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HISTORIES IN ITERATED LINE GRAPHS

Abstract. We survey the using of history, that is a useful tool for counting the distances in iterated line digraphs, iterated line graphs and iterated path graphs.

1. Introduction

This paper is devoted to the utilizying the notion of history in various metric tasks. History appeared in a natural way when we wished to determine the distances in iterated line graphs. The method of using the history was extremely powerfull in examining the radius of iterated line graphs, where it has brought highly nontrivial results.

Let Γ be the class of all graphs (digraphs), and let F be a mapping $F : \Gamma \to \Gamma$. For $G \in \Gamma$ and $i = 0, 1, 2, \ldots$ we define

$$F^i(G) = \begin{cases} G & \text{if } i = 0; \\ F(F^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

Considering the sequence

$$G, F(G), F^2(G), \ldots, F^i(G), \ldots$$

one can ask how the parameters of $F^i(G)$ depend on that of G and i. To solve problems of this type it is useful to recognize the structure of $F^i(G)$ (or at least some features of $F^i(G)$) already in G. For this purpose we introduced the concept of history, which enabled us to consider vertices of $F^i(G)$ as subgraphs of G, if F is the line digraph mapping, line graph mapping or the P_2-path graph mapping. With the help of history we are able to count the distances in $F^i(G)$ already in G.

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We remark that the notion of history was originally developed for line graphs, where it has the largest applications. Its simplifying was later used for line digraphs, although here the notion is trivial and it cannot bring much of light into the problematic. At present, a generalization of history is developed for P_2-path graphs, but here the work is not finished yet, and some problems remain unsolved.

Throughout the paper, $d_G(u,v)$ denotes the distance from u to v in G, $V(G)$ denotes the vertex set of G, and $E(G)$ denotes the edge (or arc) set of G.

The outline of the paper is as follows. In section 2 we consider line digraphs and section 3 is devoted to line graphs. Path graphs, that generalize line graphs, are considered in section 4.

2. Iterated line digraphs

Let D be a digraph, i.e., a directed graph without multiple arcs. By $L(D)$ we denote the **line digraph** of D. The vertices of $L(D)$ are just the arcs of D, and two vertices of $L(D)$, i.e. the arcs of D, say uv and xy, are joined by an arc in $L(D)$ if and only if $v = x$.

As stated above, each vertex of $L(D)$ can be viewed as a trail of length one in D. Since only adjacent vertices of $L(D)$ give rise to a vertex in $L^2(D)$, each vertex of $L^2(D)$ can be viewed as a trail of length two in D. Further on, considering a "footprint" of a vertex of $L^i(D)$ in $L^{i-j}(D)$, $0 \leq j \leq i$, we obtain the following definition.

Definition. Let D be a digraph and let u be a vertex in $L^i(D)$.

1° The 0-**history** of u, $B^0(u)$, is simply (u).

2° If $0 < j \leq i$, the j-**history** of u, $B^j(u)$, is a sequence of vertices $(x_0, x_1, x_2, \ldots, x_j)$ of $L^{i-j}(D)$, such that $(x_0x_1, x_1x_2, \ldots, x_{j-1}x_j)$ is the $(j-1)$-history of u.

Clearly, the sequence $(x_0, x_1, \ldots, x_j)$ determines a trail in $L^{i-j}(D)$, and there is a one-to-one correspondence between the j-histories (i.e., the trails of length j in $L^{i-j}(D)$) and the vertices in $L^i(D)$. This one-to-one correspondence yields some applications, in which iterated line digraphs appear. Let K_2 be a complete digraph with loops on two vertices, say 0 and 1. Then $L^i(K_2)$ is the i-dimensional de Bruijn digraph, as one can see from the correspondence between the i-histories (i.e., the 0-1 sequences of length i) and the vertices in $L^i(K_2)$, see also [9, p. 483]. We remark that de Bruijn digraphs are used as the underlying digraphs for interconnection networks. The following trivial lemma enables us to count distances in $L^i(D)$ using the distances in D.

