Orientable biembeddings of Steiner triple systems of order 15

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Abstract

A complete enumeration is given of orientable biembeddings involving five of the 80 Steiner triple systems of order 15. As a consequence, it follows that each of the 80 systems has a biembedding in an orientable surface, and precisely 78 of the systems have orientable self-embeddings.

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1 Introduction

This paper is concerned with biembeddings of Steiner triple systems. A *Steiner triple system* of order n, STS(n), is a pair $(V, \mathcal{B})$, where V is a set of n *points* and $\mathcal{B}$ is a collection of *triples*, also called *blocks*, taken from V and such that every pair of distinct points from V appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or $3 \pmod 6$ [10]. Biembeddings of such systems in orientable and nonorientable surfaces arise as follows. Consider a triangular embedding M of the complete graph K_n in the orientable surface S_g, the sphere with g handles, or in the nonorientable surface N_γ, the sphere with γ crosscaps. If the faces of M can be properly 2-coloured then the faces in each colour class form an STS(n). Euler's formula gives g (respectively γ) for triangular embeddings of K_n in an orientable (respectively nonorientable) surface, namely $g = (n-3)(n-4)/12$ ($\gamma = (n-3)(n-4)/6$). Face 2-colourability requires n to be odd. It easily follows that a necessary condition for an orientable biembedding is that $n \equiv 3$ or $7 \pmod{12}$, and a necessary condition for a nonorientable biembedding is $n \equiv 1$ or $3 \pmod 6$. These necessary conditions are also sufficient for the existence of such biembeddings, except in the case $n = 7$, where there is no biembedding of STS(7)s in the surface N_2 [12, 13, 8].

Two STS(n)s, A and B, are said to be *biembeddable* in some surface if there exists a face 2-colourable triangular embedding of K_n in that surface in which the Steiner triple systems arising from the two colour classes are isomorphic copies of A and B. The issue which then naturally arises is to determine which pairs of STS(n)s are biembeddable. For $v = 3$ there is a trivial and unique embedding of a triangle in the sphere, and this provides a biembedding of STS(3)s. The STS(7) is unique and there is precisely one biembedding of the system, this with an isomorphic copy of itself, in the torus S_1. Here and subsequently when enumerating, we refer to the number of isomorphism classes. For the STS(9), which is also unique, there is again precisely one biembedding, again with an isomorphic copy of itself, in the nonorientable surface N_5 [1, 7]. There are two STS(13)s, one is cyclic and the other is not. We will refer to these here as C and N respectively. There are 615 biembeddings of C with C, 8 539 biembeddings of C with N, and 29 454 biembeddings of N with N [9], all of these being in the nonorientable surface N_{15}.

Biembeddings of STS(15)s lie in either the orientable surface S_{11} or the nonorientable surface N_{22}. A complete enumeration of all these biembeddings is probably beyond current computational capabilities. There are 80 isomorphism classes of STS(15)s and we follow the standard numbering of these given in [11]. It was shown in [5] that every pair, including isomorphic pairs, has a biembedding in a nonorientable surface. However, it was proved in [4] that at least one pair, namely $\{\#1, \#2\}$ in the standard numbering, has no biembedding in an orientable surface. Furthermore, in [6] it was shown that there is no biembedding in an orientable surface of a pair of isomorphic copies of #2. It was known from the work of Ringel [12] that system #80 has an orientable biembedding with an isomorphic copy of itself (a *self-embedding*). This embedding was originally constructed by means of a bipartite index 3 current graph, and two further examples of this type are the self-embeddings of systems #1 and #76 given in [3]. All three of these self-embeddings have automorphism groups of order 10, with the automorphisms of even order exchanging the colour classes. The systems #1, #76 and #80 are the only three systems with an automorphism of order 5; indeed both #1 and #80 have cyclic automorphisms of order 15 and they are the only systems which do. System #1 is the point-line design of the projective geometry $\mathrm{PG}(3, 2)$ and has by far the largest automorphism group (order 20 160) of any of the STS(15)s. Additional details about the self-embedding of #1 are given in [2]. Structurally #1 is very different from #80; the former contains 105 Pasch configurations, the maximum possible, while the latter contains none. (A Pasch configuration is a set of four blocks of the form abc, xyc, xbz, ayz.) System #2 may be obtained from #1 by *switching* any Pasch configuration. This means replacing the four blocks shown by the blocks xyz, abz, ayc, xbc that cover exactly the same pairs.

