# A NOTE ON THE RADIUS OF ITERATED LINE GRAPHS

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Abstract. We prove that almost all $i$-iterated line graphs are selfcentric with radius $i+2$. This generalizes the well-known result that almost all graphs are selfcentric with radius two.



## Introduction

Let $G$ be a graph. Then by its line graph $L(G)$ we mean a graph whose nodes are the edges of $G$, and two nodes are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in $G$. We remark that if $G$ has no edges, then $L(G)$ is an empty graph. The $i$-iterated line graph of $G$, the $L^i(G)$, is $L(L^{i-1}(G))$ where $L^0(G) = G$ and $i \geq 1$. For an example of iterated line graphs see Figure 1.

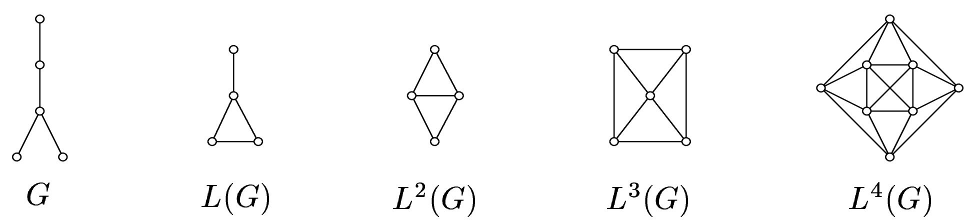


Figure 1

By $d(G)$ and $r(G)$ we denote the radius and the diameter of $D$, respectively. Let $G$ be a graph different from a path, a cycle, and a claw $K_{1,3}$. Then, as proved in [2], there are numbers $d_G$, $i_G$, $c_G$, and $c'_G$, such that

$$d(L^i(G)) = d_G + i \quad \text{for every} \quad i \geq i_G;$$
$$i - \sqrt{2\log_2 i} + c_G \leq r(L^i(G)) \leq i - \sqrt{2\log_2 i} + c'_G \quad \text{for every} \quad i \geq 0.$$

These results imply that if $G$ is not a path, a cycle, and a claw, then there is a number $s_G$ such that $d(L^i(G)) > r(L^i(G))$ for every $i \geq s_G$, i.e., the $L^i(G)$ is not selfcentric. In contrast with this we show that almost all $i$-iterated line graphs are selfcentric of radius $i+2$.

As a model of random graphs we use the well-established model of Erdős and Rényi, see [3, the model A]. In this model the node set of the graph is fixed, and

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each pair of nodes is joined by an edge with probability $p$, or left unjoined with probability $1-p$. A property is said to hold for *almost all graphs* if the limit of the probability that a random graph has the property is 1.

## Result

We will identify edges in a graph $G$ with the corresponding nodes in $L(G)$. Hence, if $u$ and $v$ are two adjacent nodes in $G$ then by $uv$ we mean an edge in $G$, as well as the node in $L(G)$ corresponding to the edge $uv$. This notation enables us to consider a node in $L^i(G)$, $i \geq 2$, as a pair of adjacent nodes in $L^{i-1}(G)$, either of these is a pair of adjacent nodes from $L^{i-2}(G)$, and so on. Furthermore, we can define each node in $L^i(G)$ using only edges of $G$, and such a definition will be called the *recursive definition of $v$ in $G$*.

Let $G$ be a graph and $v$ be a node in $L^i(G)$, $i \geq 1$. By the $j$-butt $B_j(v)$ of the node $v$ in $L^i(G)$ we mean a subgraph of $L^{i-j}(G)$ induced by the edges involved into the recursive definition of $v$. The butt we will abbreviate to $B(v)$ if $i = j$. We have:

**Lemma 1.** [2] *Let $H$ be a subgraph of a graph $G$. Then $H$ is an $i$-butt for some node in $L^i(G)$ if and only if $H$ is a connected graph with at most $i$ edges, distinct from any path with less than $i$ edges.*

The distance $d_G(H, J)$ between two subgraphs $H$ and $J$ of a graph $G$ equals to the length of a shortest path in $G$ joining a node from $H$ to a node from $J$. The following lemma enables us to compute distances between nodes in iterated line graphs:

**Lemma 2.** [2] *Let $G$ be a connected graph, and let $u$ and $v$ be distinct nodes in $L^i(G)$. Then*

(i) $d_{L^i(G)}(u, v) = i + d_G(B_i(u), B_i(v))$ *if the $i$-butts of $v$ and $u$ are edge-disjoint.*
(ii) $d_{L^i(G)}(u, v) = \max\{t$ : *$t$-butts of $u$ and $v$ are edge-disjoint*$\}$ *if $i$-butts of $u$ and $v$ have a common edge.*

