EXTREMAL GRAPHS OF DIAMETER TWO AND GIVEN MAXIMUM DEGREE, EMBEDDABLE IN A FIXED SURFACE

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Abstract. It is known that for each d there exists a graph of diameter two and maximum degree d which has at least $\lceil\frac{d}{2}\rceil\lceil\frac{d+2}{2}\rceil$ vertices. In contrast with this, we prove that for every surface S there is a constant d_S such that each graph of diameter two and maximum degree $d \geq d_S$, which is embeddable in S, has at most $\lfloor\frac{3}{2}d\rfloor + 1$ vertices. Moreover, this upper bound is best possible, and we show that extremal graphs can be found among surface triangulations.

1. Introduction

Extremal graphs of given diameter and maximum degree have been a subject of investigation for many years. Although several deep results have been obtained in the field, many questions still remain open while new ones emerge. One of the most intriguing problems is that of determining the largest number, $f(d)$, of vertices in a graph of diameter two and maximum degree d. Trivially, such a graph can have at most d^2+1 vertices. However, it is non-trivial to show (cf. [4]) that this bound is attained only for $d = 1,\ 2,\ 3,\ 7$, and possibly 57 (the corresponding graphs are known as Moore graphs of diameter two). Results of [1] show that $f(d) \leq d^2 - 1$ for the remaining values of d. So far we do not have satisfactory lower bounds (see [3] for a table of bounds on $f(d)$ for $d \leq 15$); the *general* current record construction comes from forgetting arrows in line digraphs of complete digraphs, giving $f(d) \geq \lceil\frac{d}{2}\rceil\lceil\frac{d+2}{2}\rceil$ for each d.

In view of these facts, it seems reasonable to study extremal graphs of diameter two and maximum degree d subject to further restrictions. One of the possible ways was proposed in [2] where it is proved that a *planar* graph of diameter two and maximum degree $d \geq 8$ has at most $\lfloor\frac{3}{2}d\rfloor + 1$ vertices. In addition, as shown in [5], this bound is best possible in the following (strong) sense: For each $d \geq 8$ there exists a planar *triangulation* of diameter two and maximum degree d that contains exactly $\lfloor\frac{3}{2}d\rfloor + 1$ vertices.

The aim of this paper is to generalize the above results to graphs embedded in an arbitrary (orientable or nonorientable) surface, provided that d is large enough. More specifically, we prove:

Theorem 1. *For every surface S there exists a constant d_S such that the following holds: Every graph of diameter two and maximum degree $d \geq d_S$ which is embeddable in S contains at most $\lfloor \frac{3}{2}d \rfloor + 1$ vertices.*

This is, in a sense, a surprising result. As already mentioned, the number of vertices in a largest graph of diameter two and maximum degree d is quadratic (in d). However, if restricted to graphs embeddable in a fixed surface, we have a linear upper bound which, for sufficiently large d, does not depend on the surface at all and is the same as for the plane!

We also show that the bound of Theorem 1 is best possible, even in the class of *surface triangulations*:

Theorem 2. *Every closed surface S admits a triangulation of diameter two and maximum degree $d \geq d_S$ which contains exactly $\lfloor \frac{3}{2}d \rfloor + 1$ vertices.*

Proofs are presented in Sections 3 and 4; necessary background and auxiliary results can be found in Section 2.

2. Preliminaries

Graphs in our paper are finite, undirected, without loops or parallel edges. As usual, if G is a graph, the symbols $V(G)$ and $E(G)$ are reserved for the sets of vertices and edges of G, respectively. The degree of a vertex u in G is denoted by $deg_G(u)$; the symbol $dist_G(u,v)$ stands for the distance between the vertices u and v in G.

Let S be a closed surface (i.e., compact 2-manifold), and let G be a graph embedded in S. Components of $G \setminus S$ are called *faces* of the embedding; a face is *cellular* if it is homeomorphic to an open disc. If the embedding of G in S has f cellular faces, the well known Euler-Poincaré formula yields the inequality (cf. [6])

$$(0) \qquad |V(G)| - |E(G)| + f \geq \chi(S) \ ,$$

where $\chi(S)$ is the Euler characteristic of S. (For the sake of completeness we note that $\chi(S) = 2 - 2g$ if S is an orientable surface of genus g, and $\chi(S) = 2 - h$ if S is a nonorientable surface of crosscap number h.)

