CONNECTIVITY OF ITERATED LINE GRAPHS.

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Abstract. In the paper we present lower bounds for the connectivity of the i-iterated line graph $L^i(G)$ of a graph G. We prove that if G is a connected regular graph and $i \geq 5$, then the connectivity of $L^i(G)$ is equal to the degree of $L^i(G)$, that is, the connectivity of $L^i(G)$ attains its theoretical maximum (we remark that the bound on i is best possible). Moreover, if a hypothesis on the growth of the minimum degree of the i-iterated line graph is true, then an analogous result is true for an arbitrary graph G if i is sufficiently large.

1. Introduction and Results

In recent years, the investigation of iterated line graphs has recorded a large progress. These graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0, \end{cases}$$

where L is the line graph operator. The diameter and radius of iterated line graphs are examined in [7], and [6] is devoted to the centers of these graphs. In [5], Hartke and Higgins study the growth of the maximum degree of iterated line graphs, and very recently, in [8] Xiong and Liu characterize the graphs whose i-iterated line graphs are Hamiltonian.

Even larger emphasis is devoted to iterated line digraphs, as they are well-suited for designing of interconnection networks. In [3], Fàbrega and Fiol prove that the

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(strong) connectivity of the i-iterated line digraph is equal to its minimum degree if i is sufficiently large. In fact, the result is not surprising as the minimum degree of the iterated line digraph is equal to the minimum degree of the original digraph. This is not true in the case of graphs. The minimum degree of the i-iterated line graph grows exponentially as a function of i. Nevertheless, we prove here an analogue of the above result for graphs.

Let G be a connected graph. By $\kappa(G)$ and $\lambda(G)$ we denote its vertex-connectivity and its edge-connectivity, respectively. The minimum degree of G is denoted by $\delta(G)$. It is well-known, see e.g. [4], that for every graph G we have

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Moreover, for every triple of positive integers κ, λ and δ, $\kappa \leq \lambda \leq \delta$, there is a graph $G_{\kappa,\lambda,\delta}$ such that $\kappa(G_{\kappa,\lambda,\delta}) = \kappa$, $\lambda(G_{\kappa,\lambda,\delta}) = \lambda$ and $\delta(G_{\kappa,\lambda,\delta}) = \delta$, see [1]. For the vertex-connectivity of line graph we have

$$\lambda(G) \leq \kappa(L(G)) \leq \delta(L(G)),$$

see [2]. However, the graphs $G_{\kappa,\lambda,\delta}$ of [1] satisfy also $\kappa(L(G_{\kappa,\lambda,\delta})) = \lambda$. It means that in the extremal case, for every $\delta \geq 1$ there exists a graph G such that $\kappa(G) = \kappa(L(G)) = 1$ and $\delta(G) = \delta$. It is interesting that this property cannot be extended to iterated line graphs. We have:

Theorem 1. *Let G be a connected graph with the minimum degree $\delta \geq 3$. Then $\kappa(L^2(G)) \geq \delta - 1$.*

Theorem 1 is the best possible in a sense. Let G be a connected graph with a bridge uv, such that the degrees of both u and v are equal to $\delta = \delta(G)$, see Figure 1. Denote by G_u and G_v the components of $G - \{uv\}$ containing u and v, respectively. Then it is easy to see that for every pair of edges e_u of $L(G_u)$ and e_v of $L(G_v)$, there are at most $\delta-1$ edge-disjoint paths joining the vertices of e_u with the vertices of e_v in $L(G)$, see Figure 2. Thus $\kappa(L^2(G)) \leq \delta-1$.

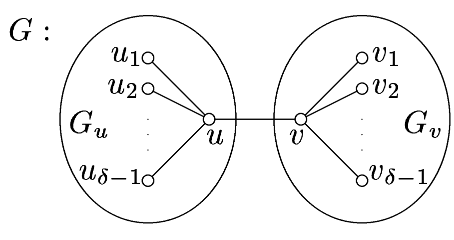

Figure 1

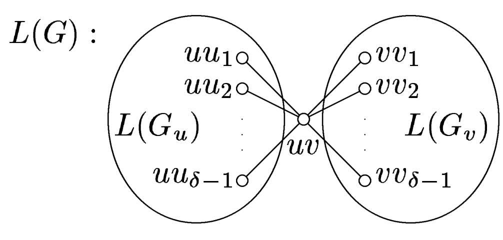

Figure 2

Let G be a graph with $\delta(G) \geq 3$. Then

$$\delta(L^i(G)) \geq 2^i(\delta(G) - 2) + 2 \tag{1}$$

for every $i \geq 0$, see [7]. Since $\kappa(L^{i+1}(G)) \geq \kappa(L^i(G))$, Theorem 1 shows that $\kappa(L^i(G))$ grows exponentially as a function of i.

Since $\delta(L^2(G)) \geq 4\delta(G)-6$ by (1), there is a gap between $\delta(L^2(G))$ and $\kappa(L^2(G))$ in Theorem 1. This gap means that the theorem guarantees the vertex-connectivity of $L^i(G)$ only (roughly speaking) at most one fourth of the minimum degree of $L^i(G)$. The following theorem gives a better bound for graphs with large vertex-connectivity.