In [6] an attempt was made to find an orientable self-embedding of each of the 80 STS(15)s. The embedding was assumed to have an involutory automorphism, with a single fixed point, that reversed the colour classes. Self-embeddings of this type were found for 78 of the 80 systems. One of the exceptions was system #2 as already noted, and the other was #79, although in this latter case the existence of an orientable self-embedding remained unsettled. However, a biembedding of #79 with #77 was discovered.

In the current paper we give all orientable biembeddings in which one of the two systems is one of #1, #2, #76, #79, #80. As noted above, all five of these systems have interesting characteristics. With the current methods of computer searching, systems with small automorphism groups are the most time consuming. The automorphism group orders of the five systems analyzed in this paper are respectively 20 160, 192, 5, 36, 60. Analysis of these systems has taken a substantial amount of computer time on existing standard PC equipment. Of the remaining 75 systems, 36 have only a trivial automorphism group and only ten have a group of order greater than 8. We intend to continue with the remaining 75 systems, but results will not be available for some time. We believe that our interim results are significant and their publication is timely, particularly as the results for #2 and #79 allow us to state the two theorems of Section 4.

2 Method

The biembeddings were found and verified by two independent computer programs. Both these programs take two systems A and B from the listing of [11]. The representation of system A is not altered, but permutations are in turn applied to the vertices of system B. For each permutation, two sets of triangles result, those from A and those from B. To form a surface embedding we have, for example, the trivial test that these sets of triangles must be disjoint. During the examination of possible permutations, the automorphism groups of the two systems are exploited. If we examine all permutations π that take a point b of B to a point a of A, then we do not need to consider further any other permutation π' that takes a point in the orbit of b under $Aut(B)$ to a point in the orbit of a under $Aut(A)$. Furthermore, if $\phi \in Aut(B)$ fixes b then checking the permutation π suffices also to check the permutation $\pi\phi$.

In the first computer program we use an enhanced procedure for the construction of the permutation. One vertex, say a_1, is chosen. Assume that the triples of A containing a_1 are $a_1a_2a_3$, $a_1a_4a_5$, $\ldots$, $a_1a_{14}a_{15}$. Then in any biembedding, in the rotation around a_1 there are pairs a_2a_3, a_4a_5, $\ldots$, $a_{14}a_{15}$. Since these pairs may be reversed and mutually interchanged, there are $6! \cdot 2^6 = 46\,080$ possible rotations at a_1, and these are considered in turn. In any biembedding of A with B, one vertex of B is mapped to a_1, and some other vertex is mapped to a_2. But as we already have the rotation at a_1, the image of the third vertex in the triple from B containing the inverse image of the pair a_1a_2 is determined. Similarly, the image of a fourth vertex determines that of a fifth vertex, and so on. In this way, only $15 \cdot 14 \cdot 12 \cdot \ldots \cdot 2 = 9\,676\,800$ permutations are constructed for each possible rotation at a_1. So, all together, for every pair of systems we examine $(46\,080) \cdot (9\,676\,800) = 445\,906\,944\,000$ possible permutations instead of $15! = 1\,307\,674\,368\,000$, that is approximately one-third. But in fact, by using the automorphism groups of A and B as described above, a much smaller number is examined. To confirm that a biembedding is formed, we simply check that the rotation at each vertex is a single cycle of length 14. To test for orientability, we fix the orientation of the triangle $a_1a_2a_3$. This implies an orientation for all the other triangles. Since every triangle is in three rotations, after assigning it an orientation, we check the two further occurrences.

