(k,l)-RADII OF PETERSEN GRAPH

Martin Knor

Abstract. Let K be a set of k vertices. The k-distance of K is the sum of all distances between pairs of vertices of K. The (k,l)-eccentricity of a set of l vertices L is the maximum k-distance over all sets K, such that $L \subseteq K$ and $|K| = k$. Finally, the (k,l)-radius of a graph is its minimum (k,l)-eccentricity. In this note we determine (k,l)-radius of Petersen graph for all possible values of k and l.

1. Introduction and results

Let $G = (V(G), E(G))$ be a graph. The **distance**, $d(u,v)$, between two vertices $u, v \in V(G)$ is the length of a shortest path connecting u with v in G, while the **diameter**, $diam(G)$, is the greatest distance in G. The **eccentricity**, $e(v)$, is the distance to a vertex farthest from v. Then the **radius**, $rad(G)$, is the smallest eccentricity in G, while the greatest eccentricity is the diameter defined above. I.e.,

$$rad(G) = \min_{v \in V(G)} e(v) = \min_{v \in V(G)} \big(\max_{u \in V(G)} d(u,v) \big), \quad diam(G) = \max_{v \in V(G)} \big(\max_{u \in V(G)} d(u,v) \big).$$

The radius and the diameter are basic invariants in theory of graphs, see [2]. They are very useful, although they do not give a full information about the graph. This defect can be reduced by inventing distance-related concepts which express the structure of G in a better way. The most reasonable thing is to consider sets of vertices instead of pairs. Let k be a number, $1 \le k \le |V(G)|$, and let K be a set of k distinct vertices in G. Then the **k-distance** of K, $d_k(K)$, is the sum of distances between all pairs of vertices of K. Observe that the usual distance is 2-distance in our new notation. The **k-diameter**, $diam_k(G)$, is the maximum k-distance in a graph, see Goddard, Swart and Swart [1]. Here we recall that, for $n = |V(G)|$, the n-distance is called the **distance** (**transmission**) of a graph, see Šoltés [6].

Now we introduce the key notion of this paper, the (k,l)-radius. Analogously as for the usual radius, we start with (k,l)-eccentricity. Let k and l be integer numbers, $0 \le l \le k \le |V(G)|$, $k > 0$, and let L be a set of l vertices in G. Then the **(k,l)-eccentricity** of L, $e_{k,l}(L)$, is the maximum k-distance of a set of k vertices K, containing L. I.e.,

$$e_{k,l}(L) = \max_{L \subseteq K \subseteq V(G)} (d_k(K);\ |K| = k).$$

Now **(k,l)-radius** of G, $rad_{k,l}(G)$, is the minimum (k,l)-eccentricity of a set of l vertices, see Horváthová [3]. Thus,

$$rad_{k,l}(G) = \min_{L \subseteq V(G)} (e_{k,l}(L);\ |L| = l) = \min_{L \subseteq V(G)} \big(\max_{L \subseteq K \subseteq V(G)} (d_k(K);\ |K| = k);\ |L| = l \big).$$

The notion of (k,l)-radius generalizes the usual diameter as $diam(G) = rad_{2,0}(G)$; further the radius $rad(G) = rad_{2,1}(G)$; the k-diameter is $rad_{k,0}(G)$; and the distance of a graph is $rad_{|V(G)|,0}(G)$.

In [4] Horváthová has shown that for almost all graphs G we have

$$rad_{k,l}(G) = 2\binom{k}{2} - \binom{l}{2}.$$

But up to now, only for complete graphs K_n (whch is trivial) and complete bipartite graphs $K_{1,n}$, $K_{2,n}$ (see Horváthová [5]) the (k,l)-radius has been determined for all admissible values of k and l. In this paper we determine (k,l)-radius for the most famous graph, the Petersen graph P, see Figure 1.

