DISTANCE INDEPENDENT DOMINATION
IN ITERATED LINE GRAPHS

MARTIN KNOR\textsuperscript{1) AND ŽUDOVÍT NIEPEL\textsuperscript{2)\footnote{Supported by VEGA grant 1/9176/02}}

\textsuperscript{1) Slovak University of Technology, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia,\textit{E-mail: knor@vox.svf.stuba.sk;}}

\textsuperscript{2) Kuwait University, Faculty of Science, Department of Mathematics \& Computer Science, P.O. box 5969 Safat 13060, Kuwait, \textit{E-mail: niepel@mcs.sci.kuniv.edu.kw.}}

\textbf{Abstract}. Let \( k \geq 1 \) be an integer and let \( G = (V, E) \) be a graph. A set \( S \) of vertices of \( G \) is \( k \)-independent if the distance between any two vertices of \( S \) is at least \( k + 1 \). We denote by \( \rho_k(G) \) the maximum cardinality among all \( k \)-independent sets of \( G \). Number \( \rho_k(G) \) is called the \( k \)-packing number of \( G \). Furthermore, \( S \) is defined to be \( k \)-dominating set in \( G \) if every vertex in \( V(G) - S \) is at distance at most \( k \) from some vertex in \( S \). A set \( S \) is \( k \)-independent dominating if it is both \( k \)-independent and \( k \)-dominating. The \( k \)-independent dominating number, \( i_k(G) \), is the minimum cardinality among all \( k \)-independent dominating sets of \( G \). We find the values \( i_k(G) \) and \( \rho_k(G) \) for iterated line graphs.

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1. Introduction and Results

Let $G$ be a graph. Its line graph $L(G)$ is defined as the graph whose vertices are the edges of $G$, with two vertices adjacent if and only if the corresponding edges are adjacent in $G$. Although the line graph operator is one of the most natural ones, only in recent years there is recorded a larger interest in studying iterated line graphs. Iterated line graphs are defined inductively as follows:

$$L^j(G) = \begin{cases} G & \text{if } j = 0, \\ L(L^{j-1}(G)) & \text{if } j > 0. \end{cases}$$

In iterated line graphs the greatest attention was devoted to Hamiltonicity. The most recent results in this area can be found in a paper by Xiong and Liu [16]. The diameter and radius of iterated line graphs are examined in [15], and [11] is devoted to the centers of these graphs. In [7] and [6], Hartke and Higgins study the growth of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [10].

We shall use the notion of domination and independency parameters as in [4]. The closed $k$-neighborhood of a vertex $v$ in a graph $G = (V(G), E(G))$ is $N_k[v] = \{ w \in V; \text{dist}_G(v, w) \leq k \}$, where $\text{dist}_G(v, w)$ denotes the distance between $v$ and $w$ in $G$. A vertex set $S$ is said to be $k$-dominating if $N_k[v] \cap S \neq \emptyset$ for every $v \in V(G)$. The set $S$ of vertices is called a $k$-packing [14] or $k$-independent [9] if $\text{dist}_G(v, w) \geq k + 1$ for each pair of distinct vertices $u$ and $v$ in $S$. By $\rho_k(G)$ we denote the $k$-packing number of $G$, that is, the maximum cardinality of a $k$-packing. Note that $\beta(G) = \rho_1(G)$, where $\beta(G)$ is the independence number of $G$; and $\rho(G) = \rho_2(G)$, where $\rho(G)$ is the packing number of $G$.

In general, a set $S$ is a maximal $k$-packing if and only if it is $k$-dominating. A vertex set $S$ is $k$-independent dominating set, if it is both $k$-packing and $k$-dominating set. The $k$-independent domination number $i_k(G)$ is the minimum cardinality of a $k$-independent dominating set. In particular, $i(G) = i_1(G)$ where $i(G)$ is the independent domination number. Thus, $i_k(G)$ and $\rho_k(G)$ are the minimum and maximum cardinalities, respectively, of any maximal $k$-packing. In [4] there are presented bounds for values of $i_k(G)$ and $\rho_k(G)$ and it is proved that the decision problem "$i_k(G) < \rho_k(G)$?" is NP-complete in general. For a survey of results concerning distance domination and independency we refer to [9] and [8].

