CHARACTERIZATION OF
MINOR-CLOSED PSEUDOSURFACES

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ABSTRACT. A pseudosurface is obtained from a collection of closed surfaces
by identifying some points. It is shown that a pseudosurface $S$ is minor-
closed if and only if $S$ consists of a pseudosurface $S^o$, having at most one
singular point, and some spheres glued to $S^o$ in a tree structure.

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1. INTRODUCTION

By a pseudosurface we understand a connected topological space result-
ing when finitely many identifications, of finitely many points each, are
made on a finite collection of closed surfaces (=compact 2-manifolds). Any
point obtained by such an identification of at least two distinct points is
called a singular point. Let $G$ be a graph and $S$ a surface or a pseudosur-
face. We say that $G$ is embeddable in $S$ if there is a continuous mapping
$\varphi : G \rightarrow S$ which maps $G$ homeomorphically onto its image $\varphi(G)$. An
embedding $\varphi : G \rightarrow S$ is called a 2-cell (or cellular) embedding if each
component of $S - \varphi(G)$, called a face, is homeomorphic to an open 2-cell.

Embeddability in a closed surface can be characterized by a finite set of
forbidden subgraphs; constructive proofs of this theorem were given
by Kuratowski for the sphere [4], Bodendiek and Wagner for orientable
surfaces [3], and by Archdeacon and Huneke for nonorientable surfaces [1].
It is natural to ask whether the same is true for pseudosurfaces. The answer
is negative in general, as shown by Širáň and Gvozdjak in [7] for 2-banana
surface, i.e. the 2-amalgamation of two spheres. However, the 2-banana
surface is not minor-closed, see [2]. We remark that a surface $S$ is minor-closed if and only if the set of graphs embeddable in $S$ is minor closed (i.e. closed under a deletion of an edge or a vertex, and under contraction of an edge).

As shown by Robertson and Seymour in [6], any minor-closed class of graphs can be characterized by a finite set of forbidden subgraphs. Thus, it seems to be reasonable to characterize minor-closed pseudosurfaces; by [6] the embeddability in such pseudosurfaces can be characterized by a finite set of forbidden subgraphs.

Let $S$ be a pseudosurface. If $S$ contains as a topological subspace a sphere $S_1$ having exactly one singular point, then $S$ is called spherically-reducible. Otherwise, $S$ is called spherically-irreducible. Clearly, from each pseudosurface $S$ we obtain a spherically-irreducible pseudosurface $S'$ by successively deleting the spheres that are "glued" to the rest of the pseudosurface in exactly one singular point. Moreover, $S'$ is determined uniquely by $S$. The main result of this paper is the following theorem:

**Theorem 1.** Let $S$ be a pseudosurface. Let $S'$ be the spherically-irreducible pseudosurface that arises from $S$ by successively deleting the spheres containing exactly one singular point. Then $S$ is minor-closed if and only if $S'$ contains at most one singular point.

### 2. Preliminaries

Let $G$ be a graph. As usual, $V(G)$ denotes the vertex set of $G$ and $E(G)$ the edge set of $G$. The degree of a vertex $u$ in $G$ is denoted by $deg_G(u)$. By $G/uv$ we denote a graph that arises from $G$ by contracting the edge $uv \in E(G)$. A cycle on $n$ vertices is denoted by $C_n$ and a path on $n$ vertices is denoted by $P_n$.

Let $S$ be a closed surface and let $G$ be a graph cellularly embedded in $S$ with $F$ faces. Then the number

$$\chi(S) = |V(G)| - |E(G)| + F$$

depends only on $S$ (and not on $G$) and is known as the **Euler characteristic** of $S$. The non-negative quantity $\epsilon(S) = 2 - \chi(S)$ is called the **Euler genus** of $S$. If $S$ is orientable, then $S$ has a **positive orientability characteristic**. Otherwise, $S$ has a **negative orientability characteristic**. We remark that $S$ is determined uniquely by $\epsilon(S)$ and the orientability characteristic.

Definitions and notations not included here can be found in White [8].

In what follows we introduce concepts of uniqueness and faithfulness due to [5].
Two embeddings \( \varphi_1, \varphi_2 : G \to S \) are said to be equivalent if there is an automorphism \( \sigma : G \to G \) and a self-homeomorphism \( h : S \to S \) with \( h \circ \varphi_1 = \varphi_2 \circ \sigma \). When there is just one equivalence class of embeddings of \( G \) in \( S \), \( G \) is said to be uniquely embeddable in \( S \).

