DISTANCE INDEPENDENT DOMINATION IN ITERATED LINE GRAPHS

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ABSTRACT. Let $k \geq 1$ be an integer and let $G = (V, E)$ be a graph. A set S of vertices of G is k-independent if the distance between any two vertices of S is at least $k+1$. We denote by $\rho_k(G)$ the maximum cardinality among all k-independent sets of G. Number $\rho_k(G)$ is called the k-packing number of G. Furthermore, S is defined to be k-dominating set in G if every vertex in $V(G) - S$ is at distance at most k from some vertex in S. A set S is k-independent dominating if it is both k-independent and k-dominating. The k-independent dominating number, $i_k(G)$, is the minimum cardinality among all k-independent dominating sets of G. We find the values $i_k(G)$ and $\rho_k(G)$ for iterated line graphs.

1991 *Mathematics Subject Classification.* 05C69, (05C70).

Key words and phrases. Domination number, independence, iterated line graph, graph dynamics, packing claws, packing number.

[1)] Supported by VEGA grant 1/9176/02

[2)] Supported by Kuwait University grant #SM 02/00

1. Introduction and results

Let G be a graph. Its line graph $L(G)$ is defined as the graph whose vertices are the edges of G, with two vertices adjacent if and only if the corresponding edges are adjacent in G. Although the line graph operator is one of the most natural ones, only in recent years there is recorded a larger interest in studying iterated line graphs. Iterated line graphs are defined inductively as follows:

$$L^j(G) = \begin{cases} G & \text{if } j = 0, \\ L(L^{j-1}(G)) & \text{if } j > 0. \end{cases}$$

In iterated line graphs the greatest attention was devoted to Hamiltonicity. The most recent results in this area can be found in a paper by Xiong and Liu [16]. The diameter and radius of iterated line graphs are examined in [15], and [11] is devoted to the centers of these graphs. In [7] and [6], Hartke and Higgins study the growth of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [10].

We shall use the notation of domination and independency parameters as in [4]. The closed k-neighborhood of a vertex v in a graph $G = (V(G), E(G))$ is $N_k[v] = \{w \in V; dist_G(v,w) \leq k\}$, where $dist_G(v,w)$ denotes the distance between v and w in G. A vertex set S is said to be k-dominating if $N_k[v] \cap S \neq \emptyset$ for every $v \in V(G)$. The set S of vertices is called a k-packing [14] or k-independent [9] if $dist_G(v,w) \geq k+1$ for each pair of distinct vertices u and v in S. By $\rho_k(G)$ we denote the k-packing number of G, that is, the maximum cardinality of a k-packing. Note that $\beta(G) = \rho_1(G)$, where $\beta(G)$ is the independence number of G; and $\rho(G) = \rho_2(G)$, where $\rho(G)$ is the packing number of G.

In general, a set S is a maximal k-packing if and only if it is k-dominating. A vertex set S is k-independent dominating set, if it is both k-packing and k-dominating set. The k-independent domination number $i_k(G)$ is the minimum cardinality of a k-independent dominating set. In particular, $i(G) = i_1(G)$ where $i(G)$ is the independent domination number. Thus, $i_k(G)$ and $\rho_k(G)$ are the minimum and maximum cardinalities, respectively, of any maximal k-packing. In [4] there are presented bounds for values of $i_k(G)$ and $\rho_k(G)$ and it is proved that the decision problem "$i_k(G) < \rho_k(G)$?" is NP-complete in general. For a survey of results concerning distance domination and independency we refer to [9] and [8].

In [2] and [3], Dutton and Brigham study the domination properties of line graphs and relationships between different parameters of domination and independence for line graphs. They proved that the domination number $\gamma(L(G))$ and independent domination number $i(L(G))$ have the same value. (Recall that $\gamma(G)$ is the minimum cardinality of a dominating set in G.) In this paper we give exact values for $\rho_k(L^j(G))$ and $i_k(L^j(G))$, providing that j is "big enough". (The proofs of our results are postponed to section 2.)