Let $D_k = D_k(x,y|t_i)$ be the graph with $V(D_k) = \{x, y, t_1, t_2, \ldots, t_k\}$ and $E(D_k) = \{xt_i, yt_i \ ; \ 1 \leq i \leq k\}$. The graph D_k will be called a *subdivided dipole* for obvious reasons: It can be obtained from a "dipole" on two vertices and k parallel edges by subdividing each edge once. If $M = \{t_i \ ; \ 1 \leq i \leq k\}$, we will occasionally use the abbreviated notation $D_k = D(x,y|M)$ if no confusion is likely.

For the purpose of our investigation, subdivided dipoles are of principal importance. We start with a simple observation.

Lemma 3. *Let G be a planar graph containing a subdivided dipole $D_k = D_k(x,y|t_i)$ as a subgraph. Let z be a vertex in $V(G) \setminus V(D_k)$ such that $dist_G(z,t_i) \leq 2$ for $1 \leq i \leq k$. If $k \geq 5$, then z is adjacent to x or to y in G.*

Proof. For the sake of simplicity we shall use the same symbols for graphs as well as for their plane embeddings. Let G be a plane embedding of our graph and let D_k be the induced embedding of the subdivided dipole. Since $k \geq 5$, there is a sequence of quadrilateral faces F_1, F_2, F_3, F_4 of D_k such that none of them is an outer (i.e., unbounded) face and, for $j = 1,2,3$, the boundary cycles of F_j and

F_{j+1} share a path of length two. (For instance, we may without loss of generality assume that the boundary cycle of F_j is xt_jyt_{j+1}, $1 \le j \le 4$.) Let z be a vertex not in the subdivided dipole, with $dist_G(z, t_i) \le 2$, $1 \le i \le k$. It is easy to check that no matter where z appears in the embedding (that is, either inside one of the quadrilaterals F_j or somewhere outside), the only way to meet the distance requirement is to have z joined either to x or to y. $\square$

As the reader may have observed, the key to the above proof is the existence of a sequence of four quadrilateral faces in a drawing of the subdivided dipole within a *bounded* planar region, such that boundary cycles of consecutive faces share a path of length two. Let us call such a configuration a *4-fan*. We now generalize this principle to graphs embedded in an arbitrary closed surface.

Lemma 4. *Let G be a graph embedded in a closed surface S, and let $k \ge 5$ if S is a sphere, and $k \ge 17 - 8\chi(S)$ otherwise. Assume that the subdivided dipole $D_k = D_k(x, y|t_i)$ is a subgraph of G. Let $z \in V(G) \setminus V(D_k)$ be such that $dist_G(z, t_i) \le 2$ for $1 \le i \le k$. Then either x or y is adjacent to z in G.*

Proof. We may assume that S is not a sphere (cf. Lemma 3). Let G be a graph with the above properties, embedded in S, and let D_k be the induced embedding of the subdivided dipole.In this embedding of D_k (which need not be cellular), let a be the total number of 2-cell faces and let a_4 be the number of 2-cell faces that are bounded by quadrilaterals. As there are $2k$ edges and $k+2$ vertices involved in D_k, by (0) we have $(2+k) - 2k + a \ge \chi(S)$; that is, $a \ge k + \chi(S) - 2$. Since the length of a boundary walk in D_k is always a multiple of four, a standard counting argument (the sum of lengths of all boundary walks gives twice the number of edges) yields $4k \ge 4a_4 + 8(a - a_4) = 8a - 4a_4 \ge 8(k + \chi(S) - 2) - 4a_4$; i.e., $a_4 \ge k + 2(\chi(S) - 2)$.

Let us now have a close look at the induced embedding D_k on S. Consider a small neighbourhood of x with a fixed local orientation. We may assume, without loss of generality, that this local orientation induces the following cyclic ordering of edges of D_k emanating from x: $(xt_1, xt_2, \ldots, xt_k)$. For $1 \le i \le k$, let F_i be the face of D_k whose facial walk contains the path t_ixt_{i+1} (indices mod k). It may happen that $F_i = F_j$ if $i \ne j$; also, some of these faces may be non-cellular (the union of all F_i's is $S \setminus D_k$). Nevertheless, a_4 of them *must be* cellular and bounded by quadrangles. Since $k > 8(2 - \chi(S))$, we have $2(\chi(S) - 2) > -\frac{1}{4}k$, and so $k + 2(\chi(S) - 2) > \frac{3}{4}k$, which (combined with the inequality at the end of the last paragraph) gives $a_4 > \frac{3}{4}k$. Applying the pigeonhole principle (note that we always have $k \ge 5$) we see that there is a collection of four consecutive cellular faces F_j, F_{j+1}, F_{j+2}, F_{j+3} (subscripts $\bmod k$), all bounded by quadrangles. The union of these four faces is homeomorphic to a (topological) disc, which gives rise to a 4-fan configuration (see the remark after Lemma 3).