Faithfulness is defined as follows. Let \( \varphi : G \to S \) be an embedding of \( G \) in \( S \). Then \( \varphi \) is said to be faithful if for any automorphism \( \sigma : G \to G \), there is a self-homeomorphism \( h : S \to S \) such that \( h \circ \varphi = \varphi \circ \sigma \). In other words, \( \varphi \) is faithful when all automorphisms of \( \varphi(G) \) extend to self-homeomorphisms of \( S \). A graph \( G \) is said to be faithfully embeddable in \( S \) if \( G \) has a faithful embedding in \( S \).

Thus, \( G \) is uniquely and faithfully embeddable in \( S \) if \( G \) has a unique embedding in \( S \) and this embedding is faithful.

For an arbitrary closed surface \( S \) there exists a graph uniquely and faithfully embeddable in \( S \) by the following lemma [5, Proposition 1.4.7]:

**Lemma 1.** Every closed surface admits an infinite number of triangulations that are uniquely and faithfully embeddable in it.

Let \( G \) be uniquely and faithfully embeddable in \( S \). Then \( G \) is uniquely embeddable in \( S \) as a labeled graph. Consider a faithful embedding of \( G \) in \( S \). Then no automorphism of \( G \) can map a vertex \( u \) of \( G \) again to \( u \) and rearrange the neighbors of \( u \). This local property of faithful embedding will often be tacitly used.

Let \( G \) be uniquely and faithfully embeddable in \( S \), and let \( H \) be a subdivision of \( G \). Then obviously, \( H \) is again uniquely and faithfully embeddable in \( S \). Moreover, we have the following lemma [5, Corollary 1.5.7]:

**Lemma 2.** Let \( G \) have a unique and faithful triangular embedding in a closed surface \( S \). If a 3-connected graph \( H \) is embeddable in \( S \) and contains a subgraph contractible to \( G \), then \( H \) is uniquely and faithfully embeddable in \( S \).

In the proof of Theorem 1 we use the following lemma:

**Lemma 3.** Every closed surface \( S \) admits infinitely many triangulations that are uniquely and faithfully embeddable in \( S \), and that cannot be embedded in \( S' \) with \( \varepsilon(S') = \varepsilon(S) \) and the opposite orientability characteristic.

**Proof.** Let \( G \) be uniquely and faithfully triangul arly embeddable in \( S \). In what follows we construct the barycentric subdivision \( G_2 \) of \( G \) and show that \( G_2 \) satisfies the conditions in Lemma 3. First subdivide all edges of \( G \) by one vertex and denote the resulting graph by \( G_1 \). Clearly, \( G_1 \) has a 2-cell embedding, say \( \varphi_1 \), in \( S \). Now insert one new vertex into each face \( f \) of \( \varphi_1 \), join it to all vertices lying on the boundary of \( f \), and denote the resulting graph by \( G_2 \).
Since $G$ triangulates $S$ and contains no loops, there are no multiple edges in $G_2$. Since $G_2$ is 3-connected and contains a subgraph contractible to $G$, $G_2$ is uniquely and faithfully embeddable in $S$, by Lemma 2. Moreover, the unique embedding of $G_2$ in $S$ is a triangulation of $S$.

Now assume that $G_2$ is embedded in $S'$ with $\epsilon(S') = \epsilon(S)$. Then $G_2$ necessarily triangulates $S'$. Clearly, each 3-cycle in $G_2$ contains exactly one vertex from $V(G)$, one vertex from $V(G_1) - V(G)$, and one vertex from $V(G_2) - V(G_1)$. Moreover, each edge of $G_2$ lies in exactly two 3-cycles. Thus, the surface admitting a triangular embedding of $G_2$ is determined uniquely, and hence $S \cong S'$.

By Lemma 1 there are infinitely many triangulations of $S$ satisfying Lemma 3. □

As a matter of fact, the graphs satisfying Lemma 1 were constructed from triangulations by means of barycentric subdivision, see [5]. Hence, they also satisfy Lemma 3.

3. Proof of the Main Result

This section is completely devoted to the proof of Theorem 1.

Proof. Let $S$ be a pseudosurface, and let $S^o$ be the spherically-irreducible pseudosurface that arises from $S$ by successively deleting the spheres containing exactly one singular point. Suppose that $S^o$ contains at most one singular point.