Theorem 2. *Let G be a graph with $\kappa(G) \geq 4$. Then $\kappa(L^2(G)) \geq 4\delta(G) - 6$.*

In [7] we present a conjecture that for every connected graph G, different from a path, a cycle and a claw $K_{1,3}$, there is an integer i_G such that for every i, $i \geq i_G$,

$$\delta(L^{i+1}(G)) = 2 \cdot \delta(L^i(G)) - 2.$$

If the conjecture is true (and we remark that it is true at least for regular graphs), then for every graph G, different from a path, a cycle and a claw, there is j_G such that for every j, $j \geq j_G$, we have

$$\kappa(L^j(G)) = \delta(L^j(G)),$$

by Theorem 1 and 2. It means that the vertex-connectivity of iterated line graphs attains its maximum. Here we have to point out that, recently Hartke and Higgins proved an analogue of the conjecture for the maximum degree of iterated line graphs, see [5].

We remark that in the pioneering work on the connectivity of iterated line graphs [2] Chartrand and Stewart give a lower bound on the connectivity of $L^i(G)$ in terms of the connectivity of G. They proved

$$\kappa(L^i(G)) \geq 2^{i-1}(\kappa(G) - 2) + 2.$$

However, by (1) and Theorem 2 we have

$$\kappa(L^i(G)) \geq 2^i(\delta(G) - 2) + 2$$

for $\kappa(G) \geq 4$ and $i \geq 2$.

Let G be a connected graph with $\delta(G) \geq 3$. Then $\delta(L^2(G)) \geq 5$ and hence, $\kappa(L^4(G)) \geq 4$, by Theorem 1. The next theorem improves this observation.

Theorem 3. *Let G be a connected graph with $\delta(G) \geq 3$. We have:*

(i) *If $\lambda(G) \geq 2$ or $\delta(G) \geq 5$, then $\kappa(L^2(G)) \geq 4$;*
(ii) *if $\lambda(G) = 1$ and $3 \leq \delta(G) \leq 4$, then $\kappa(L^3(G)) \geq 4$.*

By the example below Theorem 1, Theorem 3 is the best possible, i.e., it determines the least iteration of the line graph such that the vertex-connectivity is at least 4.

Let G be a connected graph different from a path, a cycle and a claw. Assume that $\delta(G) < 3$. Denote by d_1 (d_2; d_3) the length of a longest path in G, interior vertices of which have degrees 2, and the endvertices have degrees 1 and 3 (the endvertices of which have degrees 1 and a degree larger than 3; both endvertices have degrees at least 3). Let $k_G = \max\{d_1+1, d_2, d_3-1\}$. It is easy to see that $\delta(L^k(G)) < 3$ if $k < k_G$, and $\delta(L^{k_G}(G)) \geq 3$, see e.g. [7, Lemma 7].

Now we are able to summarize the evolution of $\kappa(L^i(G))$. Let G be a connected graph, different from a path, a cycle and a claw. If $\delta(G) < 3$ then $\kappa(G) < 3$ as well. Denote $H = L^{k_G}(G)$ if $\delta(G) < 3$, and $H = G$ if $\delta(G) \geq 3$. Then $\delta(H) \geq 3$; $\kappa(L^2(H)) \geq 2$ by Theorem 1; and $\kappa(L^3(H)) \geq 4$ by Theorem 3. (However, if $\lambda(H) \geq 2$ or if $\delta(H) \geq 5$, then even $\kappa(L^2(H)) \geq 4$.) For every $i \geq 5$ we have $\kappa(L^i(H)) \geq 4\delta(L^{i-2}(H)) - 6$ by Theorem 2, and this result is the best possible at least for regular graphs.

Finally, we remark that there is a regular graph G with $\delta(G) = 3$, such that $\kappa(L^i(G)) < \delta(L^i(G))$ if $i < 5$, and $\kappa(L^i(G)) = \delta(L^i(G))$ if $i \geq 5$. Just take a tree with a special vertex u, in which all vertices that are "not far" from u have degree 3, and the remaining vertices have degree 1. Now glue to the tree plenty of copies of K_4, each by the "middle of an edge" to one endvertex of the tree. The resulting graph G is regular of degree 3, and representing the vertices of $L^4(G)$ in $L^2(G)$ one can show that $\kappa(L^4(G)) < 18 = \delta(L^4(G))$.

All proofs and necessary notions are postponed to the next section.

2. Proofs

Throughout the paper we use the following definition of vertex-connectivity:

Definition. A graph G is k**-vertex-connected** (or simply k-connected) if it has at least $k+1$ vertices, and if for every pair u and v of non-adjacent vertices of G there are at least k internally-vertex-disjoint u–v paths in G. The **vertex-connectivity** $\kappa(G)$ is the maximum value of k such that G is a k-vertex-connected graph.

There are several definitions equivalent with the one presented here, see e.g. [4]. We use two of them:

(i) If G is a k-connected graph, then for every pair of sets of its vertices U and V such that $|U| = |V| = l \leq k$, there are l vertex-disjoint paths connecting the vertices of U with the vertices in V.

