ON FARNESS- AND RECIPROCALLY-SELFCENTRIC ANTISYMMETRIC GRAPHS

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Abstract. For every integer $k \geq 2$ we find an infinite class of graphs G for which $diam(G) = k$, the group of automorphisms of G is trivial, and the sum of distances $\sigma_G(u) = \sum_{v \in V(G)} d_G(u, v)$ (as well as the sum of reciprocals of these summands) does not depend on the choice of u.

1. Introduction

Throughout this paper, we consider graphs without loops or multiple edges. We use the standard graph terminology and notation, cf. [2].

Let $G = (V(G), E(G))$ be a graph. For two vertices, $u, v \in V(G)$, by $d_G(u, v)$ we denote their distance in G, i.e., the number of edges of a shortest path connecting u with v. The *eccentricity* $e_G(u)$ of a vertex u is $\max_{v \in V(G)}\{d_G(u, v)\}$. We recall that the maximum eccentricity in G is the diameter $diam(G)$ and the minimum eccentricity is the radius $rad(G)$. The vertices u, for which $e_G(u) = rad(G)$, are called central. For various applications, such as locations of fire-stations etc., central vertices are of

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX special importance. And, in some situations, it is comfortable if all the vertices of a graph are central. In such a case $diam(G) = rad(G)$ and G is a *selfcentric* graph.

In social networks, some authors use other notions in the place of eccentricity. The *farness* of a vertex u is $\sigma_G(u) = \sum d_G(u,v)$ and the *reciprocal distances centrality* is $\rho_G(u) = \sum \frac{1}{d_G(u,v)}$, where the sums are taken over all the vertices v of G, $v \neq u$. These invariants are frequently used in the social network analysis as a measure of the individuum position inside the network, see e.g. [4] or [6] for farness and [3] for the reciprocal distances centrality. Despite of this, only little is known about their graph-theoretical properties.

Here we are interested in graphs for which the farness (reciprocal distances centrality) of each vertex is the same. We call such graphs *farness-selfcentric* (*reciprocally-selfcentric*), due to the obvious analogy to selfcentric graphs, when one replaces the notion of eccentricity by farness (reciprocal distances centrality).

An example of a farness-selfcentric graph is a graph $C_{12}^{\times}$ obtained from a 12-cycle $(x_1, x_2, \ldots, x_{12})$ by adding the edges $x_i x_{i+2}$, where $i \in \{1, 2, 5, 6, 9, 10\}$ (see [1]). To verify that $C_{12}^{\times}$ is farness-selfcentric it suffices to examine the farness of two vertices of $C_{12}^{\times}$, due to the large automorphism group $Aut(C_{12}^{\times})$ of this graph. However, $C_{12}^{\times}$ is not reciprocally-selfcentric. In [1], there are also other examples of graphs, having special values of graph-theoretic invariants used in sociology; these invariants are discussed also for digraphs.

Obviously, if G is a vertex-transitive graph, i.e., when $Aut(G)$ acts transitively on the vertex set of G, then G is both farness- and reciprocally-selfcentric. In this note we are concerned in the other extremum, when $Aut(G)$ is trivial. In such a case, although G is both farness- and reciprocally-selfcentric, each individuum is recognisable in the network just by its role in it, regardless of the labels. Surprisingly, there exist "totally antisymmetric" farness- and reciprocally-selfcentric graphs. The next section is devoted to their construction. We show that such graphs exist for every prescribed diameter $k \geq 2$.

2. Results

By $n_G^i(u)$ we denote the number of vertices at distance i from a vertex u of G. Hence $n_G^1(u) = deg_G(u)$, i.e., $n_G^1(u)$ is the degree of u. For a set of vertices U, $U \subseteq V(G)$, by $\langle U \rangle_G$ we denote the subgraph of G induced by U.

