A NOTE ON RADIALLY MOORE DIGRAPHS

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Abstract. Let D be a regular digraph with radius s. Then D is a radially Moore digraph if it has the maximum possible number of nodes and the diameter of D does not exceed $s+1$. We show that for each s and t there exists a regular radially Moore digraph of degree t with radius s. Moreover, we give an upper bound for the number of central nodes in radially Moore digraphs with degree two.

Regular digraphs with small distances are especially well suited for designing building-block switching systems, communication networks, and distributed computer systems. An ideal structure would yield a regular digraph with prescribed diameter on the theoretically maximum possible number of nodes. However, such digraphs are very rare. Moreover, it is not always necessary to have small distances from (or to) all nodes. Hence, it seems to be reasonable to investigate regular digraphs with prescribed radius on the theoretically maximum possible number of nodes, in which a given subset of nodes lies in the center. In this paper such digraphs are constructed for one central node. Moreover, it is shown that those of degree two have less than a half of their nodes in the center.

Let D be a digraph, i.e. a directed graph without loops or multiple arcs. As usual, by $V(D)$ we denote the node set of D and by $E(D)$ the arc set of D. An arc from u to v is denoted by (u,v). By $id_D(u)$ we denote the input degree and by $od_D(u)$ the output degree of a node $u \in V(D)$. If $u, v \in V(D)$ then $d_D(u,v)$ denotes the length of a shortest path from u to v in D. Let D be a digraph and $u \in V(D)$. Then:

out-eccentricity of the node u is $\quad e_D^+(u) = \max_{v\in V(D)} (d_D(u,v))\,;$

in-eccentricity of the node u is $\quad e_D^-(u) = \max_{v\in V(D)} (d_D(v,u))\,;$

eccentricity of the node u is $\quad e_D(u) = \max\,(e_D^+(u), e_D^-(u))\,.$

Using various eccentricities we obtain various radii and various centers. The *radius* $r(D)$ (*out-radius* $r^+(D)$, *in-radius* $r^-(D)$) is the minimum value of $e_D(u)$ ($e_D^+(u)$, $e_D^-(u)$), $u \in V(D)$. The nodes with the minimum eccentricity (out-eccentricity, in-eccentricity) are called *central* (*out-central*, *in-central*); and the set of central (out central, in-central) nodes is denoted by $C(D)$ ($C^+(D)$, $C^-(D)$). We remark that the diameter of D, $d(D)$, is the maximum value of $e_D^+(u)$, $u \in V(D)$.

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A digraph D is regular of degree t if $id_D(u) = od_D(u) = t$ for each $u \in V(D)$. Clearly, a regular digraph D of degree t with diameter s contains at most

$$M_{s,t} = 1 + t + t^2 + \cdots + t^s$$

nodes. If D has exactly $M_{s,t}$ nodes, then D is called a *Moore digraph*. It is known that Moore digraphs exist only for $s = 1$ or $t = 1$, namely a complete digraph K_{t+1} and a directed cycle C_{s+1}, respectively (see [2]). The question of how "close" to the *Moore bound* $M_{s,t}$ we can go if $s > 1$ and $t > 1$ is extensively studied by several authors (see e.g. [1], [3], and [4]).

We present here another attempt to this problem. Namely, we decrease the strong condition that the diameter equals s, instead of decreasing the number of nodes. Clearly, a regular digraph of degree t with radius s contains at most $M_{s,t}$ nodes. Hence, we can ask whether there are regular digraphs of degree t on $M_{s,t}$ nodes with diameter at most $s+1$ and the radius s. If a digraph satisfy all these conditions, it will be called a *radially Moore digraph*.

In what follows we show that radially Moore digraphs exist for each positive s and t:

Theorem 1. *Let s and t be positive integers. Then there exists a radially Moore digraph of degree t with radius s.*

Proof. We construct a digraph D, and then we show that D has the required properties.