(ii) If G is a k-connected graph, then excluding $l < k$ elements of G (some of them are vertices and the other are edges) will result in a $(k-l)$-connected graph.

Let G be a graph and let u be a vertex in $L^2(G)$. By 2**-history** of u we mean the smallest subgraph U of G, such that $L^2(U)$ contains the vertex u. It is easy to see that 2-history is always a path of length two, and in fact there is a one-to-one correspondence between the vertices of $L^2(G)$ and the paths of length two in G. (We remark that interesting properties of i-histories can be found in [7].)

For simplifying the notation we adopt the following convention. We denote the vertices of $L^2(G)$ (as well as the vertices of G) by small letters u, v, ..., while their 2-histories (or simply histories) will be denoted by capital letters U, V, etc. It means that if U is a history and u is a vertex in $L^2(G)$, then U is the 2-history corresponding to the vertex u. Further, we denote a history (i.e., a path of length two in G) as a triple of vertices in parentheses, say $U = (u_0, u_1, u_2)$, where the middle vertex (u_1 in this case) has degree 2 in U. However, to distinguish the histories from other paths in G we denote the paths without parentheses; i.e., by $P = v_1, v_2, v_3$ we denote a v_1–v_3 path of length two. This enables us to write an extension of P, by v_0 in the beginning and by v_4 at the end, as v_0, P, v_4.

Observe that two distinct vertices, say u and v, in $L^2(G)$ are adjacent if and only if U and V share an edge in common. Let $P = z_0, z_1, \ldots, z_k$ be a path in G. A path $w_0, w_1, \ldots, w_{k'}$ in $L^2(G)$ is called a P**-based path** if for every i, $0 \leq i \leq k'$, W_i contains an edge of P.

Lemma 4. *Let δ be the minimum degree of a graph G, and let $P = z_0, z_1, \ldots, z_k$, $k \geq 2$, be a path in G. Then there are $\delta-1$ vertex-disjoint P-based paths in $L^2(G)$, $P_1, P_2, \ldots, P_{\delta-1}$ with $P_i = w_{i,0}, w_{i,1}, \ldots, w_{i,k_i}$, such that $W_{i,0}$ contains the edge z_0z_1 and W_{i,k_i} contains the edge $z_{k-1}z_k$, $1 \leq i \leq \delta-1$.*

Proof. We construct paths of two types: those whose vertices' histories are contained completely in P, and those who are not. Let $W_{1,j} = (z_j, z_{j+1}, z_{j+2})$, $0 \le j \le k-2$. Then P_1 is a unique "straight" P-based path among $P_1, P_2, \ldots, P_{\delta-1}$.

Denote by $x_{2,j}, x_{3,j}, \ldots, x_{\delta-1,j}$ $\delta-2$ vertices of G that are adjacent to z_j and distinct from z_{j-1} and z_{j+1}, $1 \le j \le k-1$. (Recall that the minimum degree of G is δ.) For $2 \le i \le \delta-1$ and $0 \le j \le 2(k-2)+1$ let

$$W_{i,j} = \begin{cases} (z_{\lfloor j/2 \rfloor}, z_{\lfloor j/2 \rfloor+1}, x_{i,\lfloor j/2 \rfloor+1}), & \text{if } j \text{ is even}, \\ (x_{i,\lfloor j/2 \rfloor+1}, z_{\lfloor j/2 \rfloor+1}, z_{\lfloor j/2 \rfloor+2}), & \text{if } j \text{ is odd}. \end{cases}$$

Then $P_i = w_{i,0}, w_{i,1}, \ldots, w_{i,2(k-2)+1}$ is a P-based path of the second type, and the paths $P_1, P_2, \ldots, P_{\delta-1}$ are vertex-disjoint. □

Observe that the paths $P_1, P_2, \ldots, P_{\delta-1}$ in the proof of Lemma 4 are constructed so that for every i, $1 \le i \le \delta-1$, exactly two histories of $W_{i,0}, W_{i,1}, \ldots, W_{i,k_i}$ contain the edge $z_j z_{j+1}$ if $1 \le j < k-1$; exactly one of them contains the edge $z_0 z_1$; and exactly one contains the edge $z_{k-1} z_k$.

Proof of Theorem 1. Since $\delta(G) \ge 3$, $L^2(G)$ contains two non-adjacent vertices. Let u and v be non-adjacent vertices in $L^2(G)$, $U = (u_0, u_1, u_2)$ and $V = (v_0, v_1, v_2)$. Then U and V are edge-disjoint. Denote by P' a shortest path in G joining a vertex of U with a vertex of V. Let P be a path containing P', with exactly two edges outside P', namely one edge of U and one edge of V. (The edges of U and V are the first and the last edge of P, respectively.) Clearly, the length of P is at least 2. Now denote by $P_1, P_2, \ldots, P_{\delta-1}$ the $\delta-1$ vertex-disjoint P-based paths in $L^2(G)$, guaranteed by Lemma 4. Let $1 \le i \le \delta-1$. Since the history of the first vertex of P_i contains an edge of U, this vertex is either u or it is adjacent to u. Analogously, the terminal vertex of P_i is either v or it is adjacent to v. Hence, $P_1, P_2, \ldots, P_{\delta-1}$ can be extended to $\delta-1$ internally-vertex-disjoint u–v paths. □

