DIAMETER IN ITERATED PATH GRAPHS

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ABSTRACT. If G is a graph, then its path graph, $P_k(G)$, has vertex set identical with the set of paths of length k in G, with two vertices adjacent in $P_k(G)$ if and only if the corresponding paths are "consecutive" in G. We study the behavior of $diam(P_2^i(G))$ as a function of i, where $P_2^i(G)$ is a composition $P_2(P_2^{i-1}(G))$, with $P_2^0(G) = G$.

1. INTRODUCTION

Let G be a graph and $k \geq 1$. The vertex set of the **path graph** $P_k(G)$ is the set of paths of length k in G (i.e., with $k+1$ vertices). Two vertices of $P_k(G)$ are joined by an edge if and only if one of the corresponding paths can be obtained from the other by deleting an edge from one end and adding an edge to the other end. It means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G.

Path graphs were investigated by Broersma and Hoede in [2], as a natural generalization of line graphs (observe that $P_1(G)$ is the line graph of G). In [2] and [5] P_2-path graphs are characterized, and in [7] traversability of P_2-path graphs is studied. The diameter of path graphs is studied in [1], and [3] is devoted to centers in path graphs.

In this paper we study the diameter of P_2-path graphs and their iterations. We prove:

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Theorem 1. *Let G be a connected graph. Then at most one component of $P_2(G)$ is not trivial. Let H be the nontrivial component of $P_2(G)$. Then*

$$diam(H) \leq diam(G) + 2.$$

Theorem 1 improves a result of Belan and Jurica for P_2-path graphs [1], as they need a restriction for the diameter of G.

Let G be a graph, $k \geq 1$ and $i \geq 0$. The **i-iterated path graph** of G, the $P_k^i(G)$, is

$$P_k^i(G) = \begin{cases} G & \text{if } i = 0; \\ P_k(P_k^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

If $i \geq 1$, it is easy to find a connected graph G such that $P_2^i(G)$ is not connected. However, $P_2^i(G)$ consists of a "large" component and a set of isolated vertices, by Theorem 1. For this reason, we consider the diameter of the "large" component of $P_2^i(G)$, instead of the diameter of $P_2^i(G)$.

Let G be a connected graph and let H be the nontrivial component of $P_2(G)$. By [1, Theorem 2] and Theorem 1 we have

$$diam(G) - 2 \leq diam(H) \leq diam(G) + 2.$$

Here all values from the range $[diam(G)-2, diam(G)+2]$ are attainable, see [1]. For iterated P_2-path graphs and large i this is not the case. We prove that for every connected graph G, up to a strictly determined collection of trees and unicyclic graphs, for large i we have

$$diam(H_i) = diam(H_{i-1}) + 2,$$

where H_j is the nontrivial component of $P_2^j(G)$.

We remark that the situation is analogous for line graphs and iterated line graphs, $L^i(G) = P_1^i(G)$, where

$$diam(G) - 1 \leq diam(L(G)) \leq diam(G) + 1,$$

see [4], but

$$diam(L^i(G)) = diam(L^{i-1}(G)) + 1$$

for every connected graph G, different from a path, a cycle, and a claw $K_{1,3}$, provided that i is sufficiently large, see [6].

2. Results

For convenience we adopt the following convention. We denote the vertices of $P_2(G)$ (as well as the vertices of G) by small letters $a, b, \ldots$, while the corresponding paths of length 2 in G will be denoted by capital letters $A, B, \ldots$. It means that if A is a path of length 2 in G and a is a vertex in $P_2(G)$, then a must be the vertex corresponding to the path A.

The vertices of $P_2(G)$ correspond to the paths of length two in G; and the vertices of $P_2^2(G)$ correspond to "special collections" of three paths of length two in G. If $i \geq 2$, it is not easy to represent the vertices of $P_2^i(G)$ in G. The reason is that $P_2^i(G)$ has in general much more vertices than G. However, loosing a bit of information about the vertices of $P_2^i(G)$ we can count distances between them already in G. For this reason we introduce the concept of a history.

Let G be a graph, $i \geq 0$, $k \geq 1$, and let v be a vertex in $P_k^i(G)$.

1° The 0-**history** of v, $B_k^0(v)$, is the subgraph of $P_k^i(G)$ containing only the vertex v. This vertex will be called an **out-vertex** of $B_k^0(v)$.