Clearly, each pseudosurface is closed under deletion of an edge or a vertex. Thus, it is sufficient to prove that $S$ is closed under edge contraction.

Let $G$ be a graph embeddable in $S$, and let $\varphi$ be an embedding of $G$ in $S$. Then the subgraph of $G$ embedded in $S - S^o$ in $\varphi$ is planar. Thus, $G$ is embeddable in $S^o$. Clearly, $S^o$ is closed under edge contraction, and hence, $S$ is minor-closed.

We now turn to the more difficult part of Theorem 1. The outline of the proof is as follows. Suppose that $S$ is a pseudosurface closed under edge contraction. We construct a graph $G$ embeddable in $S$ with two specified vertices $z_1$ and $z_2$ that are joined by an edge. Then we derive properties (i) - (iv) of any embedding $\varphi$ of $G/\{z_1,z_2\}$ in $S$. Finally, considering various positions of $z_1$ and $z_2$ in $G$ on $S$, and using (i) - (iv) we obtain assertions (1) - (4) that complete the proof.

Let $S$ be a pseudosurface resulting when identifications are made on a collection $S_1, S_2, \ldots, S_l$ of closed surfaces. For the sake of convenience, with $S$ we associate a bipartite multigraph $B_S$. The vertex set of $B_S$ consists of $S_i$, $1 \leq i \leq l$, and the set $P$ of singular points of $S$, and $S_i$ is joined
to \( p \in P \) by \( t \) edges if and only if \( t \) points of \( S_i \) have been identified to \( p \).
Denote \( n = |P| \), and \( n_i = \text{deg}_{S_i}(S_i) \), \( 1 \leq i \leq l \).

Let \( p \) be a singular point of \( S \). If there is \( S_i \), \( 1 \leq i \leq l \), that is joined to \( p \) by at least two edges in \( B_S \), then \( p \) is called a \textit{self-singular point}. By \( S_i^* \) we denote the topological subspace of \( S \), which had been obtained from \( S_i \), \( 1 \leq i \leq l \). More precisely, \( B_S^* \) is a subgraph of \( B_S \) induced by multiple edges incident with \( S_i \).

**The construction of \( G \)**

By Lemma 3 there are graphs \( H_i \) uniquely and faithfully triangularly embeddable in \( S_i \), \( 1 \leq i \leq l \), which cannot be embedded in \( S_i' \) with \( \epsilon(S_i') = \epsilon(S_i) \) and the opposite orientability characteristic. We can assume that each \( H_i \) has at least \( n_i \) vertices.

![Diagram](image)

*Figure 1*

Now we locally describe a construction of a graph \( H'_i \) from \( H_i \), \( 1 \leq i \leq l \).
We replace each vertex \( u \) of \( H_i \) by the Cartesian product \( C_{n \cdot \text{deg}_{H_i}(u)} \times P_{n+2} \) and each edge by \( n+1 \) independent edges as shown in Fig. 1 for \( n = 2 \).
Clearly, \( H'_i \) is 3-connected, embeddable in \( S_i \), and contains a subgraph contractible to \( H_i \). Thus, by Lemma 2 \( H'_i \) is uniquely and faithfully embeddable in \( S_i \), and the only embedding \( \varphi'_i \) of \( H'_i \) in \( S_i \) is just the one locally described above.

For every \( u \in V(H_i) \) denote by \( f_u \) the face of the embedding \( \varphi'_i \) that appears in the position of \( u \) in \( S_i \), \( 1 \leq i \leq l \) (see Fig. 1). For a moment we concentrate on \( H_1 \). Put one new vertex \( u' \) into each face \( f_u \) of \( \varphi'_1 \), and join \( u' \) to all vertices incident with \( f_u \). There are at least \( n_1 \) such added vertices \( u' \); out of them we need to distinguish \( n_1 - 1 \) vertices, say \( v_2', \ldots, v_{n_1}' \). Moreover, put one new vertex \( v_1' \) into the face where \( v_2' \) has been placed, join \( v_1' \) and \( v_2' \), and denote the resulting graph by \( G_1 \), see Fig. 2.
Similarly, for each \( i, 2 \leq i \leq l \), put one new vertex \( u' \) into each face \( f_u \) of
\( \varphi' \), join \( u' \) to all vertices incident with \( f_u \), and denote the resulting graph by \( G_i \). Denote by \( v^i_1, \ldots, v^i_{n_i} \) the \( n_i \) vertices of \( V(G_i) - V(H'_l) \), \( 2 \leq i \leq l \).