For the diameter and the radius of line graphs we have:

**Lemma 3.** [1] *Let $G$ be a connected graph such that $L(G)$ is not empty. Then*

$$d(G) - 1 \leq d(L(G)) \leq d(G) + 1 \quad \textit{and}$$
$$r(G) - 1 \leq r(L(G)) \leq r(G) + 1\,.$$

Let $H$ consists of two node-disjoint triangles. Since almost all graphs contain a prescribed graph as an induced subgraph, see [3, p. 14], the $H$ is an induced subgraph of almost all graphs. Thus, $d(L^i(G)) \geq i + 2$ for almost all graphs $G$, by Lemma 1 and Lemma 2. From the other side for almost all graphs $G$ we have $d(G) = 2$, see [3, p. 14]. Thus, by Lemma 3 $d(L^i(G)) \leq i + 2$ for almost all graphs $G$, and hence $d(L^i(G)) = i + 2$ for almost all graphs. It means that the following theorem implies that almost all $i$-iterated line graphs are selfcentric:

**Theorem 4.** *Let $i \geq 0$. Then $r(L^i(G)) = i + 2$ for almost all graphs $G$.*

*Proof.* By $V(G)$ is denoted the node set of $G$; and by $e_G(u)$ we denote the eccentricity of the node $u$ in $G$, i.e., $e_G(u) = \max\{d_G(u, v) : v \in V(G)\}$.

Let $G$ be a graph on $n$ nodes, $n$ is sufficiently large, in which each edge appears with probability $p$, $0 < p < 1$. We give an upper bound for the probability $P(r(L^i(G)) \leq i+1)$, i.e. that the radius of $L^i(G)$ does not exceed $i+1$.

Let $H$ be a subgraph of $G$ on $m$ nodes. Then $V(H)$ can be partitioned into $\lfloor \frac{m}{3} \rfloor$ sets, each consisting of at least three nodes. Thus, for the probability $P_H$ that $H$ contains no triangle we have $P_H \leq (1-p^3)^{\lfloor \frac{m}{3} \rfloor}$.

Let $u \in V(L^i(G))$ such that $e_{L^i(G)}(u) \leq i+1$. The $B(u)$ contains at most $i+1$ nodes, by Lemma 1. Let $S \supseteq V(B(u))$ such that $|S| = i+1$. Since $e_{L^i(G)}(u) \leq i+1$, there is no $v \in V(L^i(G))$ such that $d_G(B(u), B(v)) \geq 2$, by Lemma 2. In particular, there is no triangle $T$ in $G$ such that $d_G(S,T) \geq 2$. Let $v \in V(G) \setminus S$. Then the probability that $d_G(S,v) \geq 2$ equals $(1-p)^{i+1}$. Thus, we have:

$$P(e_{L^i(G)}(u) \leq i{+}1) \leq \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \Big(1-(1-p)^{i+1}\Big)^{n-i-1-j} \Big((1-p)^{i+1}\Big)^j (1-p^3)^{\lfloor \frac{j}{3} \rfloor}$$

(here $j$ denotes the number of nodes $v$ such that $d_G(S,v) \geq 2$). Further,

$$\begin{aligned} P(e_{L^i(G)}(u) \leq i+1) < {} & \frac{1}{(1-p^3)} \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \Big(1-(1-p)^{i+1}\Big)^{n-i-1-j} \Big((1-p)^{i+1}\Big)^j \sqrt[3]{1-p^3}^{\,j} = \\ & \frac{1}{(1-p^3)} \Big(1-(1-p)^{i+1} + (1-p)^{i+1}\sqrt[3]{1-p^3}\Big)^{n-i-1} = \frac{1}{(1-p^3)} a_i^{n-i-1}. \end{aligned}$$

Since $(1-(1-p)^{i+1}+(1-p)^{i+1}) = 1$ and $0 < \sqrt[3]{1-p^3} < 1$, we have $0 < a_i < 1$.

Since each $B(u)$, $u \in V(L^i(G))$, is contained in a subgraph of $G$ induced by $i+1$ nodes, we have $P(r(L^i(G)) \leq i+1) < \frac{1}{(1-p^3)}\binom{n}{i+1} a_i^{n-i-1}$. Clearly $\lim_{n\to\infty} \frac{1}{(1-p^3)}\binom{n}{i+1} a_i^{n-i-1} = 0$, and hence $r(L^i(G)) \geq i+2$ for almost all graphs $G$. Since $r(G) = 2$ for almost all graphs $G$, see [3, p. 14], by Lemma 3 we have $r(L^i(G)) \leq i+2$ for almost all graphs $G$. $\square$

## References


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