Lemma 1 [4]. *Let D be a digraph, and let u and v be vertices in $L^i(D)$. Let $x_0, x_1, \ldots, x_n$ be a shortest trail in D (if such exists) such that $(x_0, x_1, \ldots, x_i) = B^i(u)$ and $(x_{n-i}, x_{n-i+1}, \ldots, x_n) = B^i(v)$. Then*

$$d_{L^i(D)}(u, v) = n - i.$$

Moreover, $d_{L^i(D)}(u,v) = \infty$ if there is no required trail in D.

Histories in iterated line digraphs were used for determining the behavior of the radius. However, there are more definitions of radius in digraphs and we consider only three of them.

Let D be a digraph, and let u be a vertex in D. Then:

out-eccentricity of u is	$e_D^+(u) = \max\{d_D(u,v) : v \in V(D)\}$;
in-eccentricity of u is	$e_D^-(u) = \max\{d_D(v,u) : v \in V(D)\}$;
eccentricity of u is	$e_D(u) = \max\{e_D^+(u), e_D^-(u)\}$.

Using various eccentricities we obtain various radii and various centers. The **out-radius** $r^+(D)$ (**in-radius** $r^-(D)$, **radius** $r(D)$) is the minimum value of $e_D^+(u)$ ($e_D^-(u)$, $e_D(u)$) over all vertices u of D; and the **out-center** $C^+(D)$ (**in-center** $C^-(D)$, **center** $C(D)$) is the subgraph of D induced by vertices with the minimum out-eccentricity (in-eccentricity, eccentricity). We remark that the maximum value of out-eccentricity, in-eccentricity and eccentricity are equal and they are known as the **diameter** of D, $diam(D)$.

Let D' arise from D by reversing the orientation of all arcs. Then $e_{D'}^-(u) = e_D^+(u)$ for every vertex u in D, and hence, $r^+(D) = r^-(D')$. Moreover, the arcs of $C^-(D')$ are just reversed arcs of $C^+(D)$. This observation enables us to restrict the considerations to radii r^+ and r, and to centers C^+ and C, only.

When considering the out-radius, it is useful to divide digraphs into three classes, according the following theorem (see [1, Theorem 10.7.2] and [1, Theorem 10.9.1]):

Theorem 2. *Let D be a digraph.*

(i) *If D has no directed cycles, then $L^i(D)$ is an empty digraph for all i sufficiently large.*

(ii) *If D has directed cycles, no two of which are joined by a directed path, then for all sufficiently large values of i, each component of $L^i(D)$ has at most one directed cycle. Moreover, there are numbers i_D and j_D such that $L^i(D)$ is isomorphic to $L^{i+j_D}(D)$, for every $i \geq i_D$.*

(iii) *If D has two directed cycles joined by a directed path (possibly of length 0), then*

$$\lim_{i\to\infty} |V(L^i(D))| = \infty.$$

Using the notion of history it is possible to prove:

Theorem 3 [4]. *Let D be a digraph, no two of whose cycles are joined by a directed path, and assume that D contains at least one directed cycle. Then there are numbers i_D and t_D, such that for every $i \geq i_D$ we have either*

$$r^+(L^i(D)) = t_D \qquad \text{or} \qquad r^+(L^i(D)) = \infty.$$

Theorem 4 [4]. *Let D be a digraph containing two directed cycles joined by a directed path (possibly of length 0). Then either there are i_D and t_D such that*

$$r^+(L^i(D)) = i + t_D$$

for every $i \geq i_D$, or there is i_D such that for every $i \geq i_D$ we have

$$r^+(L^i(D)) = \infty.$$

Since $r(D) < \infty$ if and only if D is strongly connected, the following theorem characterizes the behavior of radius in iterated line digraphs.

Theorem 5 [4]. *Let D be a nontrivial strongly connected digraph different from a directed cycle. Then there are t_D and t'_D such that for every $i \geq 0$ we have*

$$i + t_D \leq r(L^i(D)) \leq i + t'_D.$$

We outline here the proof of the upper bound of Theorem 5. Let u be a central vertex in D, and let $\mathcal{C}$ be a directed cycle containing u (one can take a shortest one), $\mathcal{C} = (u, a_2, a_3, \ldots, a_l, u)$. The length of $\mathcal{C}$ is l. Then there is a vertex x^1 in $L^l(D)$ such that $B^l(x^1) = (u, a_2, \ldots, a_l, u)$. By Lemma 1 we have