Let $n \geq 8$. Denote by F_n a forest on n vertices, consisting of one isolated vertex and a tree obtained from a claw $K_{1,3}$ by subdividing one of its edges by $n-6$ vertices and one other edge by a single vertex. Observe that

$|Aut(F_n)| = 1$. Assume that the vertex set of F_n is $X = \{x_1, x_2, \ldots, x_n\}$.

Now let F'_n be a copy of F_n on the vertex set X, obtained from F_n by cyclic permutation of the endvertices (see Figure 1 for an example of F_8 and F'_8).

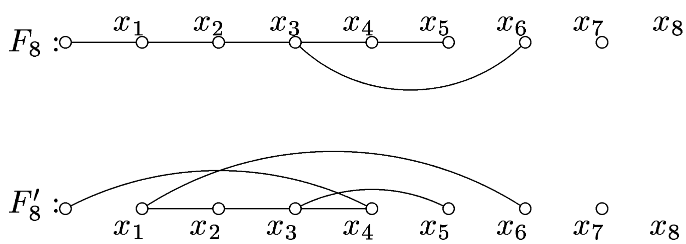

Figure 1

Subdivide all edges of F'_n, each by one vertex, denote the new vertices by $Y = \{y_1, y_2, \ldots, y_{n-2}\}$, and denote the new graph by H_{2n-2}. Finally, let B_{2n-2} be a graph obtained from a complete bipartite graph $K_{n,n-2}$, with the bipartition X and Y, by adding the edges of F_n and deleting the edges of H_{2n-2} (see Figure 2 for B_{14} obtained from Figure 1; edges that are deleted from $K_{8,6}$ are depicted by dashed lines).

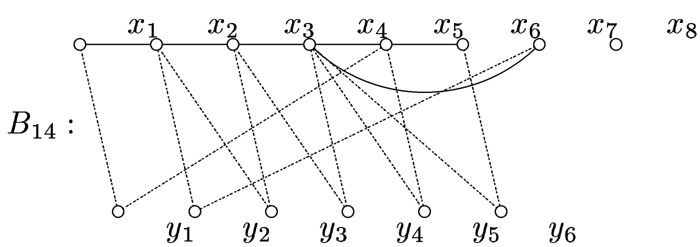

Figure 2

Theorem 1. *Let $n \geq 8$. Then B_{2n-2} is $(n-2)$-regular farness-selfcentric and reciprocally-selfcentric graph of diameter 2 with $|Aut(B_{2n-2})| = 1$.*

Proof. Observe that in the complete bipartite graph $K_{n,n-2}$ the degree of x is $n-2$ and the degree of y is n for every vertex $x \in X$ and $y \in Y$, respectively. Since the addition of F_n followed by the deletion of H_{2n-2} does not change the degree of x and it decreases the degree of y by 2, the graph B_{2n-2} is regular of degree $n-2$.

Now we show that $diam(B_{2n-2}) = 2$. If $x_a, x_b \in X$, then there is a vertex of Y which is adjacent to both x_a and x_b, since the number of deleted edges of $K_{n,n-2}$ emerging from x_a and x_b does not exceed 5 and

$|Y| = n-2 > 5$. If $y_a, y_b \in Y$, then both y_a and y_b are adjacent to the isolated vertex of F_n. Finally, suppose that $x \in X$ and $y \in Y$. If xy is not an edge of B_{2n-2} then either x is an endvertex of F_n, in which case y is connected to the neighbour of x in F_n by the construction of F'_n, or x has two neighbours in F_n, in which case y is connected to at least one of them as only two edges adjacent to y were deleted from $K_{n,n-2}$.

Since $diam(B_{2n-2}) = 2$ and B_{2n-2} is $(n-2)$-regular, for every vertex u of B_{2n-2} we have $n^1_{B_{2n-2}}(u) = n-2$ and $n^2_{B_{2n-2}}(u) = n-1$. Hence, $\sigma_{B_{2n-2}}(u) = 1 \cdot (n-2) + 2 \cdot (n-1) = 3n-4$ and $\rho_{B_{2n-2}}(u) = \frac{n-2}{1} + \frac{n-1}{2}$. Thus, B_{2n-2} is both farness- and reciprocally-selfcentric graph.