P :

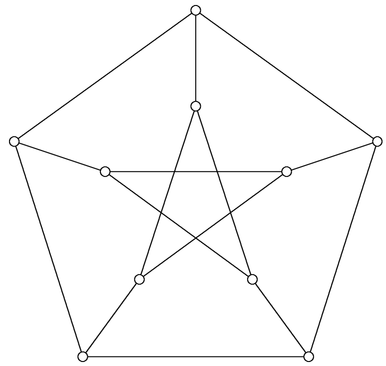

Figure 1

Theorem 1. *Let l and k be integer numbers, $0 \le l \le k \le 10$, and let $k > 0$. Then the values of (k,l)-radius of Petersen graph are listed in Table 1, where the rows correspond to values of l, and the columns correspond to k.*

$l \backslash k$	1	2	3	4	5	6	7	8	9	10
0	0	2	6	12	18	27	36	47	60	75
1	0	2	6	12	18	27	36	47	60	75
2	-	1	5	11	18	27	36	47	60	75
3	-	-	4	10	18	26	36	47	60	75
4	-	-	-	9	16	25	36	47	60	75
5	-	-	-	-	15	24	35	47	60	75
6	-	-	-	-	-	24	35	47	60	75
7	-	-	-	-	-	-	34	47	60	75
8	-	-	-	-	-	-	-	46	60	75
9	-	-	-	-	-	-	-	-	60	75
10	-	-	-	-	-	-	-	-	-	75

Table 1

As proved by Horváthová in [3], $rad_{k,l}(G) \ge rad_{k,l+1}(G)$ for every graph G. This explains the values in every column. But what is the behaviour along rows? To be able to see something, one has to use "normalized" values of $rad_{k,l}(P)$. Therefore we present the values $\binom{k}{2}^{-1} \cdot rad_{k,l}(P)$ in Table 2. We see that the function $f_l(k) = \binom{k}{2}^{-1} \cdot rad_{k,l}(P)$ is first nondecreasing and then nonincreasing. Hence, we state the following problem:

$l\backslash k$	1	2	3	4	5	6	7	8	9	10
0	0.00	2.00	2.00	2.00	1.80	1.80	1.71	1.68	1.67	1.67
1	0.00	2.00	2.00	2.00	1.80	1.80	1.71	1.68	1.67	1.67
2	-	1.00	1.67	1.83	1.80	1.80	1.71	1.68	1.67	1.67
3	-	-	1.33	1.67	1.80	1.73	1.71	1.68	1.67	1.67
4	-	-	-	1.50	1.60	1.67	1.71	1.68	1.67	1.67
5	-	-	-	-	1.50	1.60	1.67	1.68	1.67	1.67
6	-	-	-	-	-	1.60	1.67	1.68	1.67	1.67
7	-	-	-	-	-	-	1.62	1.68	1.67	1.67
8	-	-	-	-	-	-	-	1.64	1.67	1.67
9	-	-	-	-	-	-	-	-	1.67	1.67
10	-	-	-	-	-	-	-	-	-	1.67

Table 2

Problem. *What is the behaviour of* $f_{l,G}(k) = \binom{k}{2}^{-1} \cdot rad_{k,l}(G)$ *for a general graph* G*? Is it similar to that of* $f_{l,P}(k)$*?*

In the reminder of this paper we present the proof of Theorem 1.

2. Proof

Proof of Theorem 1. At first we describe all sets of vertices of Petersen's graph P. As P has 10 vertices, we have $2^{10} = 1024$ possibilities. Of course, we do not need to distinguish between such sets L and L', for which there is an automorphism of P mapping L to L'. This reduces our task considerably, since Petersen graph is 3-arc transitive. It means that for every two directed paths of length three, say (u_0, u_1, u_2, u_3) and (u'_0, u'_1, u'_2, u'_3), there is an automorphism ϕ of P such that $\phi(u_i) = u'_i$ for all i, $0 \le i \le 3$.

