A census of the orientable biembeddings of Steiner triple systems of order 15

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Abstract

A complete census is given of the orientable biembeddings of Steiner triple systems of order 15. There are 80 Steiner triple systems of order 15 and these generate a total of 9 530 orientable biembeddings.

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1 Introduction

This paper is concerned with biembeddings of Steiner triple systems. A *Steiner triple system* of order n, STS(n), is a pair $(V, \mathcal{B})$, where V is a set of n *points* and $\mathcal{B}$ is a collection of *triples*, also called *blocks*, taken from V and such that every pair of distinct points from V appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or 3 (mod 6) [9]. Biembeddings of such systems in orientable and nonorientable surfaces arise as follows. Consider a triangular embedding M of the complete graph K_n in the orientable surface S_g, the sphere with g handles, or in the nonorientable surface N_γ, the sphere with γ crosscaps. If the faces of M can be properly 2-coloured then the faces in each colour class form an STS(n). Euler's formula gives g (respectively γ) for triangular embeddings of K_n in an orientable (respectively nonorientable) surface, namely $g = (n-3)(n-4)/12$ ($\gamma = (n-3)(n-4)/6$). Face 2-colourability requires n to be odd. It easily follows that a necessary condition for an orientable biembedding is that $n \equiv 3$ or 7 (mod 12), and a necessary condition for a nonorientable biembedding is $n \equiv 1$ or 3 (mod 6). These necessary conditions are also sufficient for the existence of such biembeddings, except in the case $n = 7$, where there is no biembedding of STS(7)s in the surface N_2 [11, 12, 6].

Two STS(n)s, A and B, are said to be *biembeddable* in some surface if there exists a face 2-colourable triangular embedding of K_n in that surface in which the Steiner triple systems arising from the two colour classes are isomorphic copies of A and B. The issue which then naturally arises is to determine which pairs of STS(n)s are biembeddable. For $v = 3$ there is a trivial and unique embedding of a triangle in the sphere, and this provides a biembedding of STS(3)s. The STS(7) is unique and there is precisely one biembedding of the system, this with an isomorphic copy of itself, in the torus S_1. Here and subsequently when enumerating, we refer to the number of isomorphism classes. For the STS(9), which is also unique, there is again precisely one biembedding, again with an isomorphic copy of itself, in the nonorientable surface N_5 [1, 5]. There are two STS(13)s, one is cyclic and the other is not. We will refer to these here as C and N respectively. There are 615 biembeddings of C with C, 8 539 biembeddings of C with N, and 29 454 biembeddings of N with N [7], all of these being in the nonorientable surface N_{15}.

Biembeddings of STS(15)s lie in either the orientable surface S_{11} or the nonorientable surface N_{22}. There are 80 isomorphism classes of STS(15)s and we follow the standard numbering of these given in [10] which also gives the orders of their automorphism groups. It was shown in [3] that every pair, including isomorphic pairs, has a biembedding in a nonorientable surface, although a complete enumeration of all nonorientable biembeddings of STS(15)s is probably beyond current computational capabilities.

It was proved in [2] that at least one pair, namely $\{\#1, \#2\}$ in the standard numbering, has no biembedding in an orientable surface. In [4] a search was made to find an orientable *self-embedding*, that is a biembedding of two isomorphic copies of the same system, for each of the 80 STS(15)s. The biembedding was assumed to have an involutory automorphism, with a single fixed point, that reversed the colour classes. Self-embeddings of this type were found for 78 of the 80 systems. It was also shown in [4] that there is no orientable self-embedding of system #2, whether of this type or any other type. The other exception was system #79. In [8] it was shown that this latter system also has no orientable self-embedding. In the same paper we gave all orientable biembeddings in which one of the two systems is one of #1, #2, #76, #79, #80, these systems being of particular interest for reasons given in that paper. In the current paper we complete the determination of all the orientable biembeddings of the 80 Steiner triple systems of order 15.

