graph G is *regular of degree t*. Further, let u be a vertex of G. Then

eccentricity of u is	$e_G(u) = \max\{d_G(u,v) \mid v \in V(G)\}$;

where $d_G(x, y)$ is a distance from x to y in G. Analogously as in the case of digraphs, the *radius*, $rad(G)$, is the minimum eccentricity in G, while the *diameter*, $diam(G)$, is the maximum eccentricity.

A regular graph of degree t and with diameter s contains at most

$$MG_{s,t} = 1 + t + (t-1)t + (t-1)t^2 + \ldots + (t-1)t^{s-1}$$

vertices. If it has exactly $MG_{s,t}$ vertices, then it is called a *Moore graph.*

Moore graphs exist only for $s = 1$ (the complete graphs K_{t+1}), $t = 2$ (cycles of odd length C_{2s+1}), and some graphs of radius $t = 2$. They are the Petersen graph (for $s = 3$), Hoffman-Singleton graph (for $s = 7$), and there can possibly be a Moore graph of degree $s = 57$. Analogously as in the case of digraphs, plenty of authors relax the condition of having $MG_{s,t}$ vertices and study the problem "how close" to the Moore bound $MG_{s,t}$ we can get. But there is a possibility to relax the strong condition of having diameter equal to s. Any regular graph of degree t with radius s cannot have more then $MG_{s,t}$ vertices. If it has $MG_{s,t}$ vertices and its diameter equals $s + 1$, such graph is a *radially Moore graph.*

There are several mathematicians, who were eager to find a general construction for radially Moore graphs, but without a success. The only partial result is that for any positive integer t there exists a radially Moore graph of radius $s = 2$ with degree t, see [2]. But the problem becomes to be a really interesting one with radius 3. In this paper we describe for every t, $3 \le t \le 7$, one radially Moore graph of degree t with radius 3. Unfortunately, we are not able to give a construction covering all degrees.

2. Results

In this part we find radially Moore graph G_t of radius 3 and degree t. Let c be a vertex of this graph on which the radius is achieved, i.e., $e_G(c) = 3$. Then G_t contains a tree of radius 3 rooted in c whose all interior vertices have degrees t (see Figure 1 for the case $t = 4$). Our aim was to find a graph in which all branches emerging from c belong to one single orbit under the group of automorphisms, so that the corresponding automorphism action is a simple rotation around c. Hence, if there is an edge joining, say, the first endvertex of branch 1 with the third endvertex of branch 2, then there is also an edge joining the first endvertex of branch 2 with the third endvertex of branch 3, etc. (We remark that we do not add edges joinig endvertices of the same branch.) Then to describe our graph it suffices to give a permutation for neighbouring branches, then a permutation for branches at circular distance two, etc. However, if t is an even number, the permutation at circular distance $t/2$ must be an involution.

The problem is that if we like to run exhaustively the just described construction, then for $t = 7$ we would need to check $(36!)^3 \doteq 5.1 \cdot 10^{124}$ instances, which is simply too much. Therefore we compose every permutation of $(t-1)^2$ elements from a cyclic shift of $(t-1)$-touples and a permutation of $(t-1)$ elements. For $t = 7$ this gives only $(6 \cdot 6!)^3 \doteq 8.2 \cdot 10^{10}$ instances. Such a construction creates plenty of small circles (of length four), but surprisingly, it works. For $t \in \{3, 4, 5, 6, 7\}$ in Table 1 we present shifts with the permutations, which determine radially Moore graphs.

3:	1; (1,2)		
4:	1; (3,1,2)	0; (2,1,3)	
5:	0; (3,1,2,4)	1; (1,4,2,3)	
6:	1; (1,2,5,3,4)	1; (4,5,1,2,3)	0; (3,5,1,4,2)
7:	5; (6,1,2,3,4,5)	0; (2,1,4,3,6,5)	1; (5,2,1,4,3,6)

Table 1

Just for demonstration, in a more detailed way we describe the case $t = 4$, see Figure 1.

The adjacencies of endvertices in the neighbouring branches are encoded by 1; (3,1,2). This means that the vertices $01, 02, 03$ are joined to $14, 15, 16$ by the permutation $(3, 1, 2)$, etc. More precisely, for every $i \in \{0, 1, 2, 3\}$ there are edges $(i1, j6)$, $(i2, j4)$, $(i3, j5)$, $(i4, j9)$, $(i5, j7)$, $(i6, j8)$, $(i7, j3)$, $(i8, j1)$, $(i9, j2)$, where $j = i + 1 \pmod 4$.

The adjacencies of endvertices in the branches at circular distance 2, which are opposite in this case, are encoded by involution 0; (2,1,3). Hence, for every $i \in \{0, 1\}$ (the case $i \in \{2, 3\}$ defines the same edges as the permutation is an involution) there are edges $(i1, k2)$, $(i2, k1)$, $(i3, k3)$, $(i4, k5)$, $(i5, k4)$, $(i6, k6)$, $(i7, k8)$, $(i8, k7)$, $(i9, k9)$, where $k = i + 2$.

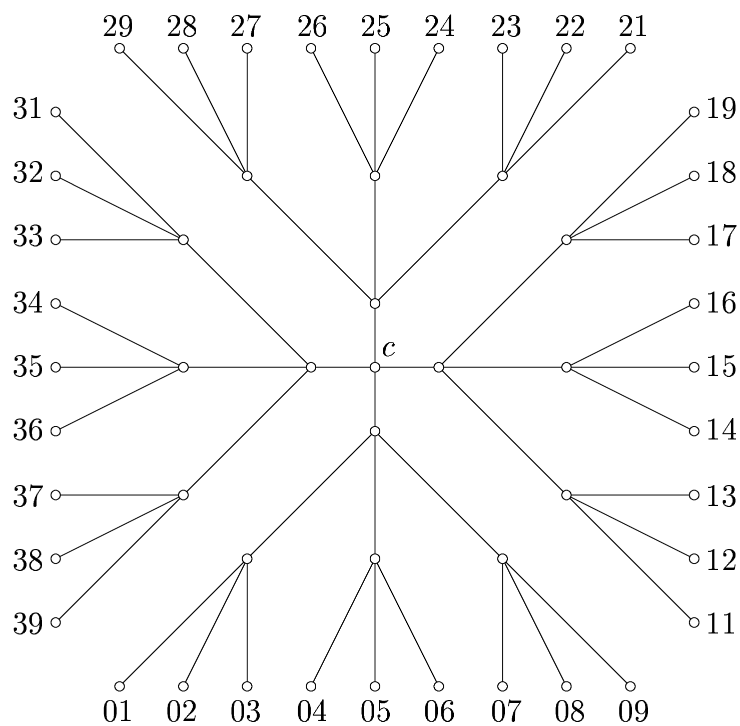

Figure 1

We remark that the cases $t = 6$ and $t = 7$ were not run exhaustively. This is because for cyclic shifts 0; 0; 0 it is too optimistic to expect a radially Moore graph. We started the search with the shifts 1; 1; 0 for the case $t = 6$ and with shifts 5; 0;

1 for the case $t = 7$. Then the results were found fast (i.e., in a few minutes).

Though computational evidence shows that there are plenty of radially Moore graphs of the type, we consider, and though we have some feeling how to choose the cyclic shifts, we have no idea how to find "good" permutations. Hence, we are not able to generalize the examples presented here. Moreover, we have no idea of how to prove that the "good" candidate has diameter only 4. Nevertheless, we think that our study is a good start to the study of radially Moore graphs.

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References

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