$$r(L^l(D)) \leq e_{L^l(D)}(x^1) \leq e_D(u) + l = r(D) + l.$$

Analogously, if $j > 1$, then $r(L^{jl}(D)) \leq j{\cdot}l + r(D)$ (take a trail going j times around $\mathcal{C}$ for $B^{jl}(x^j)$). Since $r(H) \leq r(L(H))$ holds for every nontrivial strongly connected digraph H, see [4], we have

$$r(L^i(D)) \leq r(L^{(j+1)l}(D)) \leq (j+1)l + r(D)$$

for all i, $jl < i \leq (j+1)l$. Hence, $r(L^i(D)) \leq i + (r(D)+l)$ for every $i \geq 0$.

We remark that if D is a strongly connected digraph, then $diam(L(D)) = diam(D) + 1$, see [2]. Hence, $diam(L^i(D)) = diam(D) + i$ for every $i \geq 0$.

Now we turn our attention to centers in iterated line digraphs. It is known that each (undirected) graph G can serve as a center of some graph. Just take four vertices, say a_1, a_2, b_1 and b_2, outside of G, join a_1 and b_1 to all vertices in G, add two edges a_1a_2 and b_1b_2, and denote the resulting graph by H. Then $C(H) = G$. In H the vertices a_2 and b_2 have constant distance (two) from every vertex u of G, and this distance is the largest one, a vertex u can achieve. Moreover, the distance from every vertex of $V(H) - V(G)$ to either a_2 or b_2 is larger than two. This construction can be modified to iterated line digraphs to obtain the following results:

Theorem 6 [4]. *Let D be a nontrivial strongly connected digraph. Then there is a digraph H, $H \supseteq D$, such that for every $i \geq 0$ we have*

$$C^+(L^i(H)) = L^i(D).$$

Theorem 7 [4]. *Let D be a digraph and let $j \geq 0$. If $L^j(D)$ is not empty then there is a digraph H, $H \supseteq D$, such that for every i, $0 \leq i \leq j$, we have*

$$C(L^i(H)) = L^i(D).$$

Theorems 6 and 7 are, in a sense, best possible. In [4] it is shown that some line digraphs (that are not strongly connected, of course) are not out-centers of any line digraph. Further on, there are digraphs D, such that there does not exist a digraph H, $H \supseteq D$, for which $C(L^i(D)) = L^i(D)$ for every $i \geq 0$, see [4].

We describe the digraph H of Theorem 6. Let $d = \max\{2, diam(D)\}$, and let

$$\begin{aligned} V(H) &= V(D) \cup \{a_1, b_1, \ldots, a_d, b_d\}; \\ E(H) &= E(D) \cup \{ua_1, ub_1 : u \in V(D)\} \cup \\ &\quad \cup \{a_ja_{j+1}, b_jb_{j+1} : 1 \leq j \leq d-1\} \cup \{a_da_{d-1}, b_db_{d-1}\}, \end{aligned}$$

see Figure 1. Then vertices x^i and y^i in $L^i(H)$, $B^i(x^i) = (a_d, a_{d-1}, a_d, a_{d-1}, \ldots)$ and $B^i(y^i) = (b_d, b_{d-1}, b_d, b_{d-1}, \ldots)$, play the role of a_2 and b_2 in the graph H described above.

The digraph H of Theorem 7 is constructed in a similar way.

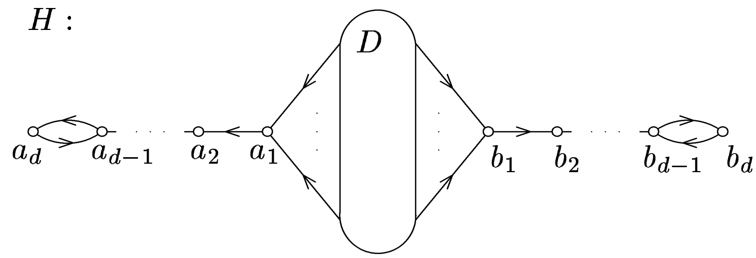


Figure 1

3. Iterated line graphs

Let G be a graph. The **line graph** of G, $L(G)$, is a graph whose vertices are the edges of G. Two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.