The sets, which do not contain isomorphic copies, are depicted in Figure 2. Their total number is 34, which is much less than 1024. Here (L^i_j, x) means that we have j-th set, we denote it by L^i_j, it contains i vertices of P and x indicates the number of edges (depicted by bold lines) in the subgraph of P induced by L^i_j.

The completeness of our list is easily verified for small numbers of vertices. Therefore, for $i \ge 6$ we made L^i_j to be a complement of L^{10-i}_{35-j}, which means that it suffices to verify the completeness of the list $L^0_1, L^1_2, \dots, L^5_{20}$. We let this as an easy exercise to the reader.

In the next we count the (k,l)-radii for all possible values of k and l. Observe that for every pair (L^i_j, x) of Figure 1 we have $d_i(L^i_j) = 2\binom{i}{2} - x$, since the diameter of Petersen graph is 2.

Obviously, for all values of l, $0 \le l \le 10$, we have $rad_{10,l}(P) = d_{10}(L^{10}_{34}) = 2\binom{10}{2} - 15 = 75$, since P has exactly 15 edges. And as P is a vertex-transitive graph, $rad_{9,l}(P) = d_9(L^9_{33}) = 2\binom{9}{2} - 12 = 60$ for all l, $0 \le l \le 9$.

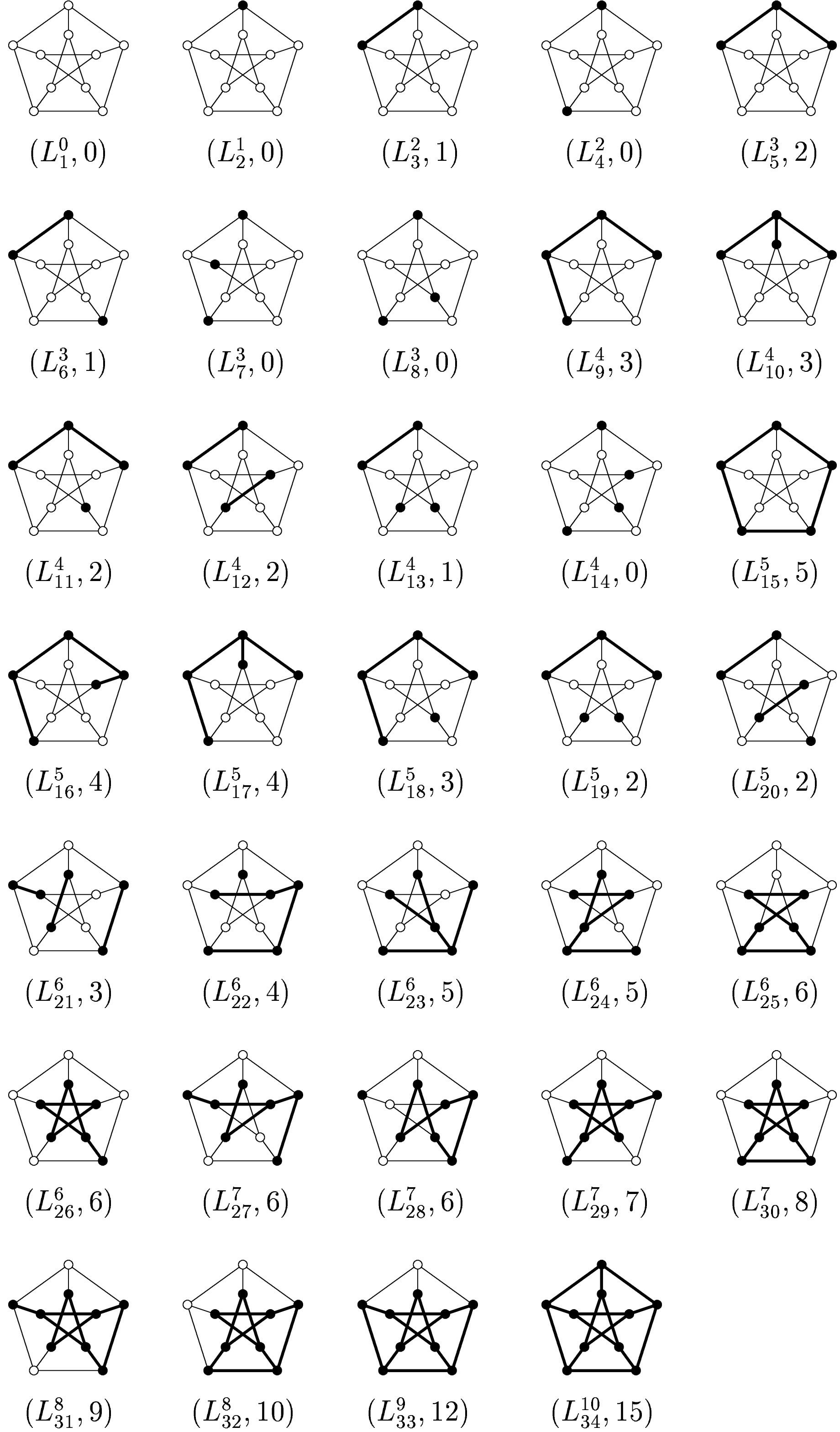

Figure 2