The rest of the proof is similar to that of Lemma 3: If $z \in V(G) \setminus V(D_k)$, no matter which "portion", F_i, of the surface S it appears in, the only way to fulfil the distance requirements is to have z joined to at least one of x, y (thanks to the existence of the planar 4-fan configuration in S). $\square$

Our last auxiliary result concerns dipoles where paths of length two are replaced by paths of length three. Formally, let $D_k^* = D_k^*(x, y|x_i, y_i)$ be the graph with the vertex set $V(D_k^*) = \{x, y, x_1, y_1, \ldots, x_k, y_k\}$ and the edge set $E(D_k^*) = \{xx_i, x_iy_i, y_iy\ ;\ 1 \le i \le k\}$; this graph will be referred to as a *twice-subdivided dipole.*

Lemma 5. *Let G be a graph embedded in a closed surface S, and let $k \geq 5$ if S is a sphere and $k \geq 17 - 8\chi(S)$ otherwise. Assume that G contains a twice-subdivided dipole $D_k^*(x, y|x_i, y_i)$ as a subgraph, and let $yx_i \notin E(G)$ for each i, $1 \leq i \leq k$. Let $z \in V(G) \setminus V(D_k^*)$ be a vertex for which $dist_G(z, x_i) \leq 2$, $1 \leq i \leq k$. Then the vertices z and x must be adjacent in G.*

Proof. Let G be embedded in S and let D_k^* be the induced embedding of the twice-subdivided dipole. We thus have a situation similar to that in the proof of Lemma 4; the only (and inessential) difference being that dipole paths have length three. The previous proof now implies the existence of four consecutive cellular faces incident to x, bounded by hexagons and successively sharing paths of length three; without loss of generality we may assume the notation to be chosen so that these faces are $F_1, \ldots, F_4$ and their face walks are $(xx_iy_iyy_{i+1}x_{i+1})$, $1 \leq i \leq 4$. Because of the distance condition for the vertex $z \in V(G) \setminus V(D_k^*)$ and the absence of edges yx_i in G, it is routine to check that the vertices x and z must be adjacent (no matter where z appears - either inside one of the F_i's or inside other faces on S). □

3. Proof of Theorem 1

Throughout, let $\sigma = 5$ if S is a sphere, and $\sigma = 17 - 8\chi(S)$ otherwise. Let $d_S = 4(\sigma - 1)^2 + 2$. Assume the contrary and let G be a graph with the following properties:

(1) G has diameter two ,
(2) the maximum degree d_G of G is at least d_S ,
(3) G is embeddable in S , and
(4) $|V(G)| \geq \lfloor \frac{3}{2} d_G \rfloor + 2$.

For each $x \in V(G)$ let $N_i(x) = \{y \in V(G) : \; dist_G(x, y) = i\}$. It follows from (4) that $|N_2(x)| \geq \lfloor \frac{1}{2} d_G \rfloor + 1$ for each $x \in V(G)$.

We first derive some useful facts about the structure of our counterexample G and then estimate its number of vertices in order to obtain a contradiction with (4).

Fix a vertex u in G of degree d_G and choose a vertex $v \in N_2(u)$ in such way that the cardinality of the set $A = N_1(u) \cap N_1(v)$ is largest possible, say, q. Consider now an arbitrary vertex $x \in N_1(u)$. Then, if $x \notin A$, there is a vertex y such that $vy, yx \in E(G)$ (because the diameter of G is two) and either $y \in N_1(u)$ or $y \in N_2(u)$. Accordingly, we partition the set $N_1(u) \setminus A$ into sets $B_1, \ldots, B_m, C_1, \ldots, C_n$ for some m and n, subject to the following requirements:

(5) for $1 \leq i \leq m$ the sets $B_i = \{b_{i,1}, \ldots, b_{i,k_i}\}$ are such that there are m distinct vertices $r_1, \ldots, r_m \in A$ for which $r_i b_{i,j} \in E(G)$, $1 \leq j \leq k_i$;
(6) for $1 \leq i \leq n$ the sets $C_i = \{c_{i,1}, \ldots, c_{i,l_i}\}$ are such that there are n distinct vertices $s_1, \ldots, s_n \in N_2(u) \cap N_1(v)$ for which $s_i c_{i,j} \in E(G)$, $1 \leq j \leq l_i$.