Since a history cannot contain a pair of non-adjacent edges, we have the following observation:

Observation 5. *Let P and P' be vertex-disjoint paths in a graph G. Then every pair of a P-based path and a P'-based one, forms a pair of vertex-disjoint paths in $L^2(G)$.*

In fact, a stronger statement is true. Let G be a graph with the minimum degree δ, and let a_1a_2 and b_1b_2 be edges of G. Further, let P_1' and P_2' be vertex-disjoint a_1–b_1 and a_2–b_2 paths, respectively. Denote $P_1 = a_2, P_1', b_2$ and $P_2 = a_1, P_2', b_1$. Then P_1 and P_2 share two edges in common, namely a_1a_2 and b_1b_2. However, the $\delta-1$ P_1-based paths and the $\delta-1$ P_2-based ones, constructed in the proof of Lemma 4, form a collection of $2\delta-2$ vertex-disjoint paths in $L^2(G)$.

Let δ be the minimum degree in a graph G, and let u be a vertex in $L^2(G)$, $U = (u_0, u_1, u_2)$. Then the degree of u is at least $4\delta - 6$. Moreover,

(i) $\delta-1$ neighbours of u have history of a form (x, u_0, u_1) for x adjacent to u_0, $x \ne u_1$;
(ii) $\delta-2$ neighbours of u have history (u_0, u_1, x), $x \ne u_2$;
(iii) $\delta-2$ neighbours of u have history (x, u_1, u_2), $x \ne u_0$;
(iv) $\delta-1$ neighbours of u have history (u_1, u_2, x).

In the proof of Theorem 2 we have to use all $4\delta-6$ neighbours of u described above. We divide the proof into three lemmas.

Lemma 6. *Let G be a graph with $\kappa(G) \geq 4$, and let δ be the minimum degree of G. Further, let u and v be vertices in $L^2(G)$, $U = (u_0, u_1, u_2)$ and $V = (v_0, v_1, v_2)$. If the distance in G between u_1 and v_1 is greater than or equal to 2, then there are $4\delta - 6$ internally-vertex-disjoint u–v paths in $L^2(G)$.*

Proof. Let x_1 be a neighbour of u_1 in G, $x_1 \notin \{u_0, u_2\}$, and let y_1 be a neighbour of v_1, $y_1 \notin \{v_0, v_2\}$. As $\kappa(G) \geq 4$, there are four vertex-disjoint paths connecting $\{u_0, u_1, u_2, x_1\}$ with $\{v_0, v_1, v_2, y_1\}$. Extending these paths to u_1 in the beginning and to v_1 at the end, we obtain a collection of four internally-vertex-disjoint u_1–v_1 paths P_1', P_2', P_3' and P_4'. Moreover, a set of vertices adjacent to u_1 in these paths is $\{u_0, u_2, x_1, x_2\}$ for a neighbour x_2 of u_1, and a set of vertices adjacent to v_1 in these paths is $\{v_0, v_2, y_1, y_2\}$ for a neighbour y_2 of v_1. Up to symmetry, there are three cases to distinguish:

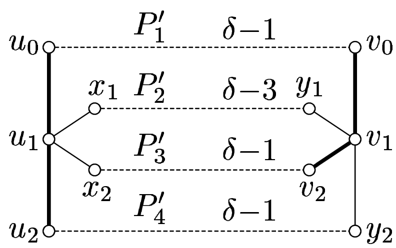

Figure 3

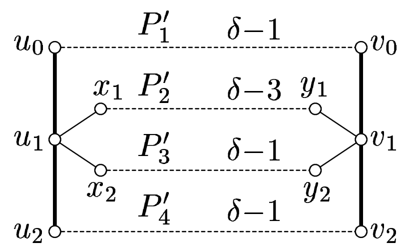

Figure 4

(1) $P_1' = u_1, u_0, \ldots, v_0, v_1$, $P_2' = u_1, x_1, \ldots, y_1, v_1$, $P_3' = u_1, x_2, \ldots, v_2, v_1$ and $P_4' = u_1, u_2, \ldots, y_2, v_1$, see Figure 3 (the edges of U and V are depicted by thick lines). We extend the paths $P_1', \ldots, P_4'$ to $P_1, \ldots, P_4$, so that the first vertices of all P_i-based paths are adjacent to u, and the last vertices of all of them are adjacent to v, $i \leq i \leq 4$; i.e., we set $P_1 = P_1'$; $P_2 = u_0, P_2', v_0$; $P_3 = u_2, P_3'$ and $P_4 = P_4', v_2$. Denote by $\mathcal{P}_i$ the set of $\delta-1$ P_i-based paths guaranteed by Lemma 4, $1 \leq i \leq 4$. By Observation 5, if we delete the first two and also the last two vertices from all paths of $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_4$, we receive a collection of vertex-disjoint paths. However, by the note preceding Lemma 6, the set of $4\delta - 4$ paths $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_4$ is not necessarily a set of vertex-disjoint paths. Two paths of $\mathcal{P}_2$ may contain vertices of two paths of $\mathcal{P}_3$ (namely the vertices with histories (u_0, u_1, x_2) and (u_2, u_1, x_1)); and two paths of $\mathcal{P}_2$ may contain vertices of two paths of $\mathcal{P}_4$ (the vertices with histories (y_1, v_1, v_2) and (y_2, v_1, v_0)). For this reason, denote by P_2^1 and P_2^2 two special P_2-based paths of $\mathcal{P}_2$. Let P_2^1 be an s_1–t_1 path, $S_1 = (u_0, u_1, u_2)$ and $T_1 = (y_2, v_1, v_0)$, and let P_2^2 be an s_2–t_2 path, $S_2 = (u_0, u_1, x_2)$ and $T_2 = (v_2, v_1, v_0)$. (Clearly, the set $\mathcal{P}_2$ can be chosen so that it contains P_2^1 and P_2^2.) Then the vertices of the paths in $\mathcal{P}_1 \cup (\mathcal{P}_2 - \{P_2^1, P_2^1\}) \cup \mathcal{P}_3 \cup \mathcal{P}_4$ are mutually distinct, and hence, these paths can be extended to $4\delta - 6$ internally-vertex-disjoint u–v paths.