2 Method

The biembeddings were found by one of the two computer programs used in [8]. This program takes two systems A and B from the listing of [10]. The representation of system B is not altered, but permutations are in turn applied to the vertices of system A. For each permutation, two sets of triangles result, those from A and those from B. To form a surface embedding we have, for example, the trivial test that these sets of triangles must be disjoint. During the examination of possible permutations, the automorphism groups of the two systems are exploited. If we examine all permutations π that take a point a of A to a point b of B, then we do not need to consider further any other permutation π' that takes a point in the orbit of a under $Aut(A)$ to a point in the orbit of b under $Aut(B)$.

The permutations were constructed as follows. One vertex, say b_1, is chosen. Assume that the triples of B containing b_1 are $b_1b_2b_3$, $b_1b_4b_5$, $\dots$, $b_1b_{14}b_{15}$. Then in any biembedding, in the rotation around b_1 there are pairs b_2b_3, b_4b_5, $\dots$, $b_{14}b_{15}$. Since these pairs may be reversed and mutually interchanged, there are $6! \cdot 2^6 = 46\,080$ possible rotations at b_1, and these are considered in turn. In any biembedding of A with B, one vertex of A is mapped to b_1, and some other vertex is mapped to b_2. But as we already have the rotation at b_1, the image of the third vertex in the triple from A containing the inverse image of the pair b_1b_2 is determined. Similarly, the image of a fourth vertex determines that of a fifth vertex, and so on. In this way, only $15 \cdot 14 \cdot 12 \cdot \ldots \cdot 2 = 9\,676\,800$ permutations are constructed for each possible rotation at b_1. So, all together, for every pair of systems we examine $(46\,080)\cdot(9\,676\,800) = 445\,906\,944\,000$ possible permutations instead of $15! = 1\,307\,674\,368\,000$, that is approximately one-third. But in fact, by using the orbits of the automorphism groups of A and B as described above, a much smaller number is examined. To confirm that a biembedding is formed, we simply check that the rotation at each vertex is a single cycle of length 14. To test for orientability, we fix the orientation of the triangle $b_1b_2b_3$. This implies an orientation for all the other triangles. Since every triangle is in three rotations, after assigning it an orientation, we check the two further occurrences.

The program described above was used in [8] and the results there were verified by a second independent program. In the current paper, as an additional check on the accuracy of the results, we computed both the biembeddings of A with B and of B with A for each pair of systems A and B, and we verified that the results were identical. Furthermore the second program from the earlier paper was used to confirm all the self-embeddings.

The programs were run on a Linux cluster, with each of the 80 systems being allocated to a separate processor. Determination of all the biembeddings of each system took between two and ten weeks; the systems having the smallest automorphism groups generally taking the longest times.

3 Results

Up to isomorphism, there are 9 530 orientable biembeddings of Steiner triple systems of order 15. Table 1 summarizes these biembeddings in the form of an 80×80 array. The entry in the (i, j) cell gives the number of orientable biembeddings of system $\#i$ with system $\#j$. A dot (.) indicates that no biembedding exists and the 26 letters a to z represent the numbers from 10 to 35 inclusive. The letter l (representing 21 biembeddings), which could be confused with the number 1, only appears in the $(10, 10)$ cell. The symbol + denotes an entry greater than 35. In fact + only appears on the diagonal and these entries are described in more detail in Table 2. Altogether in Table 1 there are $(80 \times 81)/2 = 3\,240$ cells (i, j) with $1 \leq i \leq j \leq 80$, representing distinct pairs of systems. Of these, 1 150 admit no biembedding while each of the remaining 2 090 admits at least one biembedding.

For reasons of space, it is not feasible to list all the biembeddings. However, these are available as a file from http://mcs.open.ac.uk/mjg47/emb15.txt. Each biembedding is described by a line of the file containing the system numbers i and j, the order of the automorphism group of the biembedding and followed by a permutation p. To generate the biembedding, p is applied to the standard form of system i given in [10] to produce a copy $p(i)$. The triples of the systems $p(i)$ and j are then taken as triangles and sewn together along common edges. As an example, the first line of the file is:

1 1 10 1 8 10 3 5 2 12 7 4 6 9 11 14 15 13

Here $i = j = 1$, the biembedding has 10 automorphisms, and the permutation p is taken as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 8 & 10 & 3 & 5 & 2 & 12 & 7 & 4 & 6 & 9 & 11 & 14 & 15 & 13 \end{pmatrix}$.