In [2] and [3], Dutton and Brigham study the domination properties of line graphs and relationships between different parameters of domination and independence for line graphs. They proved that the domination number $\gamma(L(G))$ and independent domination number $i(L(G))$ have the same value. (Recall that $\gamma(G)$ is the minimum cardinality of a dominating set
in $G$.) In this paper we give exact values for $\rho_k(L^j(G))$ and $i_k(L^j(G))$, providing that $j$ is “big enough”. (The proofs of our results are postponed to section 2.)

In one of the pioneering papers in graph theory [12], Kotzig proved:

**Theorem A.** Let $G$ be a connected graph with even number of edges. Then $E(G)$ can be decomposed into $\frac{1}{2} |E(G)|$ paths of length 2.

As an immediate consequence of this result (and Lemmas B and C below) we have:

**Proposition 1.** Let $G$ be a graph and $j \geq 2$. If $L^{j-2}(G)$ is connected then

$$\beta(L^j(G)) = \rho_1(L^j(G)) = \left\lfloor \frac{|E(L^{j-2}(G))|}{2} \right\rfloor.$$ 

Using Proposition 1, for independent domination number we prove:

**Theorem 2.** Let $G$ be a graph and $j \geq 3$. If $L^{j-3}(G)$ is a connected graph with $\delta(L^{j-3}(G)) \geq 3$, then

$$i(L^j(G)) = i_1(L^j(G)) = \left\lfloor \frac{|E(L^{j-2}(G))| - \frac{1}{2} |V(L^{j-2}(G))|}{2} \right\rfloor.$$ 

Here $\delta(G)$ denotes the minimum degree of a graph $G$, while $\Delta(G)$ denotes its maximum degree.

It is well-known that for a general graph $G$ the problem of finding its independence number is NP-hard, see [5]. If we consider line graphs, maximum independent sets in $L(G)$ correspond to maximum matchings in $G$. Hence, the problem of finding the independence number of $L(G)$ is polynomial, see [13]. By Proposition 1, the problem of finding the independence number of $L^2(G)$ is trivial, and by Theorem 2, if $G$ has minimum degree at least 3, then the problem of finding the independent domination number of $L^3(G)$ is trivial as well.

It is interesting that analogous straightforward formulae can be given also for $\rho_k(L^j(G))$ and $i_k(L^j(G))$, $k \geq 2$, providing that $j$ is “big enough”. We prove here:

**Theorem 3.** Let $G$ be a graph, $k \geq 2$ and $j \geq k+2$. If $L^{j-k-2}(G)$ is a connected graph with $\delta(L^{j-k-2}(G)) \geq 5$, then

$$\rho_k(L^j(G)) = \left\lfloor \frac{|E(L^{j-k-1}(G))|}{3} \right\rfloor.$$ 

For the $k$-independent domination number we have:
Theorem 4. Let $G$ be a graph, $k \geq 2$ and $j \geq k + 3$. If $L^{j-k-2}(G)$ is a connected graph with $\delta(L^{j-k-2}(G)) \geq 9k - 7$ and $\delta(L^{j-k-3}(G)) \geq 3$, then

$$i_k(L^j(G)) = \left\lceil \frac{|E(L^{j-k-1}(G))| - \frac{k}{k+1}|V(L^{j-k-1}(G))|}{k+1} \right\rceil.$$ 

For the number of vertices of iterated line graphs we have, see [15]:

$$|V(G)| \cdot \prod_{k=0}^{j-1} \left[ 2^{k-1}(\delta(G) - 2) + 1 \right] \leq |V(L^j(G))| \leq \left| V(G) \right| \cdot \prod_{k=0}^{j-1} \left[ 2^{k-1}(\Delta(G) - 2) + 1 \right].$$