For technical reasons we assume that $k_1 \geq k_2 \geq \cdots \geq k_m$ and $l_1 \geq \cdots \geq l_n$. (Note that a partition of $N_1(u) \setminus A$ with all the above properties need not be unique.)

We begin our analysis of the graph G by proving that both m and n are $\leq \sigma - 1$. Recalling that $|N_2(u)| \geq \lfloor \frac{1}{2} d_G \rfloor + 1$, let $y \in N_2(u)$, $y \neq v$. Now, if (say) $m \geq \sigma$ then Lemma 5 applied to the twice-subdivided dipole $D_m^*(u, v|b_{i,1}, r_i)$ in the graph G of diameter two shows that $yu \in E(G)$, contrary to the choice of y. Analogously, if $n \geq \sigma$ then Lemma 5 applied to $D_n^*(u, v|c_{i,1}, s_i)$ yields a contradiction. (The only case which needs care occurs when $N_2(u) = \{v\} \cup \{s_1, \ldots, s_n\}$; but then

$n \geq \frac{d_S}{2} > \sigma$ and we may apply Lemma 5 to $D^*_{n-1}(u,v|c_{i,1},s_i)$ and choose $y = s_n$.) Thus, $m \leq \sigma - 1$ and $n \leq \sigma - 1$.

As the next step we show that at least one of the numbers $q = |A|$ and k_1 is at least σ. Indeed, suppose that both are at most $\sigma - 1$. By the choice of the set A we have $q \geq l_1$ $(\geq l_2 \geq \dots)$. Therefore, taking into account that $m, n \leq \sigma - 1$, we obtain $d_G = |N_1(u)| = q + \sum_{i \leq m} |B_i| + \sum_{i \leq n} |C_i| \leq q + mk_1 + nl_1 \leq (1+n)q + mk_1 \leq (\sigma - 1)(2\sigma - 1)$, which contradicts our assumption that $d_G \geq d_S$. So, either $q \geq \sigma$ or $k_1 \geq \sigma$. We now investigate these two cases separately.

Case 1: $q \geq \sigma$. Let $A = \{a_i \ ; \ 1 \leq i \leq q\}$. Applying Lemma 4 to the subdivided dipole $D_q(u,v|A)$ we see that $yv \in E(G)$ for every $y \in N_2(u)$, $y \neq v$. Our aim is to show that in this case $l_1 \leq \sigma - 1$ whereas $k_1 \geq \sigma$.

Suppose first that $l_1 \geq \sigma$. Invoking Lemma 4 again, this time to $D_{l_1}(u, s_1|C_1)$, it follows that $ys_1 \in E(G)$ for each $y \in N_2(u)$, $y \neq v$ and $y \neq s_1$. The above facts imply that on the set $N_2(u)$ we have a subdivided dipole $D(v, s_1|N_2(u) \setminus \{v, s_1\})$ with $|N_2(u) \setminus \{v, s_1\}| \geq \lfloor \frac{1}{2} d_G \rfloor - 1 > \sigma$. Lemma 4 then implies that either $us_1 \in E(G)$ or $uv \in E(G)$, which is absurd. Thus, $l_1 \leq \sigma - 1$.

Now, suppose that $k_1 \leq \sigma - 1$. Recalling that $m, n, l_1 \leq \sigma - 1$, we have $|N_1(u) \setminus A| = \sum_{i \leq m} k_i + \sum_{i \leq n} l_i \leq mk_1 + nl_1 \leq 2(\sigma - 1)^2$. On the other hand, since $|N_2(u)| \geq \lfloor \frac{1}{2} d_G \rfloor + 1$, we have $|N_1(u) \setminus A| = deg_G(u) - q \geq deg_G(v) - q = (|N_2(u)| - 1 + q) - q \geq \lfloor \frac{1}{2} d_G \rfloor$. Combining the two inequalities for $|N_1(u) \setminus A|$ we obtain $\lfloor \frac{1}{2} d_G \rfloor \leq 2(\sigma - 1)^2$, which contradicts the choice of d_G. Therefore $k_1 \geq \sigma$, as claimed.