Let $V(D) = \{\emptyset\} \cup \{e_1e_2\dots e_{s'} : 1 \le s' \le s, \text{ and } 1 \le e_i \le t,\ 1 \le i \le s'\}$. Thus, $V(D)$ consists of strings from t symbols, each of length at most s. Hence $|V(D)| = M_{s,t}$. In what follows let $l(a)$ denote the length of string $a \in V(D)$. Moreover, we define $\bar{a} = e_2 \dots e_{s'}$ if $a = e_1e_1\dots e_1e_2\dots e_{s'}$, $e_2 \ne e_1$. Let

$$\begin{aligned} E(D) = &\{(a, ae) : l(a) \le s{-}1, \text{ and } 1 \le e \le t\} \\ &\cup \{(a, \bar{a}e) : l(a) = s,\ \bar{a} \ne \emptyset, \text{ and } 1 \le e \le t\} \\ &\cup \{(a, e) : l(a) = s,\ \bar{a} = \emptyset \text{ i.e. } a = e_1\dots e_1, \text{ and } 1 \le e \le t,\ e \ne e_1\} \\ &\cup \{(a, \emptyset) : l(a) = s, \text{ and } \bar{a} = \emptyset\}. \end{aligned}$$

Clearly, $od_D(a) = t$ if $a \in V(D)$. Let $a = e_1e_2\dots e_{s'} \in V(D)$, $a \ne \emptyset$. Then $(e_1e_2\dots e_{s'-1}, a) \in E(D)$. Moreover, $(b, a) \in E(D)$ for each $b = e\dots ee_1e_2\dots e_{s'-1}$, $e \ne e_1$, $1 \le e \le t$, and $l(b) = s$. Hence, $id_D(a) = t$ if $a \ne \emptyset$. Clearly also $id_D(\emptyset) = t$, and hence D is regular of degree t.

Consider the following system of sets of nodes of D. Let

$$S_i = \{S_i^{\triangle}(e), S_i^{-}(a) : 1 \le e \le t,\ l(a) \le s{-}i,\ a \ne \emptyset\},$$

where $0 \le i \le s$ and

$$\begin{aligned} S_i^{\triangle}(e) &= \{eb : l(b) \le i{-}1\}, \text{ and} \\ S_i^{-}(a) &= \{ab : l(b) = i\}. \end{aligned}$$

Thus, $S_i^{\triangle}(e)$ consists of all nodes of D that are at distance at most $i-1$ from e, $l(e) = 1$, and $S_i^{-}(a)$ consists of all nodes of D that are at distance exactly i

from a, $1 \le l(a) \le s - i$. Clearly, $S_0^{\triangle}(e) = \emptyset$ and $S_s^{-}(a) = \emptyset$, $a \neq \emptyset$, and hence, $S_0 = \{S_0^{-}(a) : a \in V(D) - \{\emptyset\}\} = V(D) - \{\emptyset\}$ and $S_s = \{S_s^{\triangle}(e) : 1 \le e \le t\}$.

Let $A \subseteq V(D)$. Denote $N(A) = A \cup \{b \in V(D) : (a, b) \in E(D)$ for some $a \in A\}$. We show that for each $A \in S_i$, $0 \le i < s$, there is $B \in S_{i+1}$ such that $N(A) \supseteq B$. Distinguish the following cases:

(1) $A = S_i^{\triangle}(e)$. Since $i < s$, the length of each string in A is less than s. Hence, $N(A) = S_{i+1}^{\triangle}(e)$.
(2) $A = S_i^{-}(a)$ and $l(a) < s-i$. Then $N(A) \supset S_{i+1}^{-}(a)$.
(3) $A = S_i^{-}(a)$, $l(a) = s-i$, and $a = ee\dots e$. Then $N(A) \supset S_{i+1}^{\triangle}(e')$ for each $e' \neq e$, $1 \le e' \le t$.
(4) $A = S_i^{-}(a)$, $l(a) = s-i$, and $a = e\dots ee_1\dots e_{s'}$, $e_1 \neq e$. Then $N(A) \supset S_{i+1}^{-}(\bar{a})$, $\bar{a} = e_1 \dots e_{s'}$.