We remark that if $G$ is distinct from a path, a cycle and a claw $K_{1,3}$, then there exists $j_G$ such that $\delta(L^{j_G(G)}) \geq 3$. Then $\delta(L^j(G)) \geq 2^{j-j_G} \cdot (\delta(L^{j_G(G)})-2) + 2$, so that if $j$ is “big enough”, all the assumptions of Theorems 2, 3 and 4 are fulfilled. Hence, these theorems can be applied for any graph (distinct from a path, a cycle and a claw $K_{1,3}$), providing that $j$ is “big enough”.

Observe that $|E(L^j(G))| = |V(L^{j+1}(G))|$. Hence, if $G$ is $\delta$-regular graph, $\delta \geq 3$, then we can immediately write the numbers $\rho_k(L^j(G))$ and $i_k(L^j(G))$ for $j$ “big enough”, although $|V(L^j(G))|$ grows doubly exponentially as a function of $j$.

2. Proofs

Let $G$ be a graph and let $v$ be a vertex of $L^j(G)$, $j \geq 1$. Then $v$ corresponds to an edge of $L^{j-1}(G)$, and this edge will be called the 1-history of $v$. For $t \geq 2$ we define $t$-histories recursively. The $t$-history of $v$ is a subgraph of $L^{j-t}(G)$, edges of which are induced by the vertices of $L^{j-t+1}(G)$ which are in $(t-1)$-history of $v$.

Observe that the 1-history is always an edge and the 2-history is a path of length 2. The situation is more complicated for $t$-histories when $t \geq 3$. In [15] we have the following lemma:

Lemma B. Let $G$ be a graph and let $L^j(G)$ be its $j$-iterated line graph. Further, let $0 \leq t \leq j$ and let $H$ be a subgraph of $L^{j-t}(G)$. Then $H$ is a $t$-history of some vertex of $L^j(G)$ if and only if $H$ is a connected graph with at most $t$ edges, distinct from any path with less than $t$ edges.

Also the next lemma, which is useful for calculating the distances in iterated line graphs, can be found in [15].
Lemma C. Let $G$ be a connected graph, $L^j(G)$ be its iterated line graph, and let $u$ and $v$ be distinct vertices of $L^j(G)$. Then for any $t$, $0 \leq t \leq j$, if the $t$-histories of $u$ and $v$ are edge-disjoint, then the distance between $u$ and $v$ in $L^j(G)$ equals the minimum distance between the two vertex sets of the $t$-histories in $L^{j-t}(G)$, increased by $t$. If the $t$-histories of $u$ and $v$ are not edge-disjoint, then the distance in $L^j(G)$ between $u$ and $v$ is strictly less than $t$.

As can be deduced from the lemmas above, we prove our theorems for $L^j(G)$ using histories in smaller iterations. However, first we introduce two lemmas. It is worth mentioning, that their proofs (via orientations) are similar to the proof of Theorem A in [12].

Lemma 5. Let $G$ be a connected graph. If $\delta(G) \geq 5$, then $L(G)$ contains $\left\lceil \frac{|E(L(G))|}{3} \right\rceil$ edge-disjoint copies of a claw $K_{1,3}$.

Proof. Denote $H = L(G)$. If $ab$ is an edge of $H$, then its vertices correspond to pair of adjacent edges, say $uv$ and $vw$, in $G$. In $G$ the degree of $v$ is at least 5, so that the edge $ab$ lies in a copy of a complete graph $K_5$ in $H$. Hence, every edge of $H$ lies in a copy of $K_5$.