We are now ready to finish the analysis of *Case 1.* With help of Lemma 4 applied to the subdivided dipole $D_{k_1}(u, r_1|B_1)$ we deduce that $yr_1 \in E(G)$ for each $y \in N_2(u)$. But we already know (see the beginning of *Case 1*) that $yv \in E(G)$ for each $y \in N_2(u)$, $y \neq v$. We therefore have a subdivided dipole $D(r_1, v|N_2(u) \setminus \{v\})$, with $|N_2(u) \setminus \{v\}| \geq \sigma$. Lemma 4 now implies that for each $x \in N_1(u)$, $x \neq r_1$ we have either $xv \in E(G)$ or $xr_1 \in E(G)$; in particular, $xr_1 \in E(G)$ for each $x \in N_1(u) \setminus A$. Recalling that for each $y \in N_2(u) \setminus \{v\}$ we have $yv, yr_1 \in E(G)$, we successively obtain $2d_G \geq deg_G(v) + deg_G(r_1) \geq (|N_2(u)| - 1 + q) + (|N_2(u)| + |N_1(u)| - q + 1) = |V(G)| + |N_2(u)| - 1 \geq |V(G)| + \lfloor \frac{1}{2} d_G \rfloor$. It follows that $|V(G)| \leq 2d_G - \lfloor \frac{1}{2} d_G \rfloor \leq \lfloor \frac{3}{2} d_G \rfloor + 1$, which contradicts (4).

Case 2: $k_1 \geq \sigma$; we may now assume that $q \leq \sigma - 1$. Applying Lemma 4 to the subdivided dipole $D(u, r_1|B_1)$ whose "middle layer" is formed by the k_1 vertices in B_1, we have $yr_1 \in E(G)$ for each $y \in N_2(u)$. Since $d_G \geq deg_G(r_1) \geq |N_2(u)| + k_1 + 1 \geq \lfloor \frac{1}{2} d_G \rfloor + k_1 + 2$, it follows that $|N_1(u) \setminus B_1| = d_G - k_1 \geq \lfloor \frac{1}{2} d_G \rfloor + 2$. Now, if $0 \leq k_2 \leq \sigma - 1$, then $|N_1(u) \setminus B_1| = \sum_{i=2}^{m} k_i + \sum_{j=1}^{n} l_j + q \leq (m-1)k_2 + nl_1 + q \leq (m-1)k_2 + (n+1)q \leq (m+n)(\sigma - 1) \leq 2(\sigma - 1)^2$, and so $\lfloor \frac{1}{2} d_G \rfloor + 2 \leq 2(\sigma - 1)^2$, contrary to our choice of $d_G \geq d_S$. Therefore, $k_2 \geq \sigma$.

We may now apply Lemma 4 to $D_{k_2}(u, r_2|B_2)$ which gives $yr_2 \in E(G)$ for each $y \in N_2(u)$. But then there is the subdivided dipole $D(r_1, r_2|N_2(u))$ to which Lemma 4 applies; as the result we have the fact that for each $x \in N_1(u) \setminus \{r_1, r_2\}$, either $xr_1 \in E(G)$ or $xr_2 \in E(G)$. Summing up, we obtain $2d_G \geq deg_G(r_1) + deg_G(r_2) \geq 2|N_2(u)| + (|N_1(u)| - 2) + 2 = |V(G)| + |N_2(u)| - 1 \geq |V(G)| + \lfloor \frac{1}{2} d_G \rfloor$, and hence $|V(G)| \leq 2d_G - \lfloor \frac{1}{2} d_G \rfloor \leq \lfloor \frac{3}{2} d_G \rfloor + 1$. This final contradiction completes the proof of Theorem 1. $\square$

4. Proof of Theorem 2

Consider the graph G in Fig. 1; this graph was constructed in [2] to show that $\lfloor\frac{3}{2}d_G\rfloor + 1$ is the best possible upper bound for the number of vertices in a *planar* graph of diameter two and maximum degree d_G. It follows from Theorem 1 that the very same graph can serve as extremal graph for an *arbitrary* fixed surface S, provided that $d_G \geq d_S$. However, our goal is to construct a *triangulation* of S with the required properties. This will be done by a suitable modification of the graph G. (Observe that $deg_G(b_1) = deg_G(b_2) = d_G$, and $deg_G(b_3) = d_G$ or $d_G - 1$ according as d_G is even or odd; the remaining vertices have degree two.)