It remains to show that $|Aut(B_{2n-2})| = 1$. To do this, we consider maximal independent sets S in B_{2n-2}. If $S \subseteq X$, then $|S| \le n-3$ as F_n contains at least three independent edges. If there are $x \in X$ and $y \in Y$ such that $x, y \in S$, then x is non-adjacent to at most 3 vertices of Y and y is non-adjacent to at most 2 vertices of X. Hence, in this case $|S| \le 5$. However, if $S = Y$ then $|S| = n-2 > 5$. Thus, the independence number of B_{2n-2} is $n-2$ and the unique independent set of vertices of size $n-2$ is Y. As a consequence, every automorphism of B_{2n-2} maps X to X and Y to Y.

However, $\langle X \rangle_{B_{2n-2}}$ is exactly the graph F_n which has only the trivial automorphism. Thus, every automorphism of B_{2n-2} fixes the set X. And since different vertices of Y are non-adjacent to different pairs of vertices of X, every automorphism of B_{2n-2} fixes the set Y as well. Thus $|Aut(B_{2n-2})| = 1$, as required. $\square$

In F_n, let us denote by $x_1, x_2, \ldots, x_{n-4}$ the vertices of the longest branch (see Figure 1). Then $deg_{F_n}(x_1) = 1$, $deg_{F_n}(x_{n-4}) = 3$ and in F'_n we have a path $x_2, x_3, \ldots, x_{n-5}, x_{n-4}$. However, ordering the inner vertices of this path in a different way, say $x_2, x_4, \ldots, x_3, x_{n-4}$, will not change the structure of B_{2n-2} very much, because also in this case the diameter is 2. Hence, there are $(n-7)!$ different graphs of order $2n-2$ satisfying the conclusions of Theorem 1. (Observe that possible isomorphism between pair of them maps Y to Y, and, consequently, it fixes every $x \in X$; see the proof of Theorem 1.) If we denote $m = 2n-2$, then we see that there are at least $e^{\frac{1}{2}m\ln(m)-O(m)}$ graphs of even order m satisfying Theorem 1.

Let $n \ge 9$ and let B^1_{2n-2} and B^2_{2n-2} be two non-isomorphic graphs of the form of B_{2n-2} (see the analysis above). Let C be a cycle on $2k$ vertices, $k \ge 3$, which vertices are labelled consequently by $c_0, c_1, \ldots, c_{2k-1}$. Replace c_0, c_1 and c_3 by a copy of B^1_{2n-2} and replace all the remaining c_i, $i \neq 0, 1, 3$ by a copy of B^2_{2n-2}. Denote by C_i the graph replacing c_i, $0 \le i \le 2k-1$, and join every vertex of C_i to every vertex of C_{i+1} (the indices are taken modulo $2k$). Finally, denote the resulting graph by $CB_{k,n}$.

Theorem 2. *Let $n \ge 9$ and $k \ge 3$. Then $CB_{k,n}$ is a regular farness-*

selfcentric and reciprocally-selfcentric graph of diameter k with $|Aut(CB_{k,n})| = 1$.

Proof. Obviously, the diameter of $CB_{k,n}$ is k. From the proof of Theorem 1 it follows that $CB_{k,n}$ is a regular graph of degree $(n-2)+2(2n-2) = 5n-6$. Hence, for every $u \in V(CB_{k,n})$ we have

$$\begin{aligned}
n^1_{CB_{k,n}}(u) &= (n-2) + 2(2n-2) = 5n - 6,\\
n^2_{CB_{k,n}}(u) &= (n-1) + 2(2n-2) = 5n - 5,\\
n^3_{CB_{k,n}}(u) &= n^4_{CB_{k,n}}(u) = \cdots = n^{k-1}_{CB_{k,n}}(u) = 2(2n-2),\\
n^k_{CB_{k,n}}(u) &= 2n-2.
\end{aligned}$$

It follows that $CB_{k,n}$ is both farness- and reciprocally-selfcentric with $\sigma_{CB_{k,n}}(u) = (k^2+1)(2n-2) + n - 2$.