To every edge of $H$ we assign one of the two possible orientations. In such a way we obtain from $H$ its orientation $O(H)$. Let $v$ be a vertex of $H$. Denote by $o(v)$ the number of arcs in $O(H)$ terminating at $v$, taken modulo 3. Further, denote by $o(H)$ the number of vertices $v$ with $o(v) > 0$, and assume that $O(H)$ is chosen so that $o(H)$ is the minimum possible. In the following we prove $o(H) \leq 1$.

Denote by $\overrightarrow{x\,y}$ an arc of $O(H)$ starting at $x$ and terminating at $y$. We prove that for any edge $uv \in E(H)$, such that $o(u) > 0$, there exists an orientation $O'(H)$ such that $o'(u) = 0$ and $o'(w) = o(w)$ whenever $w$ is distinct from $u$ and $v$. There are four cases to distinguish:

Case 1: $o(u) = 2$ and $\overrightarrow{uv}$ is an arc in $O(H)$. Then reversing $\overrightarrow{uv}$ we receive the required orientation $O'(H)$.

Case 2: $o(u) = 2$ and $\overrightarrow{vu}$ is an arc in $O(H)$. If there is a directed path from $u$ to $v$ in $O(H)$, then reversing its arcs we obtain the required orientation $O'(H)$. Analogously, if there is a directed path from $v$ to $u$ in $O(H)$, then reversing the arcs of this path and reversing $\overrightarrow{vu}$ we receive the required orientation $O'(H)$. Hence, we can assume that there are no directed paths of these types. Since $u$ and $v$ lie in a copy of a complete graph $K_5$, there are two vertices $x$ and $y$ with the arc $\overrightarrow{x\,y}$, such that either $\overrightarrow{ux}$, $\overrightarrow{uv}$, $\overrightarrow{vy}$, $\overrightarrow{vy}$ are in $O(H)$ or $\overrightarrow{ux}$, $\overrightarrow{ux}$, $\overrightarrow{vy}$, $\overrightarrow{vy}$ are in $O(H)$. In the first case reversing $\overrightarrow{uv}$, $\overrightarrow{ux}$, $\overrightarrow{vy}$, $\overrightarrow{vy}$ we receive the required orientation $O'(H)$, while in the second one reversing $\overrightarrow{uv}$, $\overrightarrow{vy}$, $\overrightarrow{x\,y}$, $\overrightarrow{vx}$ we get the required orientation $O'(H)$. 5
Case 3: $o(u) = 1$ and $\overrightarrow{vu}$ is an arc in $O(H)$. Then reversing $\overrightarrow{vu}$ we receive the required orientation $O'(H)$.

Case 4: $o(u) = 1$ and $\overrightarrow{uv}$ is an arc in $O(H)$. Analogously as in Case 2 we can assume that there are neither directed $u-v$ paths nor directed $v-u$ paths in $O(H)$. But then there are two vertices $x$ and $y$ with the arc $\overrightarrow{xy}$, such that either $\overrightarrow{vu}, \overrightarrow{uv}, \overrightarrow{xy}, \overrightarrow{yx}$ are in $O(H)$ or $\overrightarrow{ux}, \overrightarrow{uy}, \overrightarrow{xy}, \overrightarrow{yx}$ are in $O(H)$. In the first case reversing $\overrightarrow{vu}, \overrightarrow{uv}, \overrightarrow{xy}, \overrightarrow{yx}$ we receive the required orientation $O'(H)$, while in the second one reversing $\overrightarrow{uv}, \overrightarrow{xy}, \overrightarrow{ux}, \overrightarrow{uy}$ we obtain the required orientation $O'(H)$.