Clearly, each vertex in $L(G)$ can be viewed as a trail of length one in G. Moreover, analogously as in iterated line digraphs, each vertex in $L^2(G)$ can be viewed as a trail of length two in G. However, there are vertices in $L^3(G)$ which cannot be represented by trails of length three in G. In fact, it is not easy to find "something" in G, which is in one-to-one correspondence with the vertices in $L^i(G)$ if $i \geq 3$. But we do not need an one-to-one correspondence, if the "something" enables us to count distances in $L^i(G)$ correctly.

Definition. Let G be a graph, and let u be a vertex in $L^i(G)$.

1° The 0-**history** of u, $B^0(u)$, is a subgraph of $L^i(G)$ formed by unique vertex u.

2° If $0 < j \leq i$, the j-**history** of u, $B^j(u)$, is a subgraph of $L^{i-j}(G)$. Since every vertex of $L^{i-j+1}(G)$ corresponds to an edge in $L^{i-j}(G)$, the vertices of $B^{j-1}(u)$ correspond to edges in $L^{i-j}(G)$ and this edges form the j-history of u.

Let G be a claw, i.e., the complete bipartite graph $K_{1,3}$. Then all $L(G), L^2(G)$, $L^3(G), \ldots$ are triangles. If u is any vertex of $L^3(G)$, then $B^3(u) = G$. Hence, there is not one-to-one correspondence between the vertices in $L^3(G)$ and their 3-histories.

There arises a natural question. Which subgraphs of G form an i-history of a vertex of $L^i(G)$? The next lemma gives a complete answer.

Lemma 8 [8]. *Let G be a graph, and let H be a subgraph of G. Then H is an i-history of a vertex in $L^i(G)$ if and only if H is a connected graph with at most i edges, distinct from any path with less than i edges.*

Let H and J be two subgraphs of G. By their distance $d_G(H, J)$ we mean the length of a shortest path in G joining a vertex of H to a vertex in J. The following lemma enables us to count distances between vertices in iterated line graphs.

Lemma 9 [8]. *Let G be a graph, and let u and v be distinct vertices in $L^i(G)$. Then*

(D1) $d_{L^i(G)}(u,v) = i + d_G(B^i(u), B^i(v))$ *if the i-histories of u and v are edge-disjoint;*

(D2) $d_{L^i(G)}(u,v) = \max\{t : t\text{-histories of } u \text{ and } v \text{ are edge-disjoint}\}$ *if i-histories of u and v share an edge in common.*

Lemmas 8 and 9 have applications in determining the diameter, radius and the center in iterated line graphs. Let G be a graph and let u be a vertex in G. Then

eccentricity of u is $$e_G(u) = \max\{d_G(u,v) : v \in V(G)\}.$$

The diameter $diam(G)$ is the maximum value of $e_G(u)$, and the radius $r(G)$ is the minimum value of $e_G(u)$, respectively, over all vertices u of G. The vertices with the minimum eccentricity induce the center $C(G)$ of G.

For iterated line graphs there is an analogue of Theorem 2.

Assertion 10 [8]. *Let G be a connected graph.*

(i) *If G is a path of length j, then $L^i(D)$ is an empty graph for all $i > j$.*

(ii) *If G is a cycle, then each iterated line graph of G is isomorphic to the original cycle; and if G is a claw $K_{1,3}$ then each iterated line graph of G is a triangle.*

(iii) *If G is a connected graph different from a path, cycle and a claw, then*

$$\lim_{i\to\infty} |V(L^i(G))| = \infty.$$

Thus, it is enough to consider connected graphs different from a path, cycle and a claw. Such graphs G will be called **prolific**, since each two members of the sequence $G, L(G), L^2(G), \ldots$ are distinct. Now with the notion of history it is possible to prove:

Theorem 11 [8]. *Let G be a prolific graph. Then there are i_G and t_G such that for every $i \geq i_G$ we have*

$$diam(L^i(G)) = i + t_G.$$

Theorem 12 [8]. *Let G be a connected noncomplete graph with the minimum degree at least three. Then for every $i \geq 1$ we have*

$$i + diam(G) - 2 \leq diam(L^i(G)) \leq i + diam(G).$$

In the proof of Theorem 11 it is shown that $L^3(G)$ contains two edge-disjoint claws. This claws are i-histories of vertices of $L^{3+i}(G)$ for $i \geq 3$, by Lemma 8. Then Lemma 9 completes the proof.