Further $rad_{k,0}(P)$ is the greatest value of $d_k(L_j^k)$. Thus, $rad_{1,0}(P) = d_1(L_2^1) = 0$; $rad_{2,0}(P) = d_2(L_4^2) = 2\binom{2}{2} = 2$; $rad_{3,0}(P) = d_3(L_8^3) = 2\binom{3}{2} = 6$; $rad_{4,0}(P) = d_4(L_{14}^4) = 2\binom{4}{2} = 12$; $rad_{5,0}(P) = d_5(L_{20}^5) = 2\binom{5}{2} - 2 = 18$; $rad_{6,0}(P) = d_6(L_{21}^6) =$

$2\binom{6}{2} - 3 = 27$; $rad_{7,0}(P) = d_7(L^7_{27}) = 2\binom{7}{2} - 6 = 36$; and $rad_{8,0}(P) = d_8(L^8_{31}) = 2\binom{8}{2} - 9 = 47$. Since P is a vertex-transitive graph, $rad_{k,1}(P) = rad_{k,0}(P)$ for all k, $1 \leq k \leq 10$.

Similarly, $rad_{k,k}(P)$ is the least value of $d_k(L^k_j)$. Thus, $rad_{2,2}(P) = d_2(L^2_3) = 2\binom{2}{2} - 1 = 1$; $rad_{3,3}(P) = d_3(L^3_5) = 2\binom{3}{2} - 2 = 4$; $rad_{4,4}(P) = d_4(L^4_9) = 2\binom{4}{2} - 3 = 9$; $rad_{5,5}(P) = d_5(L^5_{15}) = 2\binom{5}{2} - 5 = 15$; $rad_{6,6}(P) = d_6(L^6_{26}) = 2\binom{6}{2} - 6 = 24$; $rad_{7,7}(P) = d_7(L^7_{30}) = 2\binom{7}{2} - 8 = 34$; and $rad_{8,8}(P) = d_8(L^8_{32}) = 2\binom{8}{2} - 10 = 46$.

Since for every $l < 8$ the complement of L^l_j contains a pair of nonadjacent vertices, L^8_{31} contains a (isomorphic) copy of L^l_j. Therefore $r_{8,l}(P) = d_8(L^8_{31}) = 47$ for all $l < 8$.