Suppose that $o(H) \geq 2$. Then there are two vertices, say $u$ and $v$, such that $o(u) > 0$ and $o(v) > 0$. Since $G$ is a connected graph, $H$ is connected as well, so that there is a path $u=x_0, x_1, \ldots, x_t=v$. Consider a sequence of orientations $O_0(H)=O(H), O_1(H), \ldots, O_t(H)$, such that $\alpha_t(x_{i-1}) = 0$ and $\alpha_t(x) = \alpha_{i-1}(x)$ whenever $x$ is distinct from $x_{i-1}$ and $x_i$. Then $\alpha_t(x_0) = \alpha_t(x_1) = \cdots = \alpha_t(x_{t-1}) = 0$, and for all $x \notin \{x_0, x_1, \ldots, x_t\}$ we have $\alpha_t(x) = o(x)$. Hence, $\alpha_t(H) < o(H)$, a contradiction.

Thus, there is an orientation $O(H)$ such that $o(H) \leq 1$. That means that up to one exception, the number of arcs directed to each vertex $v$ of $O(H)$ is a multiple of 3. Hence, these arcs can be arranged into triples to form $\left\lfloor \frac{|E(H)|}{3} \right\rfloor$ claws as required. □

**Lemma 6.** Let $G$ be a connected graph, $k \geq 2$ and $\delta(G) \geq 9k - 7$. Moreover, let $H$ be a graph obtained from $L(G)$ by deleting edges of vertex-disjoint paths, each of length at most $k$. If $|E(H)| \equiv 0 \pmod{k+1}$, then $E(H)$ can be decomposed into $\left\lfloor \frac{|E(H)|}{k+1} \right\rfloor$ stars $K_{1,k+1}$.

**Proof.** Denote by $O(H)$ an orientation of $H$. Analogously as in the proof of Lemma 5, denote by $\overrightarrow{uv}$ an arc of $O(H)$ starting at $u$ and terminating at $v$. Further, denote by $o(v)$ the number of arcs in $O(H)$ terminating at $v$, taken modulo $k+1$. Finally, denote by $o(H)$ the number of vertices $v$ with $o(v) > 0$, and assume that $O(H)$ is chosen so that $o(H)$ is the minimum possible.

In the next we prove $o(H) = 0$. To do this we prove that for any edge $uv$ of $H$, such that $o(u) > 0$, there is an orientation $O'(H)$ such that $d'(u) = 0$ and $d'(w) = o(w)$ whenever $w$ is distinct from $u$ and $v$.

If there are $o(u)$ edge-disjoint directed $v-u$ paths in $O(H)$, then reversing all their arcs we receive the required orientation $O'(H)$. Analogously, if there are $k+1-o(u)$ edge-disjoint directed $u-v$ paths in $O(H)$, then reversing all their arcs we receive the required orientation $O'(H)$. Hence, we can assume that the number of directed paths in between $u$ and $v$ does not exceed $(o(u)-1) + (k+1-o(u)-1) = k - 1$.

Since $\delta(G) \geq 9k - 7$, in $L(G)$ the edge $uv$ lies in a copy of a complete graph $K_{9k-7}$. Consider the $9k-9$ vertices of this complete graph, distinct
from $u$ and $v$. Since $\text{deg}_{L(G)}(x) - \text{deg}_H(x) \leq 2$ for every vertex $x$ of $H$, there are at most 4 vertices out of these $9k - 9$, which are not adjacent to both $u$ and $v$ in $H$. Since there are at most $k - 1$ directed paths in between $u$ and $v$, one of them obtained from the edge $uv$, there are $(9k - 13) - (k - 2) = 8k - 11$ vertices $x$ in the complete graph, such that either $\overrightarrow{ux}, \overrightarrow{xv}$ are arcs of $O(H)$ or $\overrightarrow{ux}, \overrightarrow{xv}$ are arcs of $O(H)$. Suppose that the number of vertices $x$ with arcs $\overrightarrow{ux}, \overrightarrow{xv}$ does not exceed the number of vertices $x$ with arcs $\overrightarrow{ux}, \overrightarrow{xv}$. (The other case can be solved analogously.) Then there are $4k - 5$ vertices $x$ in the complete graph, such that $\overrightarrow{ux}, \overrightarrow{xv}$ are arcs of $O(H)$. However, some edges connecting these $x$’s in $L(G)$ may be missing in $H$. But since any system of vertex-disjoint paths on $n$ vertices contains a set of independent vertices of size greater than or equal to $\left\lceil \frac{n}{2} \right\rceil$, there are $2k - 2$ vertices $x$ in $H$ which are mutually adjacent and such that $\overrightarrow{ux}, \overrightarrow{xv}$ are arcs in $O(H)$.