The proof of Theorem 12 is based on the same idea.

Theorem 13 [8]. *Let G be a prolific graph. Then there are t_G and t'_G such that for every $i \geq 0$ we have*

$$\left(i - \sqrt{2\log_2 i}\right) + t_G < r(L^i(G)) < \left(i - \sqrt{2\log_2 i}\right) + t'_G.$$

The proof of Theorem 13 is a bit more complicated than the previous ones, since it is important to find a central vertex in $L^i(G)$ for large i. Clearly, a central vertex, say u, in $L^i(G)$ has a "large" i-history in G. The best situation appears when $B^i(u)$ share an edge with every other i-history, by Lemma 9. Let $u^0, u^1, u^2, \ldots$ be a sequence of vertices such that u^i is a central vertex in $L^i(G)$. By i_j we denote the first index such that $B^{i_j-j}(u^{i_j})$ share an edge with every (i_j-j)-history of a vertex of $L^{i_j}(G)$. (Thus, $B^{i_j-j}(u^{i_j})$ is a "large" subgraph of $L^j(G)$.) In general, by Lemma 9 we have

$$\begin{array}{llll} r(L^{i_0}(G)) = i_0 - 1, & r(L^{i_0+1}(G)) = i_0, & \ldots & r(L^{i_1-1}(G)) = i_1 - 2, \\ r(L^{i_1}(G)) = i_1 - 2, & r(L^{i_1+1}(G)) = i_1 - 1, & \ldots & r(L^{i_2-1}(G)) = i_2 - 3, \\ r(L^{i_2}(G)) = i_2 - 3, & r(L^{i_2+1}(G)) = i_1 - 2, & \ldots & \end{array}$$

It means that $r(L^i(G)) = r(L^{i-1}(G)) + 1$ for every $i > i_0$, except $i = i_j$, where $r(L^i(G)) = r(L^{i-1}(G))$. Hence, it remains to compute i_j, $j \geq 0$. Since $L^j(G) - B^{i_j-j}(u^{i_j})$ is a linear forest, by Lemmas 8 and 9, we have

$$|V(L^{j+1}(G))| - |V(L^j(G))| < i_j - j < |V(L^{j+1}(G))|$$

(recall that $|V(L^{j+1}(G))| = |E(L^j(G))|$). This two bounds together with the estimation of $|V(L^i(G))|$ in [8] give the bounds of Theorem 13.

By Theorems 11 and 13, for every prolific graph G there is a number k_G, such that if $i \geq k_G$ then $L^i(G)$ is not a selfcentric graph (i.e., the radius of $L^i(G)$ is strictly less than its diameter). Clearly, almost all graphs are prolific. Therefore, the following result, a proof of which is based on the notion of history, may be surprising.

Theorem 14 [3]. *Let $i \geq 0$. Then for almost all graphs G we have*

$$diam(L^i(G)) = r(L^i(G)) = i + 2.$$

Now we introduce an analogue of Theorems 6 and 7 for iterated line graphs.

Theorem 15 [7]. *Let G be a graph and let $0 \leq j \leq 2$. If $L^j(G)$ is not empty then there is a graph H, $H \supseteq G$, such that for every i, $0 \leq i \leq j$, we have*

$$C(L^i(H)) = L^i(G).$$

Moreover, if G is triangle-free and $L^3(G)$ is not empty, then also

$$C(L^3(H)) = L^3(G).$$

Although the proof of Theorem 15 is more complicated than that of Theorem 6, it is based on the same idea. Theorem 15 is best possible in a sense, since there is a graph G such that for every $i \geq 3$ and any graph H, $H \supseteq G$, we have $C(L^i(H)) \neq L^i(G)$.

We remark that Theorem 15 characterizes the centers of line graphs, since each induced subgraph of a line graph is a line graph. It means that G is a center of some line graph if and only if G is a line graph. However, the center of i-iterated line graph is not necessary an i-iterated line graph if $i \geq 2$. Hence, the problem of characterizing the centers of i-iterated line graphs remains open for $i \geq 2$.