In the second computer program all 15! permutations are considered lexicographically. However, in the generation of these, it is possible to skip large sets of consecutive permutations for which it is clear at an early stage of partial completion that an orientable biembedding cannot result. If we have a partially specified permutation π^* and we find that a resulting triangle T from $\pi^*(B)$ either closes a rotation at some vertex "too early" (that is, creates a cycle of length less than 14), or causes nonorientability of the surface, then we may skip all fully specified permutations π that represent completions of π^*. These checks, especially that for nonorientability, are quite time consuming. Although they allow us to reject many unsuitable permutations, the running time of this second program is very similar to that of the first.

Each of the two programs was applied with B taken in turn as one of the systems #1, #2, #76, #79, #80, and with A running through all 80 systems. It is easy to determine possible isomorphisms between two biembeddings since it is only necessary to consider at most $15 \cdot 14 \cdot 2 = 420$ mappings. Determination of automorphisms is similar. After discarding isomorphic copies we were left with just 91 orientable biembeddings.

3 Results

In the following table we give all 91 orientable biembeddings involving systems #1, #2, #76, #79 and #80. These systems feature in 1, 4, 69, 14 and 3 orientable biembeddings respectively. The first two columns give the numbers of the two biembedded systems A and B from the standard listing of [11]; the second number, which corresponds to B, is one of 1, 2, 76, 79 and 80. The third column gives the order of the automorphism group of the biembedding. The remaining 15 columns specify the permutation of the vertices of B, which gives a biembedding with system A taken directly (unaltered) from the standard listing. So, for example, the second line in the table gives a biembedding of systems #11 and #2, with a trivial automorphism group, and the biembedding is realized by taking the listings of these systems from [11] and applying the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 2 & 8 & 4 & 6 & 14 & 12 & 5 & 1 & 10 & 3 & 15 & 11 & 7 & 13 & 9 \end{pmatrix}$ to system #2.

1	1,	10;	1	8	10	3	5	2	12	7	4	6	9	11	14	15	13
11	2,	1;	2	8	4	6	14	12	5	1	10	3	15	11	7	13	9
14	2,	1;	5	4	11	9	3	14	6	8	13	1	7	12	10	2	15
27	2,	1;	10	12	9	11	5	13	15	1	2	14	4	8	7	6	3
66	2,	1;	4	10	8	3	15	1	14	2	12	13	6	9	11	5	7
4	76,	1;	8	1	15	11	14	9	13	10	12	4	3	7	2	5	6
8	76,	1;	8	9	15	10	13	6	12	11	1	14	2	7	3	4	5
9	76,	1;	9	4	8	15	14	13	3	12	1	2	10	11	6	5	7
10	76,	1;	9	2	8	1	7	15	6	5	4	14	11	10	12	3	13
11	76,	1;	1	14	2	10	9	8	12	3	4	5	11	6	15	13	7
18	76,	1;	8	14	3	2	7	1	12	11	9	10	6	4	5	13	15
21	76,	1;	2	12	14	6	11	13	3	1	10	5	9	15	4	8	7
21	76,	1;	2	4	8	7	11	1	14	10	15	5	12	13	3	9	6
21	76,	1;	2	4	7	11	3	9	5	13	12	15	10	6	14	1	8
22	76,	1;	8	6	15	14	4	7	13	5	10	2	1	9	11	12	3
23	76,	1;	5	1	12	13	2	7	10	11	6	9	3	8	4	15	14
24	76,	1;	3	11	2	14	9	7	12	10	4	6	8	13	1	15	5
25	76,	1;	1	9	6	12	4	8	14	15	11	10	2	13	3	7	5
26	76,	1;	1	13	14	10	15	3	4	7	12	9	11	5	6	8	2
27	76,	1;	1	3	8	15	10	2	12	5	13	14	4	7	9	6	11