Hence, for each $A \in S_i$ there is $B \in S_{i+1}$ such that $N(A) \supseteq B$. Since $S_0 = V(D) - \{\emptyset\}$ and $N(S_s^{\triangle}(e)) = V(D)$ for each e, $1 \le e \le t$, we have $e_D^{+}(a) \le s+1$ for each $a \in V(D) - \{\emptyset\}$. Moreover, it is easy to see that $e_D^{+}(\emptyset) = s$, and hence $d(D) \le s+1$.

Finally, we show $e_D^{-}(\emptyset) = s$. Clearly, if $(b, e) \in E(D)$, $1 \le e \le t$, then $(b, \emptyset) \in E(D)$ as well. Let $a \in V(D)$. Since there is e, $1 \le e \le t$, such that $1 \le d_D(a, e) \le s$, we have $e_D^{-}(\emptyset) \le s$. Thus $e_D(\emptyset) \le s$, and hence $r(D) = s$. $\square$

We remark that in the case $s = 1$ or $t = 1$ we obtain the Moore digraphs K_{t+1} and C_{s+1}, respectively. However, in general we have $|C(D)| = 1$ and $|C^{+}(D)| = |C^{-}(D)| = t+1$, since it is easy to see that the out-neighbors of $\emptyset$ are in $C^{+}(D)$, and the in-neighbors of $\emptyset$ are in $C^{-}(D)$. It can be interesting to find radially Moore digraphs with more central nodes, since one can expect that such digraphs would have properties more close to Moore digraphs. However, we show that $|C(D)| < \frac{1}{2}|V(D)|$ if D is a radially Moore digraph with degree 2.

In what follows, by out-V we denote a digraph on three nodes, say x, y, and z, with exactly two arcs (x, y) and (x, z). By in-V is denoted a digraph obtained from out-V by reversing the arcs. In the proof of Theorem 3 we use the following lemma:

Lemma 2. *Let D be a regular digraph of degree 2 on $M_{s,2}$ nodes with out-radius s, $s > 1$. Then the subgraph of D induced by out-central nodes consists of a couple of isolated nodes and out-V's.*

Proof. Let D be a digraph satisfying the assumptions of lemma, and let $u, v \in C^{+}(D)$, $(u, v) \in E(D)$. Since D is regular of degree 2 on $M_{s,2}$ nodes and $u \in C^{+}(D)$, there is exactly one $u - x$ path of length at most s for each $x \in V(D)$. Thus, the nodes of D can be associated with 0-1 strings, each of length at most s, such that $u = \emptyset$, $v = 0$, and if $l(a) < s$ then $(a, a0), (a, a1) \in E(D)$. We remark that by $l(a)$ is denoted the length of string $a \in V(D)$.

Since $u \in C^{+}(D)$, we have $d_D(v, z) \le s - 1$ for each $z = 0a_z$, $l(0a_z) \le s$. Since $v \in C^{+}(D)$, for each node $y \in \{\emptyset\} \cup \{1b : l(b) \le s - 1\}$, there is $x = 0a$, $l(x) = s$, such that $(x, y) \in E(D)$. Since there is exactly 2^{s-1} x's and 2^s y's, there are no other arcs from the x's. Moreover, since $id_D(w) = 2$ for each $w \in V(D)$, also the nodes $1a$ of length s are joined only to nodes $\emptyset$ and $0b$, $l(b) \le s - 1$. Hence, also $1 \in C^{+}(D)$.

Suppose that there is $z \in C^{+}(D)$ such that $(z, u) \in E(D)$. Clearly $l(z) = s$. Since both 0 and 1 are in $C^{+}(D)$, we can assume that $z = 0a_z$, $l(a_z) = s-1$. Since

$(z,u) \in E(D)$, we have $d_D(z,a) \leq s$ for each $a \in V(D)$, $l(a) < s$. Thus, the second arc starting in z terminates in a node $z' = 1a_{z'}$ with length s. Moreover, the two arcs starting in z' terminate in nodes $z''_0 = 0a_{z''_0}$ and $z''_1 = 0a_{z''_1}$, both with length s, etc. Thus, there is exactly $1 + 4 + \cdots + 4^c$ nodes $1a$, $l(a) = s-1$, for some c. However, since $1 + 4 + \cdots + 4^c$ is odd, we have $s = 1$, a contradiction. Hence, the subgraph of D induced by $C^+(D)$ contains no directed path of length two.