Denote by $K$ the complete subgraph of $H$, consisting of the $2k - 2$ vertices described above. Further, denote by $O(K)$ the orientation of $K$ induced by $O(H)$. Since $(2k - 2) \cdot (k - 2) < \binom{2k - 2}{2}$, $O(K)$ contains a vertex $x_0$ with the out-degree at least $k - 1$. Hence, there are vertices $x_0, x_1, \ldots, x_{k-1}$, such that $\overrightarrow{x_0x_i}$ is an arc of $O(H)$ for $1 \leq i \leq k - 1$ and $\overrightarrow{x_jx_i}$ are arcs of $O(H)$ for $0 \leq j \leq k - 1$. Observe that $1 \leq o(u) \leq k$. Now reverse all the arcs $\overrightarrow{x_0x_i}$, $1 \leq i \leq k - 1$, $\overrightarrow{x_0x_i}, \overrightarrow{x_0x_i}$, and one arc from each pair $\overrightarrow{x_iu}, \overrightarrow{x_iu}$, so that exactly $o(u)$ arcs from $\overrightarrow{x_iu}$ will be reversed. If we denote the resulting orientation by $O'(H)$, then $o'(u) = 0$ and $o'(w) = o(w)$ whenever $w$ is distinct from $u$ and $v$.

Since $k \geq 2$, we have $\delta(G) \geq 9k - 7 \geq 5$, so that each edge of $L(G)$ lies in three distinct triangles. Hence, if $uw$ is an edge of $L(G) - H$, then there is a vertex $z$ in $H$ such that $uz$ and $zv$ are edges of $H$. Thus, since $G$ is a connected graph, so is $L(G)$ and consequently also $H$. Now proceeding analogously as at the end of the proof of Lemma 5 it can be shown that $o(H) \leq 1$. Since $|E(H)| \equiv 0 \pmod{k+1}$, the case $o(H) = 1$ is impossible, so that $o(H) = 0$. And partitioning all the arcs terminating at $v$ into $(k+1)$-tuples, we obtain the required decomposition of $E(H)$ into stars $K_{1,k+1}$. □

Now we can prove our main results. However, first we prove Theorems 3 and 4, since the proof of Theorem 2 is similar to that of Theorem 4.

**Proof of Theorem 3.** Let $u$ and $v$ be two distinct vertices of $L^j(G)$. If $\text{dist}_{L^j(G)}(u,v) > k$, then the $(k+1)$-histories of $u$ and $v$ are edge-disjoint, by Lemma C. Since every $(k+1)$-history contains at least 3 edges by Lemma B, we have $\rho_k(L^j(G)) \leq \left\lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \right\rfloor$.

By Lemma 5, there are $\left\lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \right\rfloor$ edge-disjoint copies of a claw $K_{1,3}$ in $L^{j-k-1}(G)$. These correspond to a $k$-independent set of the same size in $L^j(G)$, so that $\rho_k(L^j(G)) \geq \left\lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \right\rfloor$. □
In the proof of Theorem 4 we use the following result of Chartrand and Wall [1]:

**Theorem D.** If $G$ is a connected graph such that $\delta(G) \geq 3$, then $L^2(G)$ is Hamiltonian.

**Proof of Theorem 4.** Let $S$ be a set of vertices of $L^j(G)$ on which $i_k(L^j(G))$ is reached. Then the $(k+1)$-histories of vertices of $S$ are edge-disjoint, by Lemma C. Remove the edges of these $(k+1)$-histories from $L^{j-k-1}(G)$ and denote the resulting graph by $F$. Since the set $S$ is maximal, $F$ does not contain cycles and all vertices of $F$ have degree less than or equal to 2 by Lemma B. Hence, $F$ consists of vertex-disjoint paths, and by Lemma B, all these paths have lengths less than $k+1$. Thus, $|E(F)| \leq \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor$. As a $(k+1)$-history of a vertex contains at most $k+1$ edges, we have

$$i_k(L^j(G)) \geq \frac{|E(L^{j-k-1}(G))| - \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor}{k+1}.$$ 

On the other hand, $L^{j-k-1}(G)$ contains a Hamiltonian cycle, by Theorem D. Deleting $\left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor$ edges from this cycle we obtain a graph $F^*$ consisting of paths of lengths less than $k+1$. Obviously, $|E(F^*)| = \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor$. Let $t$ be a number, $0 \leq t < k+1$, such that

$$\left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor - |E(L^{j-k-1}(G))| \equiv t \pmod{k+1}.$$

Delete from $F^*$ exactly $t$ edges and denote the resulting graph by $F$. Then $|E(L^{j-k-1}(G))| - |E(F)| \equiv 0 \pmod{k+1}$, so that $L^{j-k-1}(G) - E(F)$ can be decomposed into $\frac{1}{k+1} (|E(L^{j-k-1}(G))| - |E(F)|)$ stars $K_{1,k+1}$, by Lemma 6. These correspond to a $k$-independent dominating set of the same size in $L^j(G)$, so that

$$i_k(L^j(G)) \leq \frac{|E(L^{j-k-1}(G))| - |E(F)|}{k+1} = \left\lfloor \frac{|E(L^{j-k-1}(G))| - \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor}{k+1} \right\rfloor.$$ 

Combining the two inequalities for $i_k(L^j(G))$ we get the result. \(\square\)

Finally, we prove the result concerning $i(L^j(G))$.

**Proof of Theorem 2.** Substituting 1 for $k$, analogously as in the proof of Theorem 4 we get

$$i_1(L^j(G)) \geq \frac{|E(L^{j-2}(G))| - \left\lfloor \frac{1}{2} |V(L^{j-2}(G))| \right\rfloor}{2}.$$
By Proposition 1, \( \beta(L^{j-1}(G)) = \left\lfloor \frac{|E(L^{j-2}(G))|}{2} \right\rfloor = \left\lfloor \frac{|V(L^{j-2}(G))|}{2} \right\rfloor \). Choose one set of \( \beta(L^{j-1}(G)) \) independent vertices of \( L^{j-1}(G) \), and denote by \( F^* \) the edges of \( L^{j-2}(G) \) which are 1-histories of vertices of the chosen set. Then \( F^* \) is a collection of independent edges. Observe that \( |E(F^*)| = \left\lfloor \frac{1}{2} |V(L^{j-2}(G))| \right\rfloor \). Now let \( t \) be a number, \( 0 \leq t < 2 \), such that
\[
\left\lfloor \frac{1}{2} |V(L^{j-2}(G))| \right\rfloor - |E(L^{j-2}(G))| \equiv t \pmod{2}.
\]
Delete from \( F^* \) exactly \( t \) edges and denote the resulting graph by \( F \). Since \( \delta(L^{j-3}(G)) \geq 3 \), each edge of \( F \) lies in a triangle in \( L^{j-2}(G) \). Hence, \( L^{j-2}(G) - E(F) \) is a connected graph with even number of edges. By Theorem A its edges can be decomposed into paths of length 2, so that
\[
i_1(L^j(G)) \leq \frac{|E(L^{j-2}(G))| - |E(F)|}{2} = \left\lfloor \frac{|E(L^{j-2}(G))| - \frac{1}{2} |V(L^{j-2}(G))|}{2} \right\rfloor.
\]

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