In one of the pioneering papers in graph theory [12], Kotzig proved:

Theorem A. *Let G be a connected graph with even number of edges. Then $E(G)$ can be decomposed into $\frac{1}{2}|E(G)|$ paths of length 2.*

As an immediate consequence of this result (and Lemmas B and C below) we have:

Proposition 1. *Let G be a graph and $j \geq 2$. If $L^{j-2}(G)$ is connected then*

$$\beta(L^j(G)) = \rho_1(L^j(G)) = \left\lfloor \frac{|E(L^{j-2}(G))|}{2} \right\rfloor.$$

Using Proposition 1, for independent domination number we prove:

Theorem 2. *Let G be a graph and $j \geq 3$. If $L^{j-3}(G)$ is a connected graph with $\delta(L^{j-3}(G)) \geq 3$, then*

$$i(L^j(G)) = i_1(L^j(G)) = \left\lceil \frac{|E(L^{j-2}(G))| - \lfloor \frac{1}{2}|V(L^{j-2}(G))| \rfloor}{2} \right\rceil.$$

Here $\delta(G)$ denotes the minimum degree of a graph G, while $\Delta(G)$ denotes its maximum degree.

It is well-known that for a general graph G the problem of finding its independence number is NP-hard, see [5]. If we consider line graphs, maximum independent sets in $L(G)$ correspond to maximum matchings in G. Hence, the problem of finding the independence number of $L(G)$ is polynomial, see [13]. By Proposition 1, the problem of finding the independence number of $L^2(G)$ is trivial, and by Theorem 2, if G has minimum degree at least 3, then the problem of finding the independent domination number of $L^3(G)$ is trivial as well.

It is interesting that analogous straightforward formulae can be given also for $\rho_k(L^j(G))$ and $i_k(L^j(G))$, $k \geq 2$, providing that j is "big enough". We prove here:

Theorem 3. *Let G be a graph, $k \geq 2$ and $j \geq k+2$. If $L^{j-k-2}(G)$ is a connected graph with $\delta(L^{j-k-2}(G)) \geq 5$, then*

$$\rho_k(L^j(G)) = \left\lfloor \frac{|E(L^{j-k-1}(G))|}{3} \right\rfloor.$$

For the k-independent domination number we have:

Theorem 4. *Let G be a graph, $k \geq 2$ and $j \geq k+3$. If $L^{j-k-2}(G)$ is a connected graph with $\delta(L^{j-k-2}(G)) \geq 9k-7$ and $\delta(L^{j-k-3}(G)) \geq 3$, then*

$$i_k(L^j(G)) = \left\lceil \frac{|E(L^{j-k-1}(G))| - \lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \rfloor}{k+1} \right\rceil.$$

For the number of vertices of iterated line graphs we have, see [15]:

$$|V(G)| \cdot \prod_{k=0}^{j-1} \left[2^{k-1} \cdot (\delta(G) - 2) + 1 \right] \leq |V(L^j(G)| \leq$$
$$|V(G)| \cdot \prod_{k=0}^{j-1} \left[2^{k-1} \cdot (\Delta(G) - 2) + 1 \right].$$

We remark that if G is distinct from a path, a cycle and a claw $K_{1,3}$, then there exists j_G such that $\delta(L^{j_G}(G)) \geq 3$. Then $\delta(L^j(G)) \geq 2^{j-j_G} \cdot (\delta(L^{j_G}(G)) - 2) + 2$, so that if j is "big enough", all the assumptions of Theorems 2, 3 and 4 are fulfilled. Hence, these theorems can be applied for any graph (distinct from a path, a cycle and a claw $K_{1,3}$), providing that j is "big enough".

Observe that $|E(L^j(G))| = |V(L^{j+1}(G))|$. Hence, if G is δ-regular graph, $\delta \geq 3$, then we can immediately write the numbers $\rho_k(L^j(G))$ and $i_k(L^j(G))$ for j "big enough", although $|V(L^j(G))|$ grows doubly exponentially as a function of j.

2. Proofs

Let G be a graph and let v be a vertex of $L^j(G)$, $j \geq 1$. Then v corresponds to an edge of $L^{j-1}(G)$, and this edge will be called the 1-history of v. For $t \geq 2$ we define t-histories recursively. The t-history of v is a subgraph of $L^{j-t}(G)$, edges of which are induced by the vertices of $L^{j-t+1}(G)$ which are in $(t-1)$-history of v.