1.........	....	....	....	....	....	....	....
....	1..1......	......1...	....	....	....	.....1....	....
..3.......	1.2.1.....	....1.1...	.11.....1.	....	.1........	....1.....	....
...7...111	11..1..1..	...131.5..	..31.1.121	2...213121	311.3.12..	..1.112251	35..11....
....1...1.	1	..1.12....	........1.	12..2.1...	...2..1...	.......11.	....
.....9...2	1.........	....112...	..11...1..	.11.11.1.1	..1..1....	......11.1	11..1.31..
......2...	....	....	....	..1.......	1.........	....	....
...1...a.1	241.1.22.2	..34123113	.4411..542	11131..212	4.1253463.	..12.12335	6a1.1111..
...11...g3	41.32....2	.483621313	154221.666	2224412841	6333521211	.213544423	52.2113...
...1.2.13l	422.4.1.1.	235472.8..	1136.3.458	3..1211975	5464396515	.211463467	32.1.141..
.111.1.244	e2.211.2.3	1324342513	1365.1.446	12124661.2	5635421112	.111444432	15..3112..
...1...412	2d..2...22	1157133522	2224.3.154	22..153133	55324523.1	422223	21222.1...
..2....1.2	..c......1	..2.31312.	..111.1111	1.....211.	1.4..111.3	...12...11	.3..1..1..
.1......3.	2..4...1..	..312.....	...1...1.1	.1...11..1	.3..13...1	...1.1.2..	....1.....
..11...124	12..91.1.2	.32324121.	1235.2.371	...2212312	43133223..	..1.143.3.	12113.1...
....	1...11....	....	....	....	........1.	....	.......1..
.......2.1	......3..1	....1..2..	.1.....121	.1.....111	1.1..1...1	....1.1..1	....
...1...2..	2..11..a.1	.11.5231..	111.212.34	2111211452	.33311.313	..1.3.21.2	31.1.1.11.
.........1	.2......61	1...1141.1	.......11.	221.1..3..	2..21..1..	....	.2.1......
....1..22.	321.2.111b	1151344212	..432..145	31.164.353	445222.123	1.2.315473	31........
.........2	11......11	e1114324..	11461.1154	35..247334	3123..21..	.113224.32	2....312..
........43	31..3..1.1	1d122462.2	26312..5.2	51.1112213	22353.22.3	125521	311.111...
....1..385	25232..1.5	11z7d83824	2948132g84	942386998d	b7ac8aab56	13312a8967	894321611.
...1...434	47.13....1	127+556926	645c62.686	8714767b75	98b97e6b1d	32.38d4669	571111521.
..1311.167	31322.1513	42d5v85734	269a22.77c	b8.36ae796	a57bb76b46	.343888e83	55222114..
...121.222	431.4..214	34858+6811	.68b11.75b	6425999a45	78698a4719	.14362b67a	65126131..
.11..2.31.	233.1..344	263656+632	3f8431.565	611558b8a8	562abe6627	1.23356298	9714.1212.
...5...138	551.2.2112	4289786y24	.b782..59b	32.1cb77a4	a88836673c	12424676a7	3b31..111.
.......11.	122.1....1	..223132g1	12.1...462	2.1.222333	1334152132	.3..2.5223	.31...12..
.......33.	32......12	.24641241j	2322.1..43	611.312554	54524182.5	.2..542321	31...121..
........11	12..1..1..	12262.3.12	c221.1.122	11.2113514	.51332.311	..1.1.1311	11.....1..
..1....451	32..2.11..	169466fb23	2x1533.a65	4342556565	57a9899656	.12.7959ac	57324.66..
..13.1.443	621.3..1.4	43459887.2	21v831.84c	121234757c	3b88b88a16	12119a8858	79242132..
...1.1.126	54115....3	618cab4812	158u41.a79	9511c55743	66b8d67254	.414c9b545	5f122.311.
.......12.	..1....2.2	12162132..	.334d.1415	3..1123122	3343221.15	.11.35.221	.22.1.....
...1....13	13..2..1..	..32211..1	1311.4.322	121.21.113	41.1322...	222132	111.......
....	..1....2..	1.2.......	....1.11..	21..1.1...	111.1.....	....2.1...	...1......
...1.1.564	41113.1.11	15g677554.	1a8a431v26	4231e94799	6986baa915	161.678cba	9g3141....
..121..465	451.7.2314	5.88756964	264712.2+9	6512995a66	5ea3a95749	2433998898	a4134121..
...1...268	64111.14.5	4246cb5b23	25c952.69v	67119b6689	87b76c775b	.8.7368655	866452331.
...21..123	121....223	3598b66326	1419312466	v343aaa28a	87768896.9	.13275b849	4a2214211.