2° If $0 < j \leq i$, we denote the j-**history** of v as $B_k^j(v)$. It is a subgraph of $P_k^{i-j}(G)$. Assume that $V(B_k^{j-1}(v)) = \{a_1, a_2, \dots, a_l\}$. Then, the vertices and edges of $B_k^j(v)$ are the vertices and edges of $A_1, A_2, \dots, A_l$. Moreover, if $a_{i_1}, a_{i_2}, \dots, a_{i_r}$ are the out-vertices of $B_k^{j-1}(v)$, then the endvertices of $A_{i_1}, A_{i_2}, \dots, A_{i_r}$ will be called **out-vertices** of $B_k^j(v)$.

We remark that if v is a vertex in the i-iterated line graph of G, then all vertices of $B_1^j(v)$ are the out-vertices, $0 \leq j \leq i$.

The unique 1-history of a vertex in $P_2(G)$ is shown in Figure 1; and all possible 2-histories of vertices in $P_2^2(G)$ are shown in Figure 2. In these pictures, the out-vertices are painted black.

Figure 1

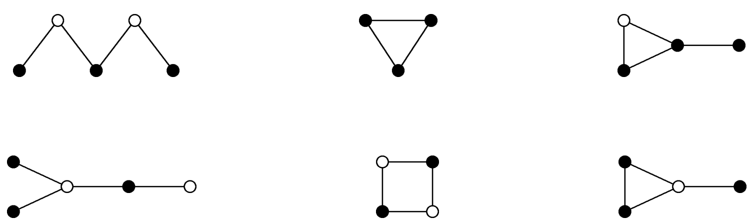

Figure 2

We will count distances in $P_2^i(G)$ using the distances between out-vertices of i-histories in G. Let G be a graph, $i \geq 0$ and $0 \leq j \leq i$. Let u and v be distinct vertices in $P_2^i(G)$, and let $B_2^j(u)$ and $B_2^j(v)$ be j-histories of u and v, respectively. Then $d_{\bullet}^j(u,v)$ denotes the shortest distance from an out-vertex of $B_2^j(u)$ to an out-vertex of $B_2^j(v)$ in $P_2^{i-j}(G)$.

Lemma 2. *Let G be a connected graph. Let u and v be non-isolated vertices in $P_2(G)$ and let $B_2^1(u)$ and $B_2^1(v)$ be edge-disjoint 1-histories. Then*

$$d_{P_2(G)}(u,v) = 2 + d_{\bullet}^1(u,v).$$

Proof. Let $\mathcal{T} = (u{=}a_0, a_1, a_2, \dots, a_{r-1}, a_r{=}v)$ be a shortest walk from u to v in $P_2(G)$. Let i_u be the last index such that A_{i_u} contains an out-vertex of U, and let i_v be the first index such that A_{i_v} contains an out-vertex of V. It is easy to see that $i_u \geq 2$ and $i_v \leq r{-}2$. Thus,

$$d_{P_2(G)}(u,v) = d_{P_2(G)}(u, a_{i_u}) + (i_v - i_u) + d_{P_2(G)}(a_{i_v}, v) \geq 4 + (i_v - i_u)$$

(we remark that $i_v - i_u = d_{P_2(G)}(a_{i_u}, a_{i_v})$ if $i_v \geq i_u$.) Let u_1 be the out-vertex of U in A_{i_u} and let v_1 be the out-vertex of V in A_{i_v}. Since all paths $U, A_1, A_2, \ldots, A_{r-1}, V$ have length 2, we have

$$d_G(u_1, v_1) \leq (i_v - i_u) + 2 \leq d_{P_2(G)}(u, v) - 2.$$

Thus,

$$d^1_{\bullet}(u, v) + 2 \leq d_{P_2(G)}(u, v).$$

Now suppose that $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$, and $d^1_{\bullet}(u, v) = d_G(u_1, v_1) = l \geq 2$. (The cases $l = 0$ and $l = 1$ are trivial and can be solved separately.) Let $(u_1, b_1, b_2, \ldots, b_{l-1}, v_1)$ be a shortest walk in G. Denote $A_2 = (u_1, b_1, b_2)$, $A_3 = (b_1, b_2, b_3)$, $\ldots$, $A_l = (b_{l-2}, b_{l-1}, v_1)$. Moreover, if $b_1 = u_2$ then $A'_2 = (u_3, b_1, b_2)$ and if $b_{l-1} = v_2$ then $A'_l = (b_{l-2}, b_{l-1}, v_3)$. Since u and v are not isolated vertices in $P_2(G)$, it is easy to check, that $d_{P_2(G)}(u, a_2) = 2$ and $d_{P_2(G)}(a_l, v) = 2$ (or $d_{P_2(G)}(u, a'_2) = 2$ if $b_1 = u_2$ and $d_{P_2(G)}(u, a_2) > 2$; and $d_{P_2(G)}(a'_l, v) = 2$ if $b_{l-1} = v_2$ and $d_{P_2(G)}(a_l, v) > 2$). Since $d_{P_2(G)}(a_2, a_l) = l - 2$, we have

$$d_{P_2(G)}(u, v) \leq 2 + (d^1_{\bullet}(u, v) - 2) + 2 = d^1_{\bullet}(u, v) + 2,$$

which completes the proof. $\square$

Now we are able to prove Theorem 1.