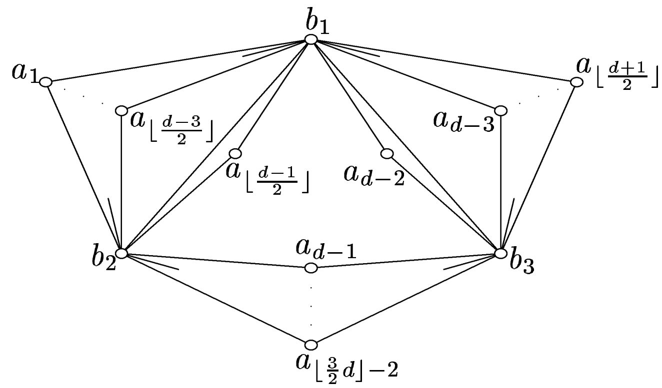

Fig. 1: The graph G.

Let S be a surface other than the sphere (for the "spherical" result we refer to [5]) and let G be as in Fig. 1 with $d_G = d \geq d_S$, where d_S is taken from the preceding proof. Let $r = 8 - 3\chi(S) = 2 + \frac{3}{8}(\sigma - 1)$. Since $\chi(S) \leq 1$, we have $\sigma \geq 9$, and then $2 + \frac{3}{8}(\sigma-1) < 2(\sigma-1)^2 - 3 < \frac{d_S-7}{2}$; hence $r < \lfloor\frac{d-7}{2}\rfloor$. Let the graph G' be obtained from G by deleting the vertices $a_1, \ldots, a_r$. Now, we add to G' the edges $a_i a_{i+1}$ for $r+2 \leq i \leq \lfloor\frac{d-5}{2}\rfloor$, $\lfloor\frac{d+1}{2}\rfloor \leq i \leq d-4$, and $d-2 \leq i \leq \lfloor\frac{3}{2}d\rfloor - 3$. Further, we add six more edges, namely, $a_{d-2}a_{\lfloor\frac{d-1}{2}\rfloor}$, $a_{\lfloor\frac{d-1}{2}\rfloor}a_{d-1}$, $a_{r+1}a_{\lfloor\frac{d+1}{2}\rfloor}$, $a_{r+1}a_{\lfloor\frac{3}{2}d\rfloor-2}$, $a_{\lfloor\frac{3}{2}d\rfloor-2}a_{\lfloor\frac{d+1}{2}\rfloor}$, and $a_{d-1}a_{d-2}$, and denote the resulting graph by H, see Fig. 2. (The added edges are drawn thin.)

It is obvious that H is a planar graph with maximum degree d, and all but one of its faces are bounded by triangles, the exceptional face being quadrilateral (the outer face in Fig. 2). Later in the construction, we shall only work with the (closed) topological disc bounded by the quadrangle $(b_1 a_{r+2} b_2 a_{r+1})$; that is, we shall remove the outer face of the embedding of H in Fig. 2. Note that $deg_H(b_1) = deg_H(b_2) = d - r$, $deg_H(b_3) \leq d$, and all remaining vertices of H have degree at most five.

Let $m = 3 - \chi(S)$ and let $D_m = D_m(x, y|c_i)$ be a subdivided dipole whose "middle layer" is formed by vertices $c_1, \ldots, c_m$. It is well known (and easy to see) that D_m has a 2-cell embedding in S with exactly one face, bounded by a closed walk of length $4m$. In terms of vertices, the boundary walk has the form $(x, c_{i_1}, y, c_{i_2}, x, c_{i_3}, y, c_{i_4}, \ldots, x, c_{i_{2m-1}} y, c_{i_{2m}})$, where $\{i_1, i_2, \ldots, i_{2m}\} = \{1, 2, \ldots, m\}$ and each i, $1 \leq i \leq m$, appears exactly twice among the subscripts i_j. We therefore represent this face as a $4m$-gon with vertices labelled x, c_{i_1}, y, c_{i_2}, etc., with the appropriate pairs of sides identified. Inside this $4m$-gon we now insert $2m+1$ new vertices $z_0, z_1, \ldots, z_{2m}$, as well as the following $10m - 1$ new edges (see Fig. 3; we remark that the new edges are again pictured with thin lines): $z_j x, z_j y$ $(0 \leq j \leq$