(2) $P_1' = u_1, u_0, \ldots, v_0, v_1$, $P_2' = u_1, x_1, \ldots, y_1, v_1$, $P_3' = u_1, x_2, \ldots, y_2, v_1$ and $P_4' = u_1, u_2, \ldots, v_2, v_1$, see Figure 4. This case can be solved analogously as the previous one. The number of P_i-based paths is indicated in the picture, $1 \leq i \leq 4$.

(3) $P_1' = u_1, u_0, \ldots, y_1, v_1$, $P_2' = u_1, x_1, \ldots, v_0, v_1$, $P_3' = u_1, x_2, \ldots, v_2, v_1$ and $P_4' = u_1, u_2, \ldots, y_2, v_1$, see Figure 5. Let $P_1 = P_1', v_0$; $P_2 = u_0, P_2'$; $P_3 = u_2, P_3'$ and $P_4 = P_4', v_2$. Denote by $\mathcal{P}_i$ the set of $\delta-1$ P_i-based paths guaranteed by Lemma 4, $1 \leq i \leq 4$. Analogously as in the case (1), the first vertices of all paths in $\mathcal{P}_i$ are adjacent to u, and the last vertices of all of them are adjacent to v. However, the paths of $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_4$ are not necessarily vertex-disjoint. In this case we delete four paths of $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_4$ and we add two different ones, to obtain a collection of $4\delta - 6$ vertex-disjoint paths.

Let Q_1 be the set of internal vertices of P_1' and P_4', and let Q_2 be the set of internal vertices of P_2' and P_3'. Since $\kappa(G) \geq 4$, there is a path P_0 in $G - \{u_1, v_1\}$, that connects a vertex of Q_1 with a vertex of Q_2. Assume that P_0 is a $z_0 - w_0$ path, where z_0 is a vertex of P_1' and w_0 is a vertex of P_2', see Figure 5. Denote by P_j^1 and P_j^2 two special P_j-based paths of $\mathcal{P}_j$, $1 \leq j \leq 2$:

(i) P_1^1 is an $a_1 - b_1$ P_1-based path containing e_1, $A_1 = (u_1, u_0, x_3)$, $B_1 = (v_2, v_1, v_0)$ and $E_1 = (z_1, z_0, z_2)$.
(ii) P_1^2 is an $a_2 - b_2$ P_1-based path containing e_2, $A_2 = (u_1, u_0, x_4)$, $B_2 = (y_2, v_1, v_0)$ and $E_2 = (z_1, z_0, z_3)$.
(iii) P_2^1 is a $c_1 - d_1$ P_2-based path containing f_1, $C_1 = (u_0, u_1, u_2)$, $D_1 = (y_3, v_0, v_1)$ and $F_1 = (w_2, w_0, w_1)$.
(iv) P_2^2 is a $c_2 - d_2$ P_2-based path containing f_2, $C_2 = (u_0, u_1, x_2)$, $D_2 = (y_4, v_0, v_1)$ and $F_2 = (w_3, w_0, w_1)$.

Clearly, the sets $\mathcal{P}_1$ and $\mathcal{P}_2$ can be chosen so that they contain P_1^1, P_1^2, P_2^1 and P_2^2. Using P_0 we construct two new paths in $L^2(G)$.

(i) P_0^1 begins with the $a_1 - e_1$ subpath of P_1^1, then it contains a P_0-based path (one among the $\delta - 1$ guaranteed by Lemma 4), and it terminates with the $f_1 - d_1$ subpath of P_2^1. (We remark that if the length of P_0 is one, we do not include a P_0-based path into P_0^1.)
(ii) P_0^2 begins with the $a_2 - e_2$ subpath of P_1^2, then it contains e_3, P_0-based path (disjoint from the one used in P_0^1), f_3, and it terminates with the $f_2 - d_2$ subpath of P_2^2, $E_3 = (z_3, z_0, z_2)$ and $F_3 = (w_2, w_0, w_3)$.