Proof of Theorem 1. At first we prove that $P_2(G)$ has at most one nontrivial component. Let u_1u_2 and v_1v_2 be edges in $P_2(G)$. We show that both of them belong to the same component. If there is a pair of edge-disjoint 1-histories among $B^1_2(u_1)$, $B^1_2(u_2)$, $B^1_2(v_1)$ and $B^1_2(v_2)$, then the edges u_1u_2 and v_1v_2 belong to the same component of $P_2(G)$, by Lemma 2. (Observe that 1-histories of adjacent vertices cannot be edge-disjoint.) Thus, suppose that such a pair does not exist. Assume that $U_1 = (x_0, x_1, x_2)$ and $U_2 = (x_1, x_2, x_3)$. Then at least one of V_1 and V_2, say V_1, contains the edge x_1x_2, so that either $d_{P_2(G)}(v_1, u_1) = 1$ or $d_{P_2(G)}(v_1, u_2) = 1$. Thus, each pair of edges of $P_2(G)$ belongs to a common component of $P_2(G)$.

Let H be the nontrivial component of $P_2(G)$, and let u and v be vertices of H such that $d_H(u, v) = diam(H)$. If $U = B^1_2(u)$ and $V = B^1_2(v)$ are edge-disjoint, then $d_H(u, v) = 2 + d_{\bullet}(u, v)$, by Lemma 2. Thus,

$$diam(H) = d_H(u, v) = 2 + d_{\bullet}(u, v) \leq diam(G) + 2.$$

Now suppose that $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$ share an edge, say u_2u_3. If $V = (u_2, u_3, v_3)$, then $1 = d_H(u, v) = diam(H) \leq diam(G) + 2$. Thus, suppose that $V = (v_1, u_2, u_3)$. If there is a vertex w_0 in G such that $w_0 \neq u_2$ and $u_3w_0 \in E(G)$, then both u and v are adjacent to w in H, $W = (u_2, u_3, w_0)$, so that $2 = diam(H) \leq diam(G) + 2$ again.

Thus, suppose that $deg_G(u_3) = 1$. This means that $deg_G(u_1) \geq 2$ and $deg_G(v_1) \geq 2$. If u_1 and v_1 are adjacent vertices in G, then (u, x, y, v) form a walk in H, $X = (u_2, u_1, v_1)$ and $Y = (u_1, v_1, u_2)$, so that $3 = diam(H) \leq diam(G) + 2$ (as $diam(G) \geq 1$).

Thus, suppose that $d_G(u_1, v_1) \geq 2$. Then $diam(G) \geq 2$. Since $deg_G(u_1) \geq 2$ and $deg_G(v_1) \geq 2$, there are vertices u_0 and v_0 in G such that $X = (u_2, u_1, u_0)$ and $Y = (u_2, v_1, v_0)$ are paths of length two in G. Then (u, x, w, y, v) form a walk in H, $W = (u_1, u_2, v_1)$, so that $4 = diam(H) \leq diam(G) + 2$. $\square$

Now we extend Lemma 2 to the case of iterated P_2-path graphs.

Lemma 3. *Let G be a connected graph, $i \geq 0$ and $0 \leq j \leq i$. Let u and v be non-isolated vertices in $P_2^i(G)$, and let $B_2^j(u)$ and $B_2^j(v)$ be edge-disjoint j-histories. Then*

$$d_{P_2^i(G)}(u,v) = 2j + d_\bullet^j(u,v).$$

Proof. For $j = 0$ the statement holds trivially, while for $j = 1$ it reduces to Lemma 2.