30	76,	1;	1	10	3	12	8	6	9	5	11	2	7	14	4	15	13
33	76,	1;	5	4	7	11	3	1	12	2	14	15	10	9	6	8	13
38	76,	1;	2	3	15	4	8	10	5	14	6	13	7	9	12	1	11
39	76,	1;	5	14	4	8	1	13	11	6	7	12	9	10	2	3	15
40	76,	1;	2	5	10	12	13	9	3	4	11	8	15	14	6	7	1
40	76,	1;	3	1	12	5	10	6	7	13	11	15	4	8	2	14	9
41	76,	1;	1	15	10	11	8	2	9	5	13	7	4	6	14	3	12
41	76,	1;	2	6	11	10	12	9	14	15	4	7	3	8	5	13	1
41	76,	1;	4	13	10	14	3	6	5	12	11	9	8	15	1	2	7
41	76,	1;	7	6	4	10	5	1	13	11	9	12	14	8	3	15	2
45	76,	1;	1	10	8	12	2	6	9	4	14	15	7	11	13	3	5
45	76,	1;	5	3	6	9	13	7	14	8	11	1	10	15	12	2	4
46	76,	1;	1	2	15	4	9	8	6	14	13	10	5	12	7	3	11
46	76,	1;	3	14	15	5	10	4	9	11	12	6	2	7	8	1	13
47	76,	1;	2	12	5	11	14	13	10	4	9	3	8	15	6	1	7
48	76,	1;	6	3	1	13	4	14	12	5	15	11	2	8	7	9	10
49	76,	1;	2	14	10	9	7	15	11	12	13	4	5	8	1	6	3
53	76,	1;	1	15	11	4	7	8	12	2	5	10	13	9	3	6	14
53	76,	1;	2	1	15	11	5	7	6	14	4	9	10	12	13	3	8
53	76,	1;	4	11	2	9	15	7	8	12	13	5	10	6	1	14	3
54	76,	1;	2	12	5	8	9	3	15	11	4	13	7	10	1	6	14
55	76,	1;	2	6	10	8	11	1	9	3	5	4	14	15	12	13	7
55	76,	1;	2	1	4	13	7	14	3	9	6	5	10	11	15	12	8
56	76,	1;	1	3	14	11	5	7	12	10	15	2	6	9	13	4	8
57	76,	1;	1	7	4	14	13	12	9	8	11	10	3	2	6	5	15
57	76,	1;	1	12	3	7	8	13	5	10	6	2	9	14	4	15	11
57	76,	1;	1	2	6	3	15	8	14	10	7	13	9	11	4	12	5
58	76,	1;	4	14	2	7	3	9	11	1	8	12	6	5	15	13	10
60	76,	1;	2	1	9	5	8	3	6	11	15	12	10	7	13	14	4
61	76,	1;	1	7	9	12	6	15	11	8	5	10	13	3	4	2	14
66	76,	1;	5	7	14	9	3	13	8	1	10	11	12	6	4	15	2
67	76,	1;	1	4	9	10	7	13	8	3	6	11	12	15	2	14	5
67	76,	1;	1	10	6	4	12	5	9	13	7	2	15	3	11	14	8
67	76,	1;	2	3	15	11	13	7	8	5	14	4	1	10	12	9	6
68	76,	1;	1	6	12	13	5	10	4	8	7	2	14	9	15	3	11
68	76,	1;	1	14	6	5	15	8	13	9	11	10	2	12	4	7	3
69	76,	1;	3	14	13	5	15	9	4	2	8	12	1	11	7	10	6
70	76,	1;	5	15	7	12	4	9	6	10	13	2	14	11	3	8	1
71	76,	1;	2	7	3	8	5	6	9	1	4	13	12	11	15	14	10
71	76,	1;	4	14	11	13	15	7	5	1	10	3	2	6	9	8	12
72	76,	1;	1	2	7	6	15	3	12	10	14	4	11	13	8	5	9
72	76,	1;	6	13	12	10	5	9	11	3	14	2	1	15	7	4	8
73	76,	1;	1	15	11	12	8	4	10	5	9	13	2	14	7	6	3
76	76,	2;	1	8	14	7	4	12	11	9	6	13	3	15	2	10	5
76	76,	2;	1	14	6	15	2	7	4	11	8	10	13	5	9	3	12
76	76,	2;	1	3	12	8	14	2	4	10	13	5	7	6	15	11	9
76	76,	2;	1	4	3	11	7	15	10	14	12	6	9	5	13	2	8
76	76,	2;	2	1	10	15	5	12	13	9	8	3	14	6	7	11	4
76	76,	5;	1	2	15	12	10	11	9	13	4	6	5	7	3	14	8
76	76,	5;	1	11	14	9	6	2	12	3	7	10	5	4	13	15	8
76	76,	10;	2	1	14	10	9	12	11	13	5	4	7	6	8	3	15
76	76,	10;	2	1	8	10	9	12	11	3	5	4	7	6	15	14	13
76	76,	10;	2	1	15	10	9	12	11	14	5	4	7	6	13	8	3
76	76,	10;	2	1	13	10	9	12	11	8	5	4	7	6	3	15	14