Observe that the 1-history is always an edge and the 2-history is a path of length 2. The situation is more complicated for t-histories when $t \geq 3$. In [15] we have the following lemma:

Lemma B. *Let G be a graph and let $L^j(G)$ be its j-iterated line graph. Further, let $0 \leq t \leq j$ and let H be a subgraph of $L^{j-t}(G)$. Then H is a t-history of some vertex of $L^j(G)$ if and only if H is a connected graph with at most t edges, distinct from any path with less than t edges.*

Also the next lemma, which is useful for calculating the distances in iterated line graphs, can be found in [15].

Lemma C. *Let G be a connected graph, $L^j(G)$ be its iterated line graph, and let u and v be distinct vertices of $L^j(G)$. Then for any t, $0 \leq t \leq j$, if the t-histories of u and v are edge-disjoint, then the distance between u and v in $L^j(G)$ equals the minimum distance between the two vertex sets of the t-histories in $L^{j-t}(G)$, increased by t. If the t-histories of u and v are not edge-disjoint, then the distance in $L^j(G)$ between u and v is strictly less than t.*

As can be deduced from the lemmas above, we prove our theorems for $L^j(G)$ using histories in smaller iterations. However, first we introduce two lemmas. It is worth mentioning, that their proofs (via orientations) are similar to the proof of Theorem A in [12].

Lemma 5. *Let G be a connected graph. If $\delta(G) \geq 5$, then $L(G)$ contains $\lfloor \frac{|E(L(G))|}{3} \rfloor$ edge-disjoint copies of a claw $K_{1,3}$.*

Proof. Denote $H = L(G)$. If ab is an edge of H, then its vertices correspond to pair of adjacent edges, say uv and vw, in G. In G the degree of v is at least 5, so that the edge ab lies in a copy of a complete graph K_5 in H. Hence, every edge of H lies in a copy of K_5.

To every edge of H we assign one of the two possible orientations. In such a way we obtain from H its orientation $O(H)$. Let v be a vertex of H. Denote by $o(v)$ the number of arcs in $O(H)$ terminating at v, taken modulo 3. Further, denote by $o(H)$ the number of vertices v with $o(v) > 0$, and assume that $O(H)$ is chosen so that $o(H)$ is the minimum possible. In the following we prove $o(H) \leq 1$.

Denote by $\overrightarrow{xy}$ an arc of $O(H)$ starting at x and terminating at y. We prove that for any edge $uv \in E(H)$, such that $o(u) > 0$, there exists an orientation $O'(H)$ such that $o'(u) = 0$ and $o'(w) = o(w)$ whenever w is distinct from u and v. There are four cases to distinguish:

Case 1: $o(u) = 2$ and $\overrightarrow{uv}$ is an arc in $O(H)$. Then reversing $\overrightarrow{uv}$ we receive the required orientation $O'(H)$.

Case 2: $o(u) = 2$ and $\overrightarrow{vu}$ is an arc in $O(H)$. If there is a directed path from u to v in $O(H)$, then reversing its arcs we obtain the required orientation $O'(H)$. Analogously, if there is a directed path from v to u in $O(H)$, then reversing the arcs of this path and reversing $\overrightarrow{vu}$ we receive the required orientation $O'(H)$. Hence, we can assume that there are no directed paths of these types. Since u and v lie in a copy of a complete graph K_5, there are two vertices x and y with the arc $\overrightarrow{xy}$, such that either $\overrightarrow{xu}$, $\overrightarrow{xv}$, $\overrightarrow{yu}$, $\overrightarrow{yv}$ are in $O(H)$ or $\overrightarrow{ux}$, $\overrightarrow{vx}$, $\overrightarrow{uy}$, $\overrightarrow{vy}$ are in $O(H)$. In the first case reversing $\overrightarrow{xu}$, $\overrightarrow{xv}$, $\overrightarrow{xy}$, $\overrightarrow{yu}$ we receive the required orientation $O'(H)$, while in the second one reversing $\overrightarrow{uy}$, $\overrightarrow{vy}$, $\overrightarrow{xy}$, $\overrightarrow{vx}$ we get the required orientation $O'(H)$.