Suppose that $j \geq 2$. Let $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$. By Lemma 2, we have $d_{P_2^i(G)}(u,v) = 2 + d_\bullet^1(u,v)$. Without loss of generality we may assume that $d_\bullet^1(u,v) = d_{P_2^{i-1}(G)}(u_1,v_1)$. Since $B_2^j(u)$ and $B_2^j(v)$ are edge-disjoint, $B_2^{j-1}(u_1)$ and $B_2^{j-1}(v_1)$ are edge-disjoint, too. By induction, we have $d_{P_2^{i-1}(G)}(u_1,v_1) = 2(j-1) + d_\bullet^{j-1}(u_1,v_1)$. Thus, $d_{P_2^i(G)}(u,v) = 2j + d_\bullet^{j-1}(u_1,v_1)$. As each out-vertex of $B_2^{j-1}(u_1)$ (of $B_2^{j-1}(v_1)$) is an out-vertex of $B_2^j(u)$ (of $B_2^j(v)$), we have $d_\bullet^{j-1}(u_1,v_1) \geq d_\bullet^j(u,v)$, and hence,

$$d_{P_2^i(G)}(u,v) \geq 2j + d_\bullet^j(u,v).$$

On the other hand, assume that $d_\bullet^j(u,v) = d_{P_2^{i-j}(G)}(p,q)$, where p is an out-vertex of $B_2^j(u)$ and q is an out-vertex of $B_2^j(v)$. As each out-vertex of $B_2^j(u)$ is an out-vertex of either $B_2^{j-1}(u_1)$ or $B_2^{j-1}(u_3)$, assume that p is an out-vertex of $B_2^{j-1}(u_1)$ and q is an out-vertex of $B_2^{j-1}(v_1)$. Then $d_\bullet^{j-1}(u_1,v_1) = d_\bullet^j(u,v)$. Since $B_2^j(u)$ and $B_2^j(v)$ are edge-disjoint, $B_2^{j-1}(u_1)$ and $B_2^{j-1}(v_1)$ are edge-disjoint, too. By induction, we have $d_{P_2^{i-1}(G)}(u_1,v_1) = 2(j-1) + d_\bullet^{j-1}(u_1,v_1) = 2(j-1) + d_\bullet^j(u,v)$. As $d_{P_2^i(G)}(u,v) \leq 2 + d_{P_2^{i-1}(G)}(u_1,v_1)$, by Lemma 2, we have

$$d_{P_2^i(G)}(u,v) \leq 2j + d_\bullet^j(u,v). \quad \square$$

Lemma 4. *Let G be a connected graph containing two edge-disjoint cycles. Then there is i_G such that for all $i \geq i_G$*

$$diam(H_i) = diam(H_{i-1}) + 2,$$

where H_j is the nontrivial component of $P_2^j(G)$, $j \geq 0$.

Proof. Since $P_2(G') \cong G'$ if G' is a cycle, $P_2^i(G)$ contains an edge for every $i \geq 0$. Let $\mathcal{C}$ and $\mathcal{D}$ be two edge-disjoint cycles in G, and let $d_G(c_0, d_0)$ be the shortest distance from a vertex of $\mathcal{C}$ to a vertex in $\mathcal{D}$, $c_0 \in V(\mathcal{C})$ and $d_0 \in V(\mathcal{D})$. We prove that for every $i \geq 0$ there are vertices c_i and d_i in $P_2^i(G)$, such that $B_2^i(c_i)$ contains only edges of $\mathcal{C}$ and c_0 is an out-vertex of $B_2^i(c_i)$, and $B_2^i(d_i)$ contains only edges of $\mathcal{D}$ and d_0 is an out-vertex of $B_2^i(d_i)$.

If $i = 0$, then c_0 and d_0 fulfill the required conditions. Let $i > 0$. Suppose that there are vertices c_{i-1} and d_{i-1} in $P_2^{i-1}(G)$ such that $B_2^{i-1}(c_{i-1})$ contains only edges of $\mathcal{C}$ and c_0 is an out-vertex of $B_2^{i-1}(c_{i-1})$, and $B_2^{i-1}(d_i)$ contains only edges of $\mathcal{D}$ and d_0 is an out-vertex of $B_2^{i-1}(d_{i-1})$. Since $B_2^{i-1}(c_{i-1})$ is a subgraph of $\mathcal{C}$, c_{i-1} is a vertex in $P_2^{i-1}(\mathcal{C})$. As $P_2^{i-1}(\mathcal{C})$ is a cycle, there are vertices x_{i-1} and y_{i-1} in $P_2^{i-1}(\mathcal{C})$, such that $C_i = (c_{i-1}, x_{i-1}, y_{i-1})$ form a path of length two in $P_2^{i-1}(\mathcal{C})$. Then c_i is a vertex in $P_2^i(G)$ such that $B_2^i(c_i)$ contains only edges of $\mathcal{C}$ and c_0 is an out-vertex of $B_2^i(c_i)$. Analogously, one can show that there is a vertex d_i in $P_2^i(G)$ such that $B_2^i(d_i)$ contains only edges of $\mathcal{D}$ and d_0 is an out-vertex of $B_2^i(d_i)$.