Finally, identify \( v^1_{j_1}, \ldots, v^1_{j_{n_1}}, v^2_{j_1}, \ldots, v^2_{j_{n_2}}, \ldots, v^l_{j_1}, \ldots, v^l_{j_{n_l}} \) into \( n \) vertices \( z_1, \ldots, z_n \) in the same way as the corresponding points of \( S_1, \ldots, S_l \) have been identified when constructing the pseudosurface \( S \), and denote the resulting graph by \( G \). More precisely, there is a one-to-one correspondence between the vertices \( v^j_i, 1 \leq i \leq l \) and \( 1 \leq j \leq n_i \), and the edges of \( B_S \) incident to \( S_i \). Identify \( v^j_i \) with \( v^j_i' \) whenever the corresponding edges of \( B_S \) are incident to the same singular point. Note that the structure of \( G \) depends on the ordering of the surfaces \( S_1, \ldots, S_l \) and the singular points of \( S \). However, the assertions \((i) - (iv)\) we are going to prove below do not depend on this ordering.

![Figure 2](image_url)

Clearly, \( G \) is embeddable in \( S \). Denote by \( \varphi \) the embedding of \( G \) in \( S \) which is determined by the embeddings \( \varphi'_i \), \( 1 \leq i \leq l \), as described above.

Denote by \( z_1 \) and \( z_2 \) the vertices of \( G \) obtained from \( v^1_1 \) and \( v^2_1 \), respectively. Assume that \( z_1 \neq z_2 \). Suppose that \( G/z_1z_2 \) is embeddable in \( S \) and denote by \( \varphi^c \) an embedding of \( G/z_1z_2 \) in \( S \). In what follows we derive some properties of \( \varphi^c \). (We remark that so far we have not had any reason to expect that the supposed embedding \( \varphi^c \) of \( G/z_1z_2 \) in \( S \) has anything in common with the original embedding \( \varphi \) of \( G \) in the same \( S \).)

**Basic properties of \( \varphi^c \)**

There are \( m \leq n \) vertices, say \( x_1, x_2, \ldots, x_m \), of \( G/z_1z_2 \) embedded in the singular points of \( S \) in \( \varphi^c \). Let \( H' \) be a subgraph of \( G/z_1z_2 \). If \( H' \) contains no vertex from \( \{ x_1, \ldots, x_m \} \), then \( H' \) is called an **unbroken** subgraph of \( G/z_1z_2 \).

For each \( i, 1 \leq i \leq l \), let us do the following. Find a connected subgraph \( H''_i \) of \( H'_i - \{ x_1, \ldots, x_m \} \) that is uniquely and faithfully embeddable in \( S_i \).
For each $u \in V(H_i)$, include to $H''_i$ all unbroken copies of $C_{n, \text{deg}H_i(u)}$ at $u$. (Since $m \leq n$, for each $u \in V(H_i)$ there are at least two copies of $C_{n, \text{deg}H_i(u)}$ in $H''_i$.) Moreover, for each $uv \in E(H_i)$, include to $H''_i$ all those unbroken copies of $P_{n+2}$ at $u$ that correspond to the unbroken copies of $P_{n+2}$ at $v$, together with the edges joining them. (Since $m \leq n$, for each edge $uv \in E(H_i)$ there is a pair of corresponding copies of $P_{n+2}$ in $H''_i$.) Finally, throw away the endvertices of $H''_i$, see Fig. 3.

Clearly, $H''_i$ contains a subgraph contractible to $H_i$. Since $H''_i$ is a subgraph of $H'_i$, $H''_i$ is embeddable in $S_i$. Moreover, from $H''_i$ we obtain a 3-connected graph by a successive contraction of edges incident with vertices of degree two. Thus, $H''_i$ is uniquely and faithfully embeddable in $S_i$, by Lemma 2 and the note before Lemma 2.

Note that each $H''_i$ is embedded in one closed surface, say $S_{i^c}$, in $\varphi^c$, since the connected graph $H''_i$ contains no vertices placed in singular points. Clearly, $\epsilon(S_i) \leq \epsilon(S_{i^c})$, since $H_i$ triangulates $S_i$ and $H''_i$ contains a subgraph contractible to $H_i$. Let $J_t = \{j : \epsilon(S_j) > t\}$, $t \geq 0$. Assume that there is $t$ such that $S_{j^c} \cong S_j$ for each $j \in J_t$ (this is certainly true for $t$ large enough). Let $j \in J_t$. Then $\varphi^c$ induces a cellular embedding of $H''_i$ in $S_{j^c}$. Thus, only planar graphs can be embedded in $S_{j^c}$ together with $H''_i$. By the finiteness of $J_t$, for each $j \in J_t$ there is $k \in J_t$ such that $S_{k^c} = S_j$ (since $t \geq 0$, $H''_k$ is not a planar graph).