It remains to show that $|Aut(CB_{k,n})| = 1$. Denote by X_i and Y_i the vertices of C_i obtained from X and Y, respectively. Then $S_0 = Y_0 \cup Y_2 \cup \cdots \cup Y_{2k-2}$ and $S_1 = Y_1 \cup Y_3 \cup \cdots \cup Y_{2k-1}$ are independent sets of vertices of size $k(n-2)$. If a vertex $v \in V(C_i)$ is in an independentent set of vertices S, then no vertex from $V(C_{i-1}) \cup V(C_{i+1})$ is in S (indices are taken modulo $2k$). This implies that the sets S_0 and S_1 are maximal. Moreover, by the proof of Theorem 1, there are no other independent sets of vertices of size $k(n-2)$. Hence, every automorphism of $CB_{k,n}$ maps $\overline{Y} = S_0 \cup S_1 = \cup_{i=0}^{2k-1} Y_i$ to itself and $\overline{X} = \cup_{i=0}^{2k-1} X_i$ to itself.

Let y_a and y_b be two vertices of $\overline{Y}$, having a common neighbour in $\overline{Y}$, and such that their neighbourhoods in $\overline{Y}$ are distinct. Assume that $y_a \in V(C_{i_a})$ and $y_b \in V(C_{i_b})$. Then $i_a \neq i_b$, and since $k \geq 3$, the common neighbours are exactly the vertices of $V(C_i)$ for some i. In this way, the sets $V(C_i)$ can be identified, so that for every automorphism ϕ of $CB_{k,n}$ there is a permutation p of the set $\{0, 1, \ldots, 2k-1\}$, such that ϕ maps $V(C_i)$ to $V(C_{p(i)})$, $0 \leq i \leq 2k-1$. As a consequence, ϕ maps every set X_i to $X_{p(i)}$. But since only $\langle X_0 \rangle_{CB_{k,n}}$, $\langle X_1 \rangle_{CB_{k,n}}$ and $\langle X_3 \rangle_{CB_{k,n}}$ are isomorphic to B^1_{2n-2}, the permutation p must be an identity. The rest follows from Theorem 1. $\square$

Analogously as above, for $m = 2k(2n-2)$ it can be shown that there are at least $e^{\frac{1}{4k} m \ln(m) - O(m)}$ non-isomorphic graphs of order m satisfying Theorem 2.

3. Concluding remarks

In the social network analysis, there are also other invariants being used for evaluating the individuals positions, see [5]. The *betweenness* of the vertex u in a graph G is $b(u) = \sum \frac{b_{v,w}(u)}{b_{v,w}}$, where $b_{v,w}$ is the number of shortest $v-w$-paths, $b_{v,w}(u)$ is the number of shortest $v-w$-paths passing through u, and the sum is taken over all the pairs of vertices v and w such that $v \neq u$ and $w \neq u$. Also for betweenness we can consider the concept of self-centricity. Again, all vertex-transitive graphs are trivially betweenness-selfcentric. On the other hand, no example of betweenness-selfcentric graph with trivial automorphism group is known.

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References

[1] K. Benková, *Matematika sociálnych sietí*, Master thesis (2004).

[2] A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, MacMillan/Elsevier, London/New York, 1976.

[3] S.P. Borgatti, M.G. Everett, L.C. Freeman, *Ucinet for Windows: Software for Social Network Analysis*, Harvard: Analytic Technologies, 2002.

[4] L.C. Freeman, *Centrality in social networks. Conceptual clarification*, Social Networks **1** (1979), 215-239.

[5] L.C. Freeman, *A set of measures of centrality based upon betweeness*, Sociometry **40** (1977), 35-41.

[6] S. Wasserman, K. Faust, *Social Network Analysis: Methods and Applications*, Cambridge University Press, Cambridge, 1994.