Since the shortest distance between $V(\mathcal{C})$ and $V(\mathcal{D})$ is realized by $d_G(c_0, d_0)$, we constructed sequences of vertices $c_0, c_1, \ldots$ and $d_0, d_1, \ldots$ such that $c_i, d_i \in V(P_2^i(G))$ for all $i \geq 0$ and

$$d_{P_2^i(G)}(c_i, d_i) = 2i + d_G(c_0, d_0),$$

by Lemma 3. However, $diam(H_i) \leq diam(H_{i-1}) + 2$, by Theorem 1. Thus, only for finitely many indices j we have $diam(H_j) < diam(H_{j-1}) + 2$, and hence, there is i_G such that for all $i \geq i_G$ we have

$$diam(H_i) = diam(H_{i-1}) + 2,$$

as required. $\square$

To prove our main result we introduce prolific graphs, kites, torn kites and kite's eggs. A connected graph G is **prolific** if there exists $i \geq 0$ such that $P_2^i(G)$ contains at least two cycles. In what follows we characterize prolific graphs.

As $P_2(G') \cong G'$ if G' is a cycle, $P_2(G)$ is a prolific graph if G is prolific. This means that to characterize prolific graphs it suffices to consider unicyclic graphs and trees.

Let G_0 and G_j be the graphs depicted in Figure 3, $j \geq 1$, where the length of $u - v$ path is $2j - 1$ in G_j.

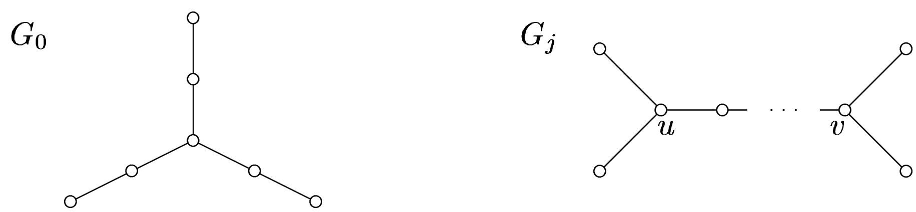

Figure 3

We have the following observation.

Observation 5. *The graph $P_2^i(G_0)$ is a cycle of length 6 for every $i \geq 1$; and if $j \geq 1$, then the nontrivial component of $P_2^i(G_j)$ is a cycle of length 4 for every $i \geq j$.*

A **kite** is a unicyclic graph composed of an even cycle $\mathcal{C}$ and a set of vertices, each joined by an edge to some vertex of $\mathcal{C}$. Moreover, each pair of vertices of a kite that have degree at least three, has an even distance (see Figure 4 for a kite with a cycle of length 8).

Lemma 6. *Let G be a connected unicyclic graph. Then G is prolific if and only if G is different from a cycle and a kite.*

Proof. It is easy to see that $P_2(G)$ is a cycle if G is a cycle. Analogously, $P_2(G)$ is a kite if G is a kite. Thus, neither a cycle, nor a kite, are prolific.

Suppose that G is different from a cycle and a kite. Let $\mathcal{C}$ be the unique cycle in G. Suppose that there is a vertex w in $\mathcal{C}$ such that $wu, uv \in E(G)$ for some $u, v \notin V(\mathcal{C})$ (i.e., G contains a path of length two glued to a vertex w in $\mathcal{C}$). Assume that

$\mathcal{C} = (w, w_1, w_2, \ldots, w_{r-2}, w_{r-1})$, and denote by H the subgraph of G with vertices $w, w_1, w_2, w_{r-2}, w_{r-1}, u, v$ and edges $ww_1, w_1w_2, ww_{r-1}, w_{r-1}w_{r-2}, wu, uv$ (we note that w_1, w_2, w_{r-2} and w_{r-1} are not necessarily distinct). By Observation 5, $P_2(H)$ contains a cycle of length six that is different from $P_2(\mathcal{C})$. Hence, G is a prolific graph.

Suppose that the cycle $\mathcal{C}$ in G has an odd length. If G is different from a cycle, there is a vertex w in C such that wv is an edge in G for some $v \notin V(\mathcal{C})$. Denote by H the subgraph of G with vertices $V(\mathcal{C}) \cup \{v\}$ and edges $E(\mathcal{C}) \cup \{wv\}$. Assume that the length of C is $2l - 1$. By Observation 5, $P_2^l(H)$ contains a cycle of length four that is different from $P_2^l(\mathcal{C})$, and hence, G is a prolific graph.