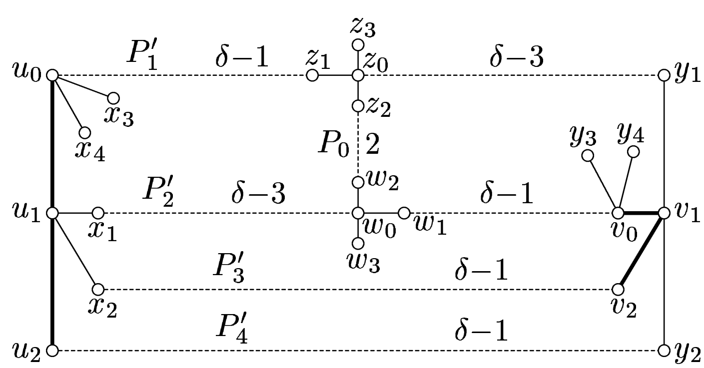

Figure 5

Now the vertices of the paths in

$$(\mathcal{P}_1 - \{P_1^1, P_1^2\}) \cup (\mathcal{P}_2 - \{P_2^1, P_2^2\}) \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \{P_0^1, P_0^2\}$$

are mutually distinct (by Observation 5, it is enough to check the first two and the last two vertices of these paths), and hence, they can be extended to $4\delta - 6$ internally-vertex-disjoint $u - v$ paths. $\square$

Lemma 7. *Let G be a graph with $\kappa(G) \geq 4$, and let δ be the minimum degree of G. Further, let u and v be non-adjacent vertices in $L^2(G)$, $U = (u_0, u_1, u_2)$ and $V = (v_0, v_1, v_2)$. If $u_1 v_1$ is an edge of G, then there are $4\delta - 6$ internally-vertex-disjoint $u - v$ paths in $L^2(G)$.*

Proof. The proof of Lemma 7 is analogous to that of Lemma 6, if the edge u_1v_1 is neither in U nor in V. Hence, suppose that $u_2 = v_1$. Since u and v are non-adjacent vertices, we have $u_1 \notin \{v_0, v_2\}$. Denote by x_1 a neighbour of u_1, $x_1 \notin \{u_0, u_2\}$. As $\kappa(G) \geq 4$, there are three vertex-disjoint paths connecting $\{u_0, u_1, x_1\}$ with $\{v_0, v_1, v_2\}$ in $G - \{u_1v_1\}$. Extending these paths to u_1 in the beginning and to v_1 at the end, we obtain (together with the u_1–v_1 path of length 1) a collection of four internally-vertex-disjoint paths P_1', P_2', P_3' and P_4' in G. Up to symmetry, there are two cases to distinguish:

(1) $P_1' = u_1, u_0, \ldots, v_0, v_1$, $P_2' = u_1, x_2, \ldots, y_1, v_1$, $P_3' = u_1, v_1$ and $P_4' = u_1, x_1, \ldots, v_2, v_1$, with $x_2 \notin \{u_0, u_2, x_1\}$ and $y_1 \notin \{v_0, v_2, u_1\}$, see Figure 6. Let $P_1 = P_1'$; $P_2 = u_0, P_2', v_0$; $P_3 = P_3', v_2$ and $P_4 = u_2, P_4'$. Denote by $\mathcal{P}_i$ the set of $\delta{-}1$ P_i-based paths guaranteed by Lemma 4, $1 \leq i \leq 4$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$. Omitting two paths of $\mathcal{P}_2$ (analogously as in the case (1) in the proof of Lemma 6) one can reduce $\mathcal{P}$ to a collection of $4\delta - 6$ vertex-disjoint paths, by Observation 5, and these paths can be extended to internally-vertex-disjoint u–v paths.

(2) $P_1' = u_1, u_0, \ldots, y_1, v_1$, $P_2' = u_1, x_2, \ldots, v_0, v_1$, $P_3' = u_1, v_1$ and $P_4' = u_1, x_1, \ldots, v_2, v_1$, with $x_2 \notin \{u_0, u_2, x_1\}$ and $y_1 \notin \{v_0, v_2, u_1\}$, see Figure 7. Let $P_1 = P_1', v_0$; $P_2 = u_0, P_2'$; $P_3 = P_3', v_2$ and $P_4 = u_2, P_4'$. Since $\kappa(G) \geq 4$, there is a path P_0 in $G - \{u_1, v_1\}$ joining an internal vertex of P_1' with an internal vertex of P_2' or P_4'. Assume that P_0 is a z_0–w_0 path, where z_0 is a vertex of P_1' and w_0 is a vertex of P_2'. Then analogously as in the case (3) in the proof of Lemma 6, one can construct $4\delta - 6$ internally-vertex-disjoint u–v paths in $L^2(G)$. □