....21.12.	22.1..1121	51478412.1	1325.21257	3g2.811414	3653334312	..2125a136	371.1.11..
.....1112.	1......11.	..21.21.11	.411.1.311	42c11.111.	1121..1.23	111.23.321	11..2.5...
.......341	2...2..1.1	.1343551..	22211..121	3.1j462373	3233365122	1311324554	31.1..2...
...221.142	41..2..216	2187695c23	153c121e99	a814+88465	7785499956	.232c73666	741312.2..
...1.1..11	65.11..1.4	4166a98b21	154521.99b	a1.68x8e58	5945896c27	13419a7636	86613211..
...31...21	63212..1..	7297e9b722	36753.1456	a11288q891	776i82371b	..5289b63a	39131113..
...1.1.289	111.3.1433	329b7a8735	555711.7a6	24134e8z67	a97595a458	.4135aa98d	7b45212...
...2...147	.31.1.15.5	318794aa35	167421.968	81176596r9	aa39499579	.233294a45	79333133..
...1.1.215	23.12.12.3	43d5658434	45c323.969	a4.358179v	8d9c68a922	.221b59899	78114.21..
...3..1465	551.4.1.24	32b9a75a15	.536341658	8313757aa8	vd958a4526	.13.7ad797	a7113.52.1
..11....34	65.33..3.4	1278586834	57b63119e7	76127979ad	d+aa4a4a34	.1319d6c25	68.24.12.1
...1.1.136	334.1.13.5	23ab762835	1a8b4.18ab	7523846739	9as6d9a419	.535a49866	3312635...
....2..234	52..3..322	35c9b9a842	398831.637	631355i59c	5a6r3776.5	.2.5675c85	a5233145..
...3...553	44.13..112	.387b8b314	38bd231ba6	83.3488946	84d3w29538	1212475b78	862222212.
.....1.329	25132.11.2	..ae7ae651	298622.a9c	83.6992598	aa972oe838	...5777d3d	62132122..
...11..416	121.2.....	22a6646628	.98712.a57	9415963a9a	44a79eq538	.141444ba5	56133322..
...2...625	131.3..311	12bbb76712	36a2...977	63.19c7459	5a46585u13	141.447g63	452.21231.
.......311	1....1.1.2	..5141233.	15151..145	.122521572	231.333192	.111236442	372...1...
........15	2131..13.3	.36d697c25	16645..59b	923267b892	649588832u	.125843789	951.4113..
....	1	..13..11..	..1....12.	..11.1....	....1..1..	11...2..2.	2....1....
........22	1.........	1.3231.232	.1241..648	1.1323.422	11522.1411	1512123121	.2..1.11..
...1...111	1...1..1.2	1.3.4424..	12111..13.	3211345132	333.1.4112	.181553421	.3.1..2...
.......231	1.11......	3.133332..	..14....37	21.1212331	.155251.15	.21d114243	12.21.1...
..11....54	442.1.13.3	2128863425	179c322693	7223c9852b	79a6474428	.151+68757	4522..311.
.1.1...146	42.14....1	22ad8256.4	.9a952.796	55327a9a95	ad47774434	22516+8765	34314132..
...2.1.243	42..3.12.5	45848b6752	158b.21888	ba.437ba49	d695574763	.33488m557	86135331..
...211.344	42.2...1.4	.596e62623	398521.c86	81356669a8	7c8cbdbg47	.142775p73	751.3222..
...51..326	321.3....7	3266879a22	1a5423.b95	4325633849	926873a648	22245657t8	35113133..
...1.1.537	231...12.3	21793a8731	1c8512.a85	961466ad59	75658d5329	.11375738s	971321.2..
...3.1.653	12..1..3.3	23855693.3	1575.1.9a8	4313783777	a63a865439	2..1438739	v5243211..
...5.1.a22	513.2..121	.197557b31	179f21.g46	a711469b98	7835626575	.232546557	5u.11242..
.......1..	.2..1.....	.14121131.	.32121.316	21..161431	1.12211221	231111	2.6121....
........21	.2..1..11.	..312241..	.242..1134	2..1313531	1223233...	..12213.13	41161.1...
...1.1.11.	32113.....	.12126....	.4221..445	112.131234	34632232.4	.1.1.45332	3121t.51..
...1...111	1.....1...	3111111..1	..1....112	4...22111.	..312131.1	1....13211	221..b....
.....3.134	11..1.....	1165132112	.633....23	2152.11232	5154222211	.12133323.	14.15.h21.
.....1.1.1	2.1..1.1..	2.12411121	1621....13	11..213.31	22.51223.3	.1..121232	12..1.28..
....	.......1..	..11..21..	...1.....1	1.........	....2..1..	....1.....	......1...
....	....	....	....	....	11........	....	1