Finally, suppose that the cycle $\mathcal{C}$ in G has an even length. Moreover, suppose that there are vertices w_1 and w_2 in $\mathcal{C}$ such that $w_1v_1, w_2v_2 \in E(G)$ for some $v_1, v_2 \notin V(\mathcal{C})$ and $d_G(w_1, w_2) = 2l - 1$ for some $l \geq 1$. Let $\mathcal{T} = (w_1, u_1, u_2, \ldots, u_{2l-2}, w_2)$ be a shortest walk in G, and let u_0 and u_{2l-1} be vertices in $\mathcal{C}$ adjacent to w_1 and w_2, respectively, $u_0, u_{2l-1} \notin V(\mathcal{T})$. Denote by H the subgraph of G with vertices $V(\mathcal{T}) \cup \{v_1, v_2, u_0, u_{2l-1}\}$ and edges $E(\mathcal{T}) \cup \{w_1v_1, w_2v_2, w_1u_0, w_2u_{2l-1}\}$. By Observation 5, $P_2^l(H)$ contains a cycle of length four that is different from $P_2^l(\mathcal{C})$, and hence, G is a prolific graph. $\square$

Lemma 7. *Let G be a tree and $i > 0$. If $P_2^i(G)$ contains a cycle then G contains a subgraph isomorphic to G_j for some j, $0 \leq j \leq i$.*

Proof. First suppose that $P_2(G)$ contains a cycle $\mathcal{C}$. Since G is a tree, 1-histories of every pair of adjacent vertices in $\mathcal{C}$ form a path of length three in G. Moreover, there are three subsequential vertices a, b, c on $\mathcal{C}$ such that $A = (u_1, u, v)$, $B = (u, v, v_1)$ and $C = (u_2, u, v)$ for some $u_1, u_2, u, v, v_1 \in V(G)$. Let d be the vertex on $\mathcal{C}$ following c. Clearly, $d \neq a$. If $D = (u, v, v_2)$ for some vertex v_2 in G, then G contains G_1. Thus, suppose that $D = (w_2, u_2, u)$. Let x be a vertex of $\mathcal{C}$ preceding a. Clearly, $x \neq d$. If $X = (u, v, v_2)$, then G again contains a copy of G_1. On the other hand, if $X = (w_1, u_1, u)$, then G contains a copy of G_0.

Now suppose that H is a tree and $P_2(H)$ contains a copy of G_0 with central vertex a. Assume that $A = (u_1, u, v)$. As there are three vertices adjacent to a in G_0, without loss of generality we may assume that 1-histories of two of them, say b_1 and b_2, do not contain u_1. Hence, $B_1 = (u, v, v_1)$ and $B_2 = (u, v, v_2)$ for some vertices v_1, v_2 in H. Let c_1 be the endvertex of G_0 adjacent to b_1 and let c_2 be the endvertex of G_0 adjacent to b_2. If both C_1 and C_2 do not contain u, then H contains a copy of G_0. On the other hand, if this is not the case, H contains a copy of G_1. Hence, if G is a tree and $P_2^{i-1}(G)$ contains a copy of G_0, $i \geq 2$, then $P_2^{i-1}(G)$ contains a cycle.

Finally, suppose that H is a tree and $P_2(H)$ contains a copy of G_j, $j \geq 1$. Denote by a and b the vertices of degree three in G_j. Moreover, suppose that $P_2(H)$ is a tree. If two vertices, say u and v, are adjacent in $P_2(H)$, then the distance from any out-vertex of U to any out-vertex of V is odd. Since $P_2(H)$ is a tree, we can extend this observation to any pair of vertices in $P_2(H)$. Thus, if u and v are vertices with an odd distance in $P_2(H)$, then the distance from any out-vertex of U to any out-vertex of V is odd. As a and b have degree three, at least one out-vertex of A, say x, and at least one out-vertex of B, say y, have degree at least three in H. Since the distance from a to b is odd in $P_2(H)$, x and y have an odd distance in H. Moreover, since the distance from a to b is $2j - 1$ in $P_2(H)$, and $d_{P_2(H)}(a, b) = 2 + d_\bullet^1(a, b)$, by Lemma 2, we have $d_H(x, y) \leq 2 + d_\bullet^1(a, b) + 2 = 2j + 1$. Thus, H contains a copy of G_k, $1 \leq k \leq j+1$. Hence, if G is a tree and $P_2^i(G)$ contains a cycle, then G contains a copy of G_j, $0 \leq j \leq i$, as required. $\square$