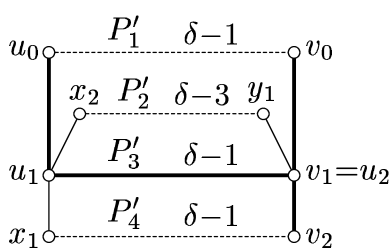

Figure 6

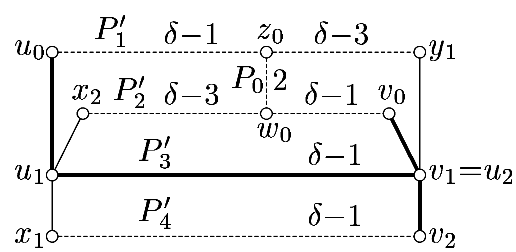

Figure 7

Lemma 8. *Let G be a graph with $\kappa(G) \geq 4$, and let δ be the minimum degree of G. Further, let u and v be non-adjacent vertices in $L^2(G)$, $U = (u_0, u_1, u_2)$ and $V = (v_0, v_1, v_2)$. If $u_1 = v_1$, then there are $4\delta - 6$ internally-vertex-disjoint u–v paths in $L^2(G)$.*

Proof. Since u and v are non-adjacent vertices, $\{u_0, u_2\} \cap \{v_0, v_2\} = \emptyset$, see Figure 8. As $\kappa(G) \geq 4$, there are two vertex-disjoint paths connecting $\{u_0, u_2\}$ with $\{v_0, v_2\}$ in $G - \{u_1\}$. Extending these paths to u_1 and v_1 we obtain two walks P_1 and P_2 in G. Assume that $P_1 = u_1, u_0, \ldots, v_0, v_1$ and $P_2 = u_1, u_2, \ldots, v_2, v_1$. Let $P_3 = u_0, u_1, v_0$ and $P_4 = u_2, u_1, v_2$. Denote by $\mathcal{P}_i$ the set of $\delta{-}1$ P_i-based paths guaranteed by Lemma 4, $1 \leq i \leq 4$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$. Omitting two paths of $\mathcal{P}_2$ (analogously as in the case (1) in the proof of Lemma 6) one can reduce $\mathcal{P}$ to a collection of $4\delta - 6$ vertex-disjoint paths, by Observation 5, and these paths can be extended to internally-vertex-disjoint u–v paths. □

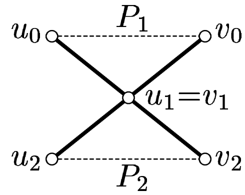

Figure 8

Now Theorem 2 is a straightforward consequence of Lemmas 6, 7 and 8.

Lemma 9. *Let G be a graph with $\lambda(G) \geq 2$ and $\delta(G) \geq 3$. Then $\kappa(L^2(G)) \geq 4$.*

Proof. Let u and v be non-adjacent vertices in $L^2(G)$, $U = (u_0, u_1, u_2)$ and $V = (v_0, v_1, v_2)$. Let P be a $u_1{-}v_1$ path in G and let x and y be vertices of P. By $x \preceq^P y$ we denote that x is a vertex of the $u_1{-}y$ subpath of P.

At first suppose that the distance in G from u_1 to v_1 is at least 2, see Figure 9. Let P_1 and P_2 be two edge-disjoint $u_1{-}v_1$ paths, such that their union does not contain a union of two edge-disjoint $u_1{-}v_1$ paths as a proper subgraph. Suppose that x and y are vertices of both P_1 and P_2, $x \neq y$, such that $x \preceq^{P_1} y$ and $y \preceq^{P_2} x$. Denote by P_1' a path composed of the $u_1{-}x$ subpath of P_1 and the $x{-}v_1$ subpath of P_2. Analogously, denote by P_2' a path composed of the $u_1{-}y$ subpath of P_2 and the $y{-}v_1$ subpath of P_1. Then P_1' and P_2' are edge-disjoint $u_1{-}v_1$ paths, and their union is a proper subgraph of $P_1 \cup P_2$, which contradicts the choice of P_1 and P_2. Hence, for the vertices in the intersection of P_1 and P_2 we have $x \preceq^{P_1} y$ if and only if $x \preceq^{P_2} y$.

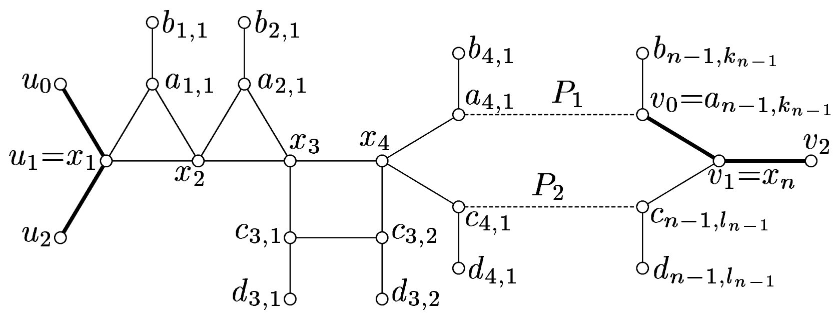

Figure 9

Let $x_1, x_2, \ldots, x_n$ be the vertices of $P_1 \cap P_2$. Assume that $x_1 \preceq^{P_1} x_2 \preceq^{P_1} \cdots \preceq^{P_1} x_n$. Clearly, $x_1 = u_1$ and $x_n = v_1$. Let

$$\begin{aligned} P_1 &= x_1, a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, x_2, \quad \ldots, \quad x_{n-1}, a_{n-1,1}, a_{n-1,2}, \ldots, a_{n-1,k_{n-1}}, x_n; \\ P_2 &= x_1, c_{1,1}, c_{1,2}, \ldots, c_{1,l_1}, x_2, \quad \ldots, \quad x_{n-1}, c_{n-1,1}, c_{n-1,2}, \ldots, c_{n-1,l_{n-1}}, x_n. \end{aligned}$$