Now consider $l = 2$. For $k = 3$, L^2_3 is contained in L^3_5 and L^3_6, while L^2_4 is in all L^3_5, L^3_6, L^3_7 and L^3_8. Therefore $rad_{3,2}(P) = e_{3,2}(L^2_3) = d_3(L^3_6) = 2\binom{3}{2} - 1 = 5$. Analogously, $rad_{4,2}(P) = e_{4,2}(L^2_3) = d_4(L^4_{13}) = 2\binom{4}{2} - 1 = 11$. However, for $k > 4$ all L^k_j contain both L^2_3 and L^2_4, so that $rad_{k,2}(P) = rad_{k,0}(P)$ when $k > 4$.

Let $l = 3$. For $k = 4$, L^3_5 is contained only in L^4_9, L^4_{10} and L^4_{11}, while L^3_6, L^3_7 and L^3_8 are contained in L^4_{13}. Therefore $rad_{4,3}(P) = e_{4,3}(L^3_5) = d_4(L^4_{11}) = 2\binom{4}{2} - 2 = 10$. Since L^3_5 and L^3_8 are contained in L^5_{19} and L^3_6 and L^3_7 are contained in L^5_{20}, we have $rad_{5,3}(P) = d_5(L^5_{19}) = d_5(L^5_{20}) = 18$. For $k = 6$, L^3_5 is contained only in L^6_j for $22 \leq j \leq 26$, while L^3_6, L^3_7 and L^3_8 are contained in L^6_{21}. Thus, $rad_{6,3}(P) = e_{6,3}(L^3_5) = d_6(L^6_{22}) = 2\binom{6}{2} - 4 = 26$. Finally, as all L^3_j, $5 \leq j \leq 8$, are contained in L^7_{27}, we have $rad_{7,3}(P) = d_7(L^7_{27}) = 36$.

Consider $l = 4$. For $k = 5$, L^4_{10} is contained only in L^5_{17}, while L^4_9 and L^4_{11} are contained in L^5_{18}, L^4_{12} and L^4_{13} are contained in L^5_{20} and L^4_{14} is contained in L^5_{19}. Thus, $rad_{5,4}(P) = e_{5,4}(L^4_{10}) = d_5(L^5_{17}) = 2\binom{5}{2} - 4 = 16$. For $k = 6$, L^4_{10} is contained only in L^6_j, $23 \leq j \leq 26$, while the remaining L^4_j, $j \neq 10$, are contained in L^6_{22}. Therefore $rad_{6,4}(P) = e_{6,4}(L^4_{10}) = d_6(L^6_{23}) = 2\binom{6}{2} - 5 = 25$. Finally, for $k = 7$ all L^4_j, $9 \leq j \leq 14$, are contained in L^7_{27}, so that $rad_{7,4}(P) = d_7(L^7_{27}) = 36$.

Now suppose that $l = 5$. Since L^5_{15} is contained only in L^6_{26} for $k = 6$, we have $rad_{6,5}(P) = e_{6,5}(L^5_{15}) = d_6(L^6_{26}) = 24$. For $k = 7$ L^5_{15} is contained only in L^7_{29} and L^7_{30}, while L^5_j, $16 \leq j \leq 20$, are contained in L^7_{27}. Therefore $rad_{7,5}(P) = e_{7,5}(L^5_{15}) = d_7(L^7_{29}) = 2\binom{7}{2} - 7 = 35$.

As the last case consider $l = 6$ and $k = 7$. Observe that L^6_{26} is contained only in L^7_{29} and L^7_{30}. However, L^6_{25} is contained in L^7_{28}, L^6_{24} is contained in L^7_{27}, L^6_{23} is contained in L^7_{29}, L^6_{22} is contained in L^7_{28} and L^6_{21} is contained in L^7_{27}. Therefore $rad_{7,6}(P) = e_{7,6}(L^6_{26}) = d_7(L^7_{29}) = 35$. □

Acknowledgement This paper was supported by grant VEGA 1/2004/05.

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Autor:
Doc. RNDr. Martin Knor, Dr., Katedra matematiky a deskriptívnej geometrie, Stavebná fakulta STU, Radlinského 11, 813 68 Bratislava,
e-mail: knor@math.sk