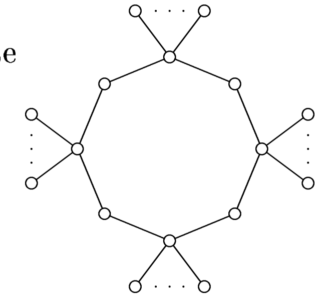

Figure 4

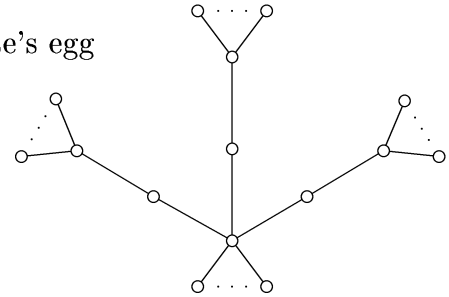

Figure 5

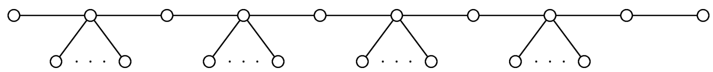

Figure 6

A **torn kite** is a tree composed of a diametric path $\mathcal{T}$ and a set of vertices, each joined by an edge to some vertex of $\mathcal{T}$. Moreover, each pair of vertices of a torn kite that have degree at least three, has an even distance (see Figure 6 for a torn kite with the diametric path of length 9).

Corollary 8. *Let G be a tree. Then $P_2^i(G)$ does not contain a cycle for all $i \geq 0$ if and only if G is a torn kite.*

Proof. By Observation 5, if $P_2^i(G)$ does not contain a cycle, then G does not contain G_j, $0 \leq j \leq i$. Since $P_2^i(G)$ does not contain a cycle for all $i \geq 0$, G does not contain G_j for all $j \geq 0$, and hence, G is a torn kite.

Now suppose that G is a torn kite. Then G does not contain G_j for all $j \geq 0$, and hence, $P_2^i(G)$ does not contain a cycle for all $i \geq 0$, by Lemma 7. $\square$

A **kite's egg** is a tree composed of a copy of G_0 and a set of vertices, each joined by an edge either to the central vertex or to some endvertex of G_0 (see Figure 5 for a kite's egg).

In the following theorem a characterization of prolific graphs is given.

Theorem 9. *Let G be a connected graph. Then G is prolific if and only if G is different from a cycle, a kite, a torn kite, a kite's egg, and G_j, $j \geq 1$.*

Proof. By Lemma 6, if G is a cycle or a kite, then G is not a prolific graph. Analogously, G is not a prolific graph if G is a torn kite, by Corollary 8. Since the P_2-path graph of kite's egg is a kite, kite's egg is not a prolific graph, too. Finally, since $P_2^i(G_j)$ is a 4-cycle for all $j \geq 1$ and $i > j$, by Observation 5, the graphs G_j are not prolific.

Now suppose that G is not a prolific graph. Then G does not contain two cycles. If G is a unicyclic graph, then G is either a cycle or a kite, by Lemma 6. Hence, suppose that G is a tree. If $P_2^i(G)$ does not contain any cycle for all $i \geq 0$, then G is a torn kite, by Corollary 8. Thus, suppose that $P_2^i(G)$ contains a cycle for some $i > 0$. Then G contains a copy of G_j, $0 \leq j \leq i$, by Lemma 7. Consider two cases.

(1) G contains a copy of G_j, $j \geq 1$.

Let k be the smallest index, $k \geq 1$, such that G contains a copy of G_k. Suppose that G is different from G_k. Since k is the smallest index, $k \geq 1$, such that G contains a copy of G_k, there are two cases to consider. Either there is a vertex from $V(G) - V(G_k)$ adjacent to an endvertex of G_k, or there is a vertex from $V(G) - V(G_k)$ adjacent to a vertex of degree three in G_k.

At first suppose that there is a vertex, say x_1, from $V(G) - V(G_k)$ adjacent to an endvertex of G_k in G. Then $P_2(G)$ contains a copy of G_{k-1} and a vertex, say x_2, adjacent to an endvertex of G_{k-1}. Thus, $P_2^{k-1}(G)$ contains a copy of G_1 and a vertex, say x_k, adjacent to an endvertex of G_1. Hence, $P_2^k(G)$ contains a cycle of length four, and a path of length two glued by one endvertex to the cycle. Since $P_2^k(G)$ is a unicyclic graph different from a cycle and a kite, G is a prolific graph, by Lemma 6, which contradicts our assumptions.

Now suppose that there is a vertex, say x_1, from $V(G) - V(G_k)$ adjacent to a vertex of degree three in G_k. Then $P_2(G)$ contains a copy of G_{k-1} and a vertex, say x_2, adjacent to a vertex of degree three in G_{k-1}. Thus, $P_2^{k-1}(G)$ contains a copy of G_1 and a vertex, say x_k, adjacent to a vertex of degree three in G_1. Hence, $P_2^k(G)$ contains a copy of $K_{3,2}$, that has two cycles. This means that G is a prolific graph which contradicts our assumptions.