As $\delta(G) \geq 3$, there is an edge incident to $a_{i,j}$, say $a_{i,j}b_{i,j}$, which is not in P_1, $1 \leq i < n$ and $1 \leq j \leq k_i$. Analogously, there is an edge, say $c_{i,j}d_{i,j}$, lying outside P_2, $1 \leq i < n$ and $1 \leq j \leq l_i$. Now we construct four vertex-disjoint paths P_1^*, P_2^*, P_3^* and P_4^* in $L^2(G)$. In what follows, histories of vertices of these paths are listed:

P_1^* : $(x_1, a_{1,1}, a_{1,2})$, $(a_{1,1}, a_{1,2}, a_{1,3})$, $\ldots$, $(a_{1,k_1-1}, a_{1,k_1}, x_2)$, $(a_{1,k_1}, x_2, a_{2,1})$, $(x_2, a_{2,1}, a_{2,2})$, $\ldots$, $(a_{n-1,k_{n-1}-1}, a_{n-1,k_{n-1}}, x_n)$;

P_2^* : $(x_1, c_{1,1}, c_{1,2})$, $(c_{1,1}, c_{1,2}, c_{1,3})$, $\ldots$, $(c_{1,l_1-1}, c_{1,l_1}, x_2)$, $(c_{1,l_1}, x_2, c_{2,1})$, $(x_2, c_{2,1}, c_{2,2})$, $\ldots$, $(c_{n-1,l_{n-1}-1}, c_{n-1,l_{n-1}}, x_n)$;

$P_3^*: (x_1, a_{1,1}, b_{1,1}),\quad (b_{1,1}, a_{1,1}, a_{1,2}),\quad (a_{1,1}, a_{1,2}, b_{1,2}),\quad \ldots,\quad (b_{1,k_1}, a_{1,k_1}, x_2),$
$(a_{1,k_1}, x_2, c_{2,1}),\ (x_2, c_{2,1}, d_{2,1}),\ (d_{2,1}, c_{2,1}, c_{2,2}),\ \ldots\ ;$
$P_4^*: (x_1, c_{1,1}, d_{1,1}),\quad (d_{1,1}, c_{1,1}, c_{1,2}),\quad (c_{1,1}, c_{1,2}, d_{1,2}),\quad \ldots,\quad (d_{1,k_1}, c_{1,k_1}, x_2),$
$(c_{1,k_1}, x_2, a_{2,1}),\ (x_2, a_{2,1}, b_{2,1}),\ (b_{2,1}, a_{2,1}, a_{2,2}),\ \ldots\ .$

I.e., P_1^* and P_2^* are "straight" P_1-based and P_2-based paths, respectively, while P_3^* and P_4^* are paths containing alternatively P_1-based and P_2-based parts. Clearly, P_1^*, P_2^*, P_3^* and P_4^* are vertex-disjoint, and it is a matter of routine to check that they can be extended to four internally-vertex-disjoint u–v paths in $L^2(G)$.

Now suppose that u_1v_1 is an edge of G. This case can be solved analogously as the previous one if u_1v_1 is neither in U nor in V. Hence, suppose that $u_2 = v_1$. Let P_1 be a shortest u_1–v_1 path in $G - \{u_1v_1\}$, and let $P_2 = u_1, v_1$. Since u and v are non-adjacent vertices, $u_1 \notin \{v_0, v_2\}$, see Figure 10. As there are just two vertices in the intersection of P_1 and P_2, we can construct P_1^* and P_2^* analogously as the paths P_1^* and P_3^* above (i.e., both P_1^* and P_2^* are P_1-based paths). Further, let P_3^* and P_4^* be one-vertex paths, with histories of the vertices (u_1, v_1, v_0) and (u_1, v_1, v_2). Clearly, these four paths can be extended to internally-vertex-disjoint u–v paths.

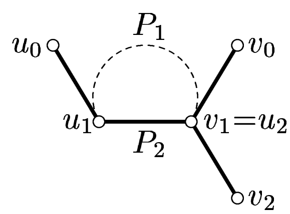

Figure 10

Finally, suppose that $u_1 = v_1$. Since u and v are non-adjacent vertices, we have $\{u_0, u_2\} \cap \{v_0, v_2\} = \emptyset$, and it is easy to see that there are four u–v paths of length 2 in $L^2(G)$. $\square$

Proof of Theorem 3. If $\lambda(G) \geq 2$ then $\kappa(L^2(G)) \geq 4$, by Lemma 9. Similarly, if $\delta(G) \geq 5$ then $\kappa(L^2(G)) \geq 4$, by Theorem 1. Thus, suppose that $\lambda(G) = 1$ and $3 \leq \delta(G) \leq 4$. Since $\delta(L(G)) \geq 4$ and each edge of $L(G)$ lies in a triangle, $\lambda(L(G)) \geq 2$. Now applying Lemma 9 to $L(G)$ we obtain $\kappa(L^3(G)) \geq 4$. $\square$

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