Table 1. The number of biembeddings of the 80 STS(15)s.

The diagonal entries of Table 1 represent 1 587 self-embeddings. There is a general tendency for the diagonal entry to be the greatest entry in the corresponding row (and column). Table 2 lists the diagonal entries from Table 1 and also gives the row (and column) sums which represent the total number of biembeddings of each system. In Table 2, the first column in each section gives the system number i, the second gives the number of self-embeddings s, and the third gives the total number of biembeddings t.

i	s	t	i	s	t	i	s	t	i	s	t
1	1	1	21	14	129	41	31	352	61	1	25
2	0	4	22	13	133	42	16	194	62	5	108
3	3	14	23	35	422	43	12	85	63	8	116
4	7	79	24	57	413	44	19	162	64	13	121
5	1	19	25	31	413	45	39	369	65	43	347
6	9	37	26	40	370	46	33	365	66	41	366
7	2	4	27	46	361	47	26	352	67	22	370
8	10	143	28	34	371	48	35	401	68	25	381
9	16	196	29	16	122	49	27	362	69	29	338
10	21	227	30	19	165	50	31	369	70	28	352
11	14	189	31	12	98	51	31	387	71	31	318
12	13	157	32	33	358	52	40	404	72	30	349
13	12	63	33	31	354	53	28	385	73	6	90
14	4	37	34	30	373	54	27	375	74	6	100
15	9	122	35	13	119	55	32	371	75	29	156
16	1	5	36	4	79	56	24	387	76	11	69
17	3	26	37	1	22	57	26	345	77	17	147
18	10	101	38	31	395	58	30	330	78	8	95
19	6	39	39	43	404	59	9	143	79	0	14
20	11	151	40	31	412	60	30	343	80	1	3

Table 2. Numbers of self-embeddings and biembeddings.

Overall, the value of t varies from 1 to 422. Examining the results in more detail, we observe that the systems having the largest automorphism groups have, in general, the smallest total number of biembeddings t. However, if we exclude the exceptional projective system #1, the value of the product of t with the order of the automorphism group varies from 264 to 1344, a factor of approximately 5. For the 36 systems with the trivial automorphism group, this product ($= t$, the total number of biembeddings) lies in the narrow range from 318 to 422.

Most of the 9 530 biembeddings have a trivial automorphism group and no biembedding has an automorphism group of order greater than 10. We include as automorphisms of a biembedding mappings that reverse the orientation and, for self-embeddings, those that reverse the colour classes. Table 3 analyzes the biembeddings by the order of their automorphism group Γ.

$\lvert\Gamma\rvert$	1	2	3	4	5	6	7	8	9	10
# biembeddings	8 097	1 371	22	0	2	32	0	0	0	6

Table 3. Automorphism group orders.

The six biembeddings having $|\Gamma| = 10$ comprise one self-embedding of system #1, four self-embeddings of system #76, and one self-embedding of system #80. These are all described in [8].

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