Since there is not a vertex in $V(G)-V(G_k)$ adjacent to a vertex of degree two in G_k, G is isomorphic to G_k.

(2) G contains a copy of G_0, but it does not contain a copy of G_j, $j \geq 1$.

If there is a path of length two, glued by one endvertex to G_0, then $P_2(G)$ contains a unicyclic graph different from a cycle and a kite. Thus, G is a prolific graph, by Lemma 6, which contradicts our assumptions. Hence, G consists of a copy of G_0 and a collection of vertices adjacent to vertices of G_0. Moreover, as G does not contain a copy of G_1, G is a kite's egg. □

By Theorem 9, a connected graph is either a prolific graph, or a cycle, a kite, a torn kite, a kite's egg, or the graph G_j, $j \geq 1$.

Theorem 10. *Let G be a connected graph. Denote by H_j the unique nontrivial component of $P_2^j(G)$ (if it exists), $j \geq 0$.*

1. *If G is a prolific graph, then there is i_G such that for all $i \geq i_G$ we have $diam(H_i) = diam(H_{i-1}) + 2$;*
2. *if G is a cycle, or a kite, a kite's egg, or the graph G_j, $j \geq 1$, then there is i_G such that for all $i \geq i_G$ we have $diam(H_i) = diam(H_{i-1})$;*
3. *if G is a torn kite, then there is i_G such that for all $i \geq i_G$ the graph $P_2^i(G)$ is empty.*

Proof. Let G be a prolific graph. Then there is $i' \geq 0$ such that $P_2^{i'}(G)$ contains two cycles, say $\mathcal{C}$ and $\mathcal{D}$. However, $\mathcal{C}$ and $\mathcal{D}$ are not necessarily edge-disjoint. Let $\mathcal{T}$ be a longest path that have $\mathcal{C}$ and $\mathcal{D}$ in common, and let l be the length of $\mathcal{T}$. Since $P_2(\mathcal{T})$ is a path of length $l-2$ (or an empty graph if $l \leq 1$), the length of a longest path that have $P_2(\mathcal{C})$ and $P_2(\mathcal{D})$ in common is $l-2$. Hence, $H_{\lceil l/2 \rceil + i'}$ is a connected graph containing two edge-disjoint cycles. By Lemma 4, there is i_G such that for all $i \geq i_G$ $diam(H_i) = diam(H_{i-1}) + 2$.

If G is a cycle, then $P_2^i(G) \cong G$, so that $diam(P_2^i(G)) = diam(P_2^{i-1}(G))$ for all $i \geq 1$. By Observation 5, if $j \geq 1$ then $P_2^i(G_j)$ is a cycle of length four for every

$i > j$, and hence, $diam(P_2^i(G_j)) = diam(P_2^{i-1}(G_j))$ for all $i > j+1$. Now suppose that G is a kite with a cycle $\mathcal{C}$ of length $2l$. Since the case when G is a cycle is already solved, assume that there is a vertex v in G such that $v \notin V(\mathcal{C})$. As G is a kite, there is a vertex w in $\mathcal{C}$ adjacent to v. Let $\mathcal{C} = (u_1, u_2, w, u_4, u_5, \dots)$. Then a_1 and a_2 are endvertices in $P_2(G)$, $A_1 = (u_2, w, v)$ and $A_2 = (u_4, w, v)$. Moreover, a_1 is adjacent to b_1 in $P_2(\mathcal{C})$, $B_1 = (u_1, u_2, w)$, a_2 is adjacent to b_2 in $P_2(\mathcal{C})$, $B_2 = (u_5, u_4, w)$, and $d_{P_2(G)}(b_1, b_2) = 2$. Thus, $P_2^{l-1}(G)$ (and $P_2^i(G)$, $i \geq l$ as well) is a kite with a cycle of length $2l$, in which there are l vertices of degree two and l vertices of degree at least three. Moreover, these vertices are distributed alternatively around the cycle. Hence, $diam(H_i) = diam(H_{i-1})$ for all $i \geq l$. Finally, as $P_2(G)$ is a kite if G is a kite's egg, there is i_G such that $diam(H_i) = diam(H_{i-1})$ for all $i \geq i_G$.

Let G be a torn kite with a diametric path of length l. Since $P_2(G)$ is a torn kite with a diametric path of length $l-2$ (or an empty graph if $l \leq 1$), $P_2^i(G)$ is an empty graph if $i \geq \lfloor \frac{l}{2} \rfloor + 1$. $\square$

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