A note on 2-subset-regular self-complementary 3-uniform hypergraphs

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Abstract

We show that a 2-subset-regular self-complementary 3-uniform hypergraph with \( n \) vertices exists if and only if \( n \geq 6 \) and \( n \) is congruent to 2 modulo 4.

1 Introduction

A \( k \)-uniform hypergraph of order \( n \) is an ordered pair \( \Gamma = (V, E) \), where \( V = V(\Gamma) \) is an arbitrary set of size \( n \), and \( E = E(\Gamma) \) is a subset of \( V^{(k)} = \{e \subseteq V : |e| = k\} \). Note that the notion of a 2-uniform hypergraph coincides with the usual notion of a simple graph. We shall call a \( k \)-uniform hypergraph simply a \( k \)-hypergraph.

A \( k \)-hypergraph \( \Gamma \) is self-complementary if it is isomorphic to its complement \( \Gamma^C \), defined by \( V(\Gamma^C) = V(\Gamma) \) and \( E(\Gamma^C) = V(\Gamma)^{(k)} \setminus E(\Gamma) \). Equivalently, \( \Gamma = (V, E) \) is self-complementary whenever there exists a permutation \( \tau \in \text{Sym}(V) \), called the antimorphism of \( \Gamma \), such that for all \( e \in V^{(k)} \) the equivalence \( e \in E \iff e^\tau \not\in E(\Gamma) \) holds. Antimorphisms of uniform hypergraphs were characterized in terms of their cyclic decompositions by Wojda in [7].

A \( k \)-hypergraph \( \Gamma \) is \( t \)-subset-regular if there exists an integer \( \lambda \), also called the \( t \)-valence of \( \Gamma \), such that each \( t \)-element subset of \( V(\Gamma) \) is a subset of exactly \( \lambda \) elements of \( E(\Gamma) \). Clearly \( t \)-subset-regular \( k \)-hypergraphs generalize the notion of regular graphs, and can also be viewed as a bridge between graph theory and design theory. Namely, a \( t \)-subset-regular \( k \)-hypergraph of \( \lambda \)-valence and order \( n \) is simply a \( t \)-\((n, k, \lambda) \) design. Moreover, such a \( k \)-hypergraph \( \Gamma \) is self-complementary if and only if the pair \( \{\Gamma, \Gamma^C\} \) forms a large set of \( t \)-designs \( \text{LS}[2](t, n, k) \) with the additional property that the two designs constituting the large set are isomorphic (see [1] for the definition of a large set of designs).

Here a question of existence of a self-complementary \( t \)-subset-regular hypergraph with prescribed parameters \( n, k, \) and \( t \) arises naturally. An easy counting argument shows that whenever a self-complementary \( t \)-subset-regular \( k \)-hypergraph on \( n \) vertices exists, then \( \binom{n-i}{k-i} \) is even for all \( i = 0, \ldots, t \).

It can be seen that the above divisibility conditions can in fact be expressed in terms of certain congruence conditions on \( n \) modulo an appropriate power of 2 (see [4]). For example if \( k = 2^\ell \) or \( k = 2^\ell + 1 \) for some positive integer \( \ell \), then \( n \) is congruent to one of \( t, \ldots, k-1 \)

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modulo $2^{t+1}$. In particular, if $k = 2$ and $t = 1$, then $n \equiv 1 \pmod{4}$; if $k = 3$ and $t = 1$, then $n \equiv 1 \pmod{2}$; if $k = 3$ and $t = 2$, then $n \equiv 2 \pmod{4}$.

In [5] the following question, strengthening Hartman’s conjecture [3] about existence of large sets of (not necessarily isomorphic) designs, was raised:

**Question.** [5] Is it true that for every triple of integers $t < k < n$ such that $\binom{n-i}{k-i}$ is even for all $i = 0, \ldots, t$, there exists a self-complementary $t$-subset-regular $k$-hypergraph of order $n$?