18	79,	1;	10	8	9	2	13	14	3	1	12	5	6	11	7	15	4
23	79,	1;	9	2	1	14	11	7	5	12	13	8	3	10	15	6	4
24	79,	1;	8	2	11	7	15	13	3	9	14	12	1	6	4	10	5
27	79,	1;	4	1	14	12	11	8	15	9	3	5	6	7	13	10	2
27	79,	1;	10	2	14	9	7	3	1	12	8	4	15	5	13	6	11
28	79,	1;	4	1	13	2	12	10	8	9	3	7	5	6	14	15	11
34	79,	1;	3	1	6	12	11	15	4	9	7	14	5	10	13	8	2
40	79,	1;	15	2	1	9	6	10	8	12	14	13	7	11	5	4	3
41	79,	1;	5	1	3	7	8	15	4	11	9	14	10	6	12	13	2
55	79,	1;	3	1	15	10	11	2	6	14	13	8	12	4	9	7	5
55	79,	1;	13	2	9	3	14	4	6	8	7	15	1	11	5	10	12
58	79,	1;	11	1	3	8	13	5	4	10	7	12	9	2	15	14	6
65	79,	1;	6	2	15	12	9	7	8	4	1	14	10	11	13	3	5
77	79,	3;	8	1	3	14	2	10	12	9	15	7	11	6	5	4	13
51	80,	1;	1	15	11	6	8	3	7	12	10	4	13	9	14	5	2
52	80,	1;	1	15	12	14	5	6	4	7	9	13	11	3	8	2	10
80	80,	10;	1	3	11	10	5	9	7	6	4	15	13	12	8	14	2

4 Conclusion

The enumeration results given in this paper provide the final pieces in determining the answers to two questions concerning the biembeddability of STS(15)s. These are: which of the 80 nonisomorphic systems can be biembedded with some system in an orientable surface, and which have self-embeddings? We can state the following two theorems.

Theorem 4.1 *Each of the 80 nonisomorphic STS*(15)*s has a biembedding with some STS*(15) *in an orientable surface.*

Theorem 4.2 *Of the 80 nonisomorphic STS*(15)*s, 78 have a self-embedding in an orientable surface. The two exceptions which have no such self-embeddings are #2 and #79 in the standard listing.*

The results also enable us to identify many pairs of STS(15)s which are <u>not</u> biembeddable in an orientable surface. As we remarked in Section 2, previously the only pairs which were known not to be biembeddable were #1 with #2 and #2 with itself. In fact, system #1 can <u>only</u> be biembedded with itself and this biembedding is unique [2]. Systems #2 and #80 also have relatively few biembeddings.

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