Case 3: $o(u) = 1$ and $\overrightarrow{vu}$ is an arc in $O(H)$. Then reversing $\overrightarrow{vu}$ we receive the required orientation $O'(H)$.

Case 4: $o(u) = 1$ and $\overrightarrow{uv}$ is an arc in $O(H)$. Analogously as in Case 2 we can assume that there are neither directed $u - v$ paths nor directed $v - u$ paths in $O(H)$. But then there are two vertices x and y with the arc $\overrightarrow{xy}$, such that either $\overrightarrow{xu}$, $\overrightarrow{xv}$, $\overrightarrow{yu}$, $\overrightarrow{yv}$ are in $O(H)$ or $\overrightarrow{ux}$, $\overrightarrow{vx}$, $\overrightarrow{uy}$, $\overrightarrow{vy}$ are in $O(H)$. In the first case reversing $\overrightarrow{xu}$, $\overrightarrow{xv}$, $\overrightarrow{xy}$, $\overrightarrow{yv}$ we receive the required orientation $O'(H)$, while in the second one reversing $\overrightarrow{uy}$, $\overrightarrow{vy}$, $\overrightarrow{xy}$, $\overrightarrow{ux}$ we obtain the required orientation $O'(H)$.

Suppose that $o(H) \geq 2$. Then there are two vertices, say u and v, such that $o(u) > 0$ and $o(v) > 0$. Since G is a connected graph, H is connected as well, so that there is a path $u{=}x_0, x_1, \ldots, x_t{=}v$. Consider a sequence of orientations $O_0(H){=}O(H)$, $O_1(H)$, $\ldots$,$O_t(H)$, such that $o_i(x_{i-1}) = 0$ and $o_i(x) = o_{i-1}(x)$ whenever x is distinct from x_{i-1} and x_i. Then $o_t(x_0) = o_t(x_1) = \cdots = o_t(x_{t-1}) = 0$, and for all $x \notin \{x_0, x_1, \ldots, x_t\}$ we have $o_t(x) = o(x)$. Hence, $o_t(H) < o(H)$, a contradiction.

Thus, there is an orientation $O(H)$ such that $o(H) \leq 1$. That means that up to one exception, the number of arcs directed to each vertex v of $O(H)$ is a multiple of 3. Hence, these arcs can be arranged into triples to form $\lfloor \frac{|E(H)|}{3} \rfloor$ claws as required. $\square$

Lemma 6. *Let G be a connected graph, $k \geq 2$ and $\delta(G) \geq 9k - 7$. Moreover, let H be a graph obtained from $L(G)$ by deleting edges of vertex-disjoint paths, each of length at most k. If $|E(H)| \equiv 0 \pmod{k{+}1}$, then $E(H)$ can be decomposed into $\frac{|E(H)|}{k+1}$ stars $K_{1,k+1}$.*

Proof. Denote by $O(H)$ an orientation of H. Analogously as in the proof of Lemma 5, denote by $\overrightarrow{uv}$ an arc of $O(H)$ starting at u and terminating at v. Further, denote by $o(v)$ the number of arcs in $O(H)$ terminating at v, taken modulo $k{+}1$. Finally, denote by $o(H)$ the number of vertices v with $o(v) > 0$, and assume that $O(H)$ is chosen so that $o(H)$ is the minimum possible.

In the next we prove $o(H) = 0$. To do this we prove that for any edge uv of H, such that $o(u) > 0$, there is an orientation $O'(H)$ such that $o'(u) = 0$ and $o'(w) = o(w)$ whenever w is distinct from u and v.

If there are $o(u)$ edge-disjoint directed $v - u$ paths in $O(H)$, then reversing all their arcs we receive the required orientation $O'(H)$. Analogously, if there are $k{+}1 - o(u)$ edge-disjoint directed $u - v$ paths in $O(H)$, then reversing all their arcs we receive the required orientation $O'(H)$. Hence, we can assume that the number of directed paths in between u and v does not exceed $(o(u){-}1) + (k{+}1{-}o(u){-}1) = k - 1$.