It is well known (see [6]) that a regular self-complementary graph on $n$ vertices exists if and only if $n$ is congruent to 1 modulo 4, showing that the answer to the above question is affirmative for $k = 2$ and $t = 1$. Recently, the answer was proved to be affirmative also for the case $k = 3$ and $t = 1$ (see [5]). The aim of this note is to show that the answer to the question above is affirmative also for the remaining case of 3-hypergraphs, namely for the case $k = 3$, $t = 2$. More precisely, in Section 2 we present a construction which proves the following:

**Theorem 1** If $n \geq 6$ and $n$ is congruent to 2 modulo 4, then there exists a 2-subset-regular self-complementary 3-hypergraph on $n$ vertices.

## 2 Construction

Let $n = 4k + 2$ for some integer $k$. For $i = 0, 1$, let $V_i = \{0, 1, \ldots, (2k)\}$ be a copy of the the ring $\mathbb{Z}_{2k+1}$. Define $\Gamma_n$ to be the 3-hypergraph with vertex set $V = V_0 \cup V_1$ and edge set $E = E_1 \cup E_2 \cup E_3$, where

\[
E_1 = V_0^{(3)},
\]

\[
E_2 = \{\{a_0, b_0, c_1\} : a, b \in \mathbb{Z}_{2k+1}, a \neq b, c = \frac{a + b}{2}\},
\]

\[
E_3 = \{\{a_0, b_1, c_1\} : a, b, c \in \mathbb{Z}_{2k+1}, a \neq \frac{b + c}{2}\}.
\]

Note that 2 is invertible in $\mathbb{Z}_{2k+1}$, hence dividing by 2 in the definitions of $E_2$ and $E_3$ is well defined.

First we show that $\Gamma_n$ is 2-subset-regular, i.e. we show that each pair of vertices is contained in exactly $(n-2)/2 = 2k$ edges. There are four types of pairs of vertices to consider:

(a) A pair $a_0, b_0$, where $a, b \in \mathbb{Z}_{2k+1}, a \neq b$. This pair is contained in $2k-1$ edges of $E_1$ and in a unique edge in $E_2$. As it is contained in none of the edges of $E_3$, the pair is in total of $2k$ edges.

(b) A pair $a_1, b_1$, where $a, b \in \mathbb{Z}_{2k+1}, a \neq b$. This pair appears only in $(2k + 1) - 1 = 2k$ edges of $E_3$.

(c) A pair $a_0, a_1$, where $a \in \mathbb{Z}_{2k+1}$. This pair also appears only in the edges of $E_3$. In fact, it appears precisely in the $2k$ edges of the form $\{a_0, a_1, b\}, b \in \mathbb{Z}_{2k+1} \setminus \{a\}$.

(d) A pair $a_0, c_1$, where $a, c \in \mathbb{Z}_{2k+1}, a \neq c$. This pair is contained in the edge $\{a_0, c_1, (2c - a)_0\}$ of $E_2$, and in the $2k - 1$ edges of the form $\{a_0, c_1, b\}$ where $b \in \mathbb{Z}_{2k+1} \setminus \{2a - c\}$, of $E_3$. Hence this pair is in exactly $2k$ edges of $\Gamma_n$.

This proves that $\Gamma_n$ is a 2-subset-regular hypergraph. To prove that it is self-complementary, note that the mapping $\phi : V \to V$ defined by $\phi(a_i) = a_{i+1}$, for $i = 0, 1$, with addition in the subscript being modulo 2, is an antimorphism of $\Gamma_n$. \hfill \square
We remark that $\Gamma_n$ is not a vertex-transitive hypergraph if $n > 6$ (check the complete sub-hypergraphs of order $n/2$). However, in $\Gamma_6$ every pair of vertices appears in exactly two edges. Hence $\Gamma_6$ can be considered as a triangular embedding of a complete graph $K_6$ into a surface (see [2] for a detailed account on graph embeddings). In fact, $\Gamma_6$ represents a regular triangulation of the projective plane by $K_6$. As a consequence, $\Gamma_6$ is vertex-transitive.

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References