4. Iterated P_2-path graphs

Let G be a graph, $k \geq 1$, and let $\mathcal{P}_k$ be the set of all subgraphs of G which form a path of length k (i.e., with $k+1$ vertices). The **P_k-path graph** $P_k(G)$ of G has vertex set $\mathcal{P}_k$. Let $A, B \in \mathcal{P}_k$. The vertices of $P_k(G)$ that correspond to A and B are joined by an edge in $P_k(G)$ if and only if the edges of $A \cap B$ form a path of length $k-1$ and $A \cup B$ is either a path of length $k+1$ or a cycle of length $k+1$.

Path graphs generalize line graphs, as $P_1(G)$ is a line graph of G. However, there is a stronger connection between path graphs and the line graphs. In what follows, the vertices of a path graph $P_k(G)$ (as well as the vertices of G) are denoted by small letters a, b, ..., while the corresponding paths of length k in G we denote by capital letters A, B, ... It means that if A is a path of length k in G and a is a vertex in $P_k(G)$, then a must be the vertex corresponding to the path A. We have the following theorem:

Theorem 16 [6]. *Let G be a graph and $k \geq 2$. Then there is a unique embedding $\varphi : P_k(G) \to L^k(G)$ such that for every vertex u in $P_k(G)$, the path U and the k-history $B^k(\varphi(u))$, we have*

$$U = B^k(\varphi(u)).$$

It means that the P_k-path graph $P_k(G)$ is a subgraph of $L^k(G)$, and there is a unique embedding of $P_k(G)$ into $L^k(G)$ preserving histories.

Now we turn our attention to histories in iterated path graphs. Since using the histories we wish to count distances between vertices in $P_k^i(G)$ already in G, it is not enough to consider the "footprint" of a vertex in $P_k^i(G)$ as a subgraph of G. In these subgraphs we need to distinguish special vertices, called out-vertices.

Definition. Let G be a graph, $k \geq 1$, and let u be a vertex in $P_k^i(G)$.

1° The 0-**history** of u, $B_k^0(u)$, is a subgraph of $P_k^i(G)$ formed by unique vertex u, and this vertex is defined as **out-vertex** of $B_k^0(u)$.

2° If $0 < j \leq i$, the j-**history** of u, $B_k^j(u)$, is a subgraph of $P_k^{i-j}(G)$. Assume that $V(B_k^{j-1}(u)) = \{a_1, a_2, \ldots, a_l\}$. The vertices and edges of $B_k^j(u)$ are the vertices and edges of $A_1, A_2, \ldots, A_l$. Moreover, if $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ are the out-vertices of $B_k^{j-1}(u)$, then the endvertices of $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ are defined as **out-vertices** of $B_k^j(u)$.

We remark, that if u is a vertex in the i-iterated line graph of G, then all vertices of $B_1^j(u)$ are out-vertices, $0 \leq j \leq i$. In Figure 2 we picture all 2-histories of vertices in $P_2^2(G)$. In these pictures, out-vertices are painted black.

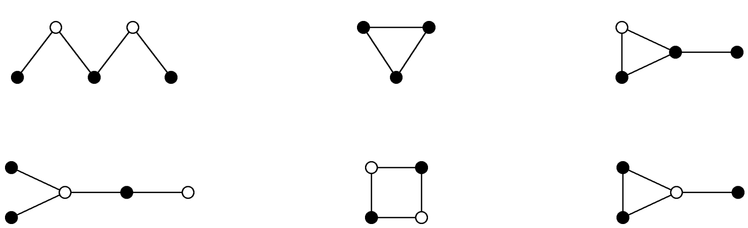


Figure 2

At present, we have not a characterization of i-histories of vertices in $P_k^i(G)$ for arbitrary i if $k \geq 2$. However, if $k = 2$, we are able to count distances between vertices in $P_2^i(G)$ using their i-histories. Let G be a graph, and let u and v be vertices in $P_2^i(G)$. Then $d_\bullet^j(u,v)$ denotes the shortest distance between out-vertices of $B_2^j(u)$ and out-vertices of $B_2^j(v)$ in $P_2^{i-j}(G)$.