$2m$); $z_0 c_{i_1}$; $z_1 z_j, z_j c_{i_j}$ $(2 \leq j \leq 2m)$; $z_j z_{j+1}$ $(2 \leq j \leq 2m-1)$. We thus obtain an embedding of a graph (say, K) in S where all faces are triangular, except one (the shaded quadrangle in Fig. 3). Note that K contains neither parallel edges nor loops, $|V(K)| = 3m+3$, $deg_K(x) = deg_K(y) = 3m+1$, $deg_K(z_1) = 2m+1$, and all other vertices of K have degree at most 6.

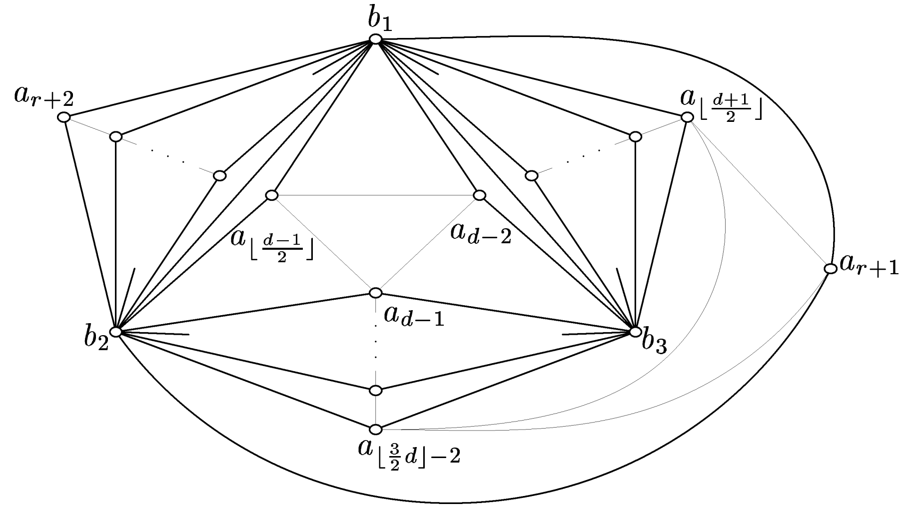

Fig. 2: The disc embedding of the graph H.

As the last step, we replace the shaded quadrangle in Fig. 3 by the topological disc from Fig. 2 bounded by the quadrilateral $(b_1 a_{r+2} b_2 a_{r+1})$ in such way that we identify vertices $b_1, a_{r+2}, b_2, a_{r+1}$ with x, z_0, y, z_1, respectively (the corresponding edges are identified as well). We thus obtain a triangulation T of the surface S, with no loops or parallel edges. It is easy to check that the diameter of T is two. The degree of the vertex $b_1 \equiv x$ in T is equal to $deg_H(b_1) + deg_K(x) - 2 = (d-r) + (3m+1) - 2 = d$; the same holds for the vertex $b_2 \equiv y$. The degree of $z_1 \equiv a_{r+1}$ in T does not exceed $deg_H(a_{r+1}) + deg_K(z_1) - 2 = 2m+3 < d$, and the remaining vertices have degree ≤ 6 in T. Finally, $|V(T)| = |V(H)| + |V(K)| - 4 = |V(G)| - r + |V(K)| - 4 = (\lfloor \frac{3}{2}d \rfloor + 1) - (3m-1) + (3m+3) - 4 = \lfloor \frac{3}{2}d \rfloor + 1$, and hence T is a triangulation of S with the required properties. $\square$

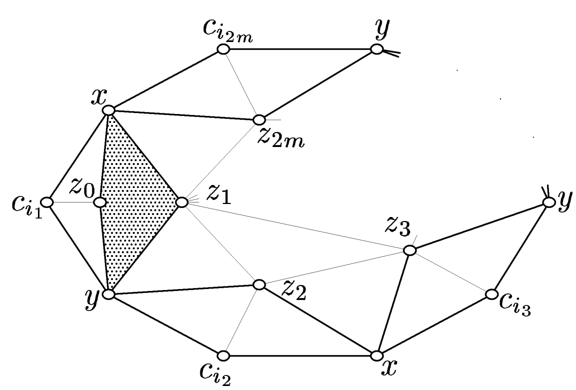

Fig. 3: The graph K.

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