Since $\delta(G) \geq 9k - 7$, in $L(G)$ the edge uv lies in a copy of a complete graph K_{9k-7}. Consider the $9k - 9$ vertices of this complete graph, distinct from u and v. Since $deg_{L(G)}(x) - deg_H(x) \leq 2$ for every vertex x of H, there are at most 4 vertices out of these $9k-9$, which are not adjacent to both u and v in H. Since there are at most $k-1$ directed paths inbetween u and v, one of them obtained from the edge uv, there are $(9k-13)-(k-2) = 8k-11$ vertices x in the complete graph, such that either $\overrightarrow{xu}, \overrightarrow{xv}$ are arcs of $O(H)$ or $\overrightarrow{ux}, \overrightarrow{vx}$ are arcs of $O(H)$. Suppose that the number of vertices x with arcs $\overrightarrow{ux}, \overrightarrow{vx}$ does not exceed the number of vertices x with arcs $\overrightarrow{xu}, \overrightarrow{xv}$. (The other case can be solved analogously.) Then there are $4k-5$ vertices x in the complete graph, such that $\overrightarrow{xu}, \overrightarrow{xv}$ are arcs of $O(H)$. However, some edges connecting these x's in $L(G)$ may be missing in H. But since any system of vertex-disjoint paths on n vertices contains a set of independent vertices of size greater than or equal to $\lceil \frac{n}{2} \rceil$, there are $2k-2$ vertices x in H which are mutually adjacent and such that $\overrightarrow{xu}, \overrightarrow{xv}$ are arcs in $O(H)$.

Denote by K the complete subgraph of H, consisting of the $2k-2$ vertices described above. Further, denote by $O(K)$ the orientation of K induced by $O(H)$. Since $(2k-2) \cdot (k-2) < \binom{2k-2}{2}$, $O(K)$ contains a vertex x_0 with the out-degree at least $k-1$. Hence, there are vertices $x_0, x_1, \ldots, x_{k-1}$, such that $\overrightarrow{x_0 x_i}$ is an arc of $O(H)$ for $1 \leq i \leq k-1$ and $\overrightarrow{x_j u}, \overrightarrow{x_j v}$ are arcs of $O(H)$ for $0 \leq j \leq k-1$. Observe that $1 \leq o(u) \leq k$. Now reverse all the arcs $\overrightarrow{x_0 x_i}$, $1 \leq i \leq k-1$, $\overrightarrow{x_0 u}, \overrightarrow{x_0 v}$, and one arc from each pair $\overrightarrow{x_i u}, \overrightarrow{x_i v}$, so that exactly $o(u)$ arcs from $\overrightarrow{x_i u}$ will be reversed. If we denote the resulting orientation by $O'(H)$, then $o'(u) = 0$ and $o'(w) = o(w)$ whenever w is distinct from u and v.

Since $k \geq 2$, we have $\delta(G) \geq 9k-7 \geq 5$, so that each edge of $L(G)$ lies in three distinct triangles. Hence, if uv is an edge of $L(G) - H$, then there is a vertex z in H such that uz and zv are edges of H. Thus, since G is a connected graph, so is $L(G)$ and consequently also H. Now proceeding analogously as at the end of the proof of Lemma 5 it can be shown that $o(H) \leq 1$. Since $|E(H)| \equiv 0 \pmod{k+1}$, the case $o(H) = 1$ is impossible, so that $o(H) = 0$. And partitioning all the arcs terminating at v into $(k+1)$-touples, we obtain the required decomposition of $E(H)$ into stars $K_{1,k+1}$. $\square$

Now we can prove our main results. However, first we prove Theorems 3 and 4, since the proof of Theorem 2 is similar to that of Theorem 4.

Proof of Theorem 3. Let u and v be two distinct vertices of $L^j(G)$. If $dist_{L^j(G)}(u, v) > k$, then the $(k+1)$-histories of u and v are edge-disjoint, by Lemma C. Since every $(k+1)$-history contains at least 3 edges by Lemma B, we have $\rho_k(L^j(G)) \leq \lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \rfloor$.