Lemma 17 [5]**.** *Let G be a graph, and let u and v be non-isolated vertices in $P_2^i(G)$. Moreover, let $B_2^j(u)$ and $B_2^j(v)$ be edge-disjoint j-histories, $0 \leq j \leq i$. Then*

$$d_{P_2^i(G)}(u,v) = 2j + d_\bullet^j(u,v).$$

Lemma 17 was used for determining the diameter in iterated P_2-path graphs. However, since $P_2(G)$ may be disconnected even if G is a connected graph, we consider the diameter of such a component of $P_2^i(G)$ which contains an edge (if it exists). In [5] it is proved that at most one component of $P_2(G)$ contains an edge if G is connected.

First we introduce some definitions. By G_j, $j \geq 1$, we denote a tree composed of two paths of length two, central vertices of which are joined by a path of length $2j-1$. A **dragon** is a unicyclic graph composed of an even cycle $\mathcal{C}$ and a set of vertices, each joined by an edge to some vertex of $\mathcal{C}$. Moreover, each pair of vertices of a dragon that have degree at least three, has an even distance (see Figure 3 for a dragon with cycle of length 8). **Broken dragon** is a tree composed of a diametric path $\mathcal{T}$ and a set of vertices, each joined by an edge to some vertex of $\mathcal{T}$. Moreover, each pair of vertices of a broken dragon that have degree at least three, has an even distance (see Figure 5 for a broken dragon with diametric path of length 9). A **dragon's egg** is a tree composed of a claw $K_{1,3}$ in which each edge was subdivided by one vertex, and a set of vertices, each joined by an edge either to the central vertex or to some endvertex of the subdivided claw (see Figure 4 for a dragon's egg). If a connected graph G is different from a cycle, dragon, broken dragon, dragon's egg and the graph G_j, $j \geq 1$, then G is called **prolific**.

We have an analogue of Assertion 10 and Theorem 11 for iterated P_2-path graphs.

Theorem 18 [5]**.** *Let G be a graph with a unique nontrivial component. Denote by H_j a nontrivial component of $P_2^j(G)$ (if it exists), $j \geq 0$.*

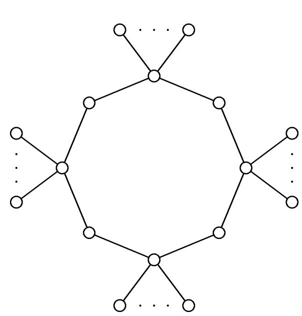
(i) *If G is a broken dragon, then there is i_G such that for every $i \geq i_G$ the graph $P_2^i(G)$ is empty.*

(2) *If G is a cycle, or a dragon, dragon's egg, or the graph G_j, $j \geq 1$, then there are i_G and t_G such that for every $i \geq i_G$ we have*

$$diam(H_i) = t_G.$$

(3) *If G is a prolific graph, then there are i_G and t_G such that for every $i \geq i_G$ we have*

$$diam(H_i) = 2i + t_G.$$



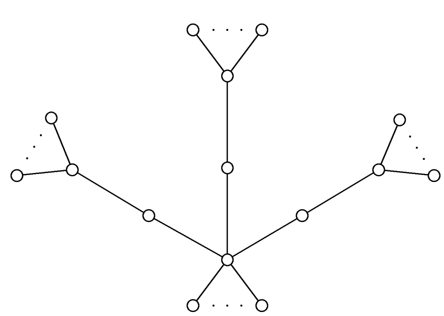


Figure 3 Figure 4

broken dragon

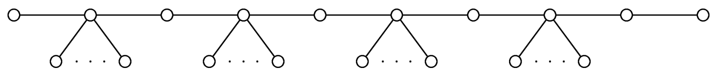


Figure 5

We remark that $\lim_{i\to\infty} |V(H_i)| = \infty$ even if G is a dragon different from a cycle.

In the proof of Theorem 18 we used the fact that $P_2^i(G)$ contains two edge-disjoint cycles, if G is a prolific graph and i is large enough. This cycles are histories of vertices of $P_2^{i+j}(G)$ for large j. Thus, Lemma 17 and the inequality $diam(H_i) \leq diam(H_{i-1}) + 2$, see [5], give the result (iii) of Theorem 18.

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