Suppose that there is $z \in C^+(D)$ such that $(z,v) \in E(D)$. Since $v = 0$, we have $z = 1a_z$ and $l(z) = s$. As shown above, also the second arc starting in z terminates in a node, say z', such that $z' = 0a_{z'}$. However, it means that there are at least two different paths from z to z', both with length at most s. Hence, $z \notin C^+(D)$, a contradiction. Hence, the subgraph of D induced by $C^+(D)$ consists of isolated nodes and out-V's. $\square$

Although the digraphs constructed in the proof of Theorem 1 have only one central node, it is not complicated to find a radially Moore digraph of degree 2 with radius 2, containing exactly two central nodes. However, we have the following theorem:

Theorem 3. *Let D be a radially Moore digraph of degree 2 with radius s, $s > 1$. Then $|C(D)| < \frac{1}{2}|V(D)|$.*

Proof. Let D be a digraph and let $A \subseteq V(D)$. Then by $\langle A \rangle$ is denoted the subgraph of D induced by A.

Let D be a digraph satisfying the assumptions of theorem. By Lemma 2 the $\langle C^+(D) \rangle$ consists of a collection of isolated nodes and out-V's. By reversing the arcs in the proof of Lemma 2, it can be shown that $\langle C^-(D) \rangle$ consists of a collection of isolated nodes and in-V's. Hence, $\langle C(D) \rangle$ consists of a collection of isolated nodes and arcs.

Let c_1 denotes the number of isolated nodes in $\langle C(D) \rangle$, and let $\frac{1}{2}c_2$ denotes the number of isolated arcs in $\langle C(D) \rangle$. Then $c = |C(D)| = c_1 + c_2$. Let $b = |V(D) - C(D)|$, and let a denotes the number of arcs that have one endnode in $C(D)$ and the second in $V(D) - C(D)$. Clearly, $a = 4 \cdot c_1 + 6 \cdot \frac{c_2}{2}$. Moreover $a \leq 4 \cdot b$, since there are exactly four arcs incident with each node in D. In what follows we give a better upper bound for a.

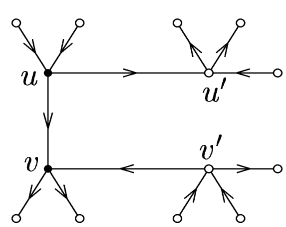

Figure 1

Let $u, v \in C(D)$, $(u,v) \in E(D)$. Then there are $u', v' \in V(D)$, different from u and v, such that $(u,u'), (v',v) \in E(D)$, and $u' \in C^+(D)$ and $v' \in C^-(D)$, by Lemma 2 (see Figure 1). Since $s > 1$ we have $u' \neq v'$ and $u', v' \notin C(D)$. By Lemma 2 the nodes adjacent with u' that are different from u, are not in $C^+(D)$, and hence, they are not in $C(D)$. Analogously, the nodes adjacent with v' that are different from v, are not in $C(D)$.

For each (u,v) in $\langle C(D)\rangle$, denote by $E_{(u,v)}$ those arcs incident with u' or v', that are not incident with u or v. The arcs from E's are in $\langle V(D)-C(D)\rangle$, and hence, we can subtract their number when bounding a. Clearly, $|E_{(u,v)}|\geq 4$ for each (u,v) in $\langle C(D)\rangle$. Moreover, each arc belongs to at most two E's, since one endnode can be in $C^+(D)$, while the other one can be in $C^-(D)$. Thus, we have $a\leq 4\cdot b-\frac{1}{2}\cdot 4\cdot\frac{c_2}{2}$. As shown above we have $a=4\cdot c_1+6\cdot\frac{c_2}{2}$, and hence $c_1+c_2\leq b$. However, since $|V(D)|=b+c$ is odd, we have $c<b$ as required. $\square$

It remains an open problem to bound the number of central nodes in radially Moore digraphs of degree greater than two.

Concluding remarks

The author would like to dedicate this paper to memory of prof. Štefan Znám, who suggested the radially Moore digraphs just a week before he tragically died, in July 1993.

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