By Lemma 5, there are $\lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \rfloor$ edge-disjoint copies of a claw $K_{1,3}$ in $L^{j-k-1}(G)$. These correspond to a k-independent set of the same size in $L^j(G)$, so that $\rho_k(L^j(G)) \geq \lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \rfloor$. $\square$

In the proof of Theorem 4 we use the following result of Chartrand and Wall [1]:

Theorem D. *If G is a connected graph such that $\delta(G) \geq 3$, then $L^2(G)$ is Hamiltonian.*

Proof of Theorem 4. Let S be a set of vertices of $L^j(G)$ on which $i_k(L^j(G))$ is reached. Then the $(k+1)$-histories of vertices of S are edge-disjoint, by Lemma C. Remove the edges of these $(k+1)$-histories from $L^{j-k-1}(G)$ and denote the resulting graph by F. Since the set S is maximal, F does not contain cycles and all vertices of F have degree less than or equal to 2 by Lemma B. Hence, F consists of vertex-disjoint paths, and by Lemma B, all these paths have lengths less than $k+1$. Thus, $|E(F)| \leq \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor$. As a $(k+1)$-history of a vertex contains at most $k+1$ edges, we have

$$i_k(L^j(G)) \geq \frac{|E(L^{j-k-1}(G))| - \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor}{k+1}.$$

On the other hand, $L^{j-k-1}(G)$ contains a Hamiltonian cycle, by Theorem D. Deleting $\left\lceil \frac{1}{k+1} |V(L^{j-k-1}(G))| \right\rceil$ edges from this cycle we obtain a graph F^* consisting of paths of lengths less than $k+1$. Obviously, $|E(F^*)| = \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor$. Let t be a number, $0 \leq t < k+1$, such that

$$\left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor - |E(L^{j-k-1}(G))| \equiv t \pmod{k+1}.$$

Delete from F^* exactly t edges and denote the resulting graph by F. Then $|E(L^{j-k-1}(G))| - |E(F)| \equiv 0 \pmod{k+1}$, so that $L^{j-k-1}(G) - E(F)$ can be decomposed into $\frac{1}{k+1}\big(|E(L^{j-k-1}(G))| - |E(F)|\big)$ stars $K_{1,k+1}$, by Lemma 6. These correspond to a k-independent dominating set of the same size in $L^j(G)$, so that

$$i_k(L^j(G)) \leq \frac{|E(L^{j-k-1}(G))| - |E(F)|}{k+1} =$$
$$\left\lceil \frac{|E(L^{j-k-1}(G))| - \left\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \right\rfloor}{k+1} \right\rceil.$$

Combining the two inequalities for $i_k(L^j(G))$ we get the result. $\square$

Finally, we prove the result concerning $i(L^j(G))$.

Proof of Theorem 2. Substituting 1 for k, analogously as in the proof of Theorem 4 we get

$$i_1(L^j(G)) \geq \frac{|E(L^{j-2}(G))| - \left\lfloor \frac{1}{2} |V(L^{j-2}(G))| \right\rfloor}{2}.$$

By Proposition 1, $\beta(L^{j-1}(G))=\lfloor\frac{|E(L^{j-3}(G))|}{2}\rfloor=\lfloor\frac{|V(L^{j-2}(G))|}{2}\rfloor$. Choose one set of $\beta(L^{j-1}(G))$ independent vertices of $L^{j-1}(G)$, and denote by F^* the edges of $L^{j-2}(G)$ which are 1-histories of vertices of the chosen set. Then F^* is a collection of independent edges. Observe that $|E(F^*)| = \lfloor\frac{1}{2}|V(L^{j-2}(G))|\rfloor$. Now let t be a number, $0 \le t < 2$, such that

$$\lfloor\tfrac{1}{2}|V(L^{j-2}(G))|\rfloor - |E(L^{j-2}(G))| \equiv t \pmod 2.$$

Delete from F^* exactly t edges and denote the resulting graph by F. Since $\delta(L^{j-3}(G)) \ge 3$, each edge of F lies in a triangle in $L^{j-2}(G)$. Hence, $L^{j-2}(G) - E(F)$ is a connected graph with even number of edges. By Theorem A its edges can be decomposed into paths of length 2, so that

$$i_1(L^j(G)) \le \frac{|E(L^{j-2}(G))| - |E(F)|}{2} = \left\lceil\frac{|E(L^{j-2}(G))| - \lfloor\frac{1}{2}|V(L^{j-2}(G))|\rfloor}{2}\right\rceil. \quad \square$$

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