Suppose that $\epsilon(S_i) = t$. If $S_i$ is not a sphere, then $H''_i$ is not a planar graph, and hence $i^c \notin J_t$. Thus, $\epsilon(S_{i^c}) \leq \epsilon(S_i)$ and hence $\epsilon(S_{i^c}) = \epsilon(S_i)$. Moreover, since $H_i$ is not embeddable in $S'$ with $\epsilon(S') = \epsilon(S_i)$ and the opposite orientability characteristic, we have $S_{i^c} \cong S_i$. Hence, $S_{j^c} \cong S_j$ for each $j \in J_{t-1}$. Thus:

(i) For each $i$, $1 \leq i \leq l$, $\varphi^c$ induces an embedding of $H''_i$ in $S_{i^c}$, $1 \leq i^c \leq l$. If $S_i$ is not a sphere, we have $S_{i^c} \cong S_i$. Moreover, if $i_1 \neq i_2$, and $S_{i_1}$ and $S_{i_2}$ are not spheres, then $S_{i_1^c} \neq S_{i_2^c}$.

\[H''_i \quad \text{and} \quad G''_i\]

Figure 3

Clearly, each vertex from $V(G_i) - V(H_i')$ (except $v_{i_1}^1$) is joined to $H''_i$ by 7
at least \( n+1 \) vertex-disjoint paths in \( G_i \). Since \( v^1_1 \) and \( v^2_1 \) are identified into a single vertex in \( G/z_1z_2 \), we have:

(ii) All vertices of \( G/z_1z_2 \) that have been obtained by the identification of some vertices from \( V(G_i) - V(H'_i) \), and possibly some other vertices, lie in \( S_i^* \) in \( \varphi^c \), \( 1 \leq i \leq l \).

Suppose that \( S_i \) is a sphere, but \( S_{ic} \) is not. Then there is a \( j \) such that \( S_j \) is not a sphere and \( S_{jc} = S_{ic} \), by (i) (the second part). Moreover, \( \varphi^c \) induces a cellular embedding of \( H''_j \) in \( S_{ic} \), and hence, \( H''_1 \) is embedded in one cell of the embedding of \( H''_j \) in \( S_{ic} \). Thus, \( \varphi^c \) induces an embedding of \( H''_j \) in \( S_{ic} \) that arises from the embedding of \( H_j \) in \( S_i \), since \( H''_j \) is uniquely and faithfully embeddable in the sphere (we do not distinguish the exterior face of the embedding of \( H''_j \) in the cell). Analogously, if \( S_i \) is not a sphere, or if \( S_{ic} \) is a sphere, then \( \varphi^c \) induces an embedding of \( H''_i \) in \( S_{ic} \) that arises from the embedding of \( H_i \) in \( S_i \), by (i).

Let \( V = \{ v^1_1, \ldots, v^i_1, \ldots, v^i_n \} \setminus \{ v^1_1 \} \). Denote by \( f'_u \) the face of the embedding of \( H''_i \) in \( S_{ic} \) that corresponds to the face \( f_u \) in \( \varphi^c \), see Fig. 3. Since \( m \leq n \), for all pairs \( u, v \in V(H_i) \) there are at least four vertex-disjoint cycles in the embedding of \( H''_i \) in \( S_{ic} \) that separate \( f'_u \) from \( f'_v \), namely the copies of \( C_{n, \text{deg} H_i}(u) \) and \( C_{n, \text{deg} H_i}(v) \). Suppose that \( u, v \in V \) have been identified to \( z \) in \( G/z_1z_2 \). Since there are at least two vertex-disjoint cycles separating \( u \) from \( v \) in \( S_{ic} \) in \( \varphi^c \) (the exterior ones, see Fig. 3), \( z \) is placed in a self-singular point of \( S_{ic}^* \) in \( \varphi^c \). Analogously, we have:

(iii) Let \( v_1, \ldots, v_a \in V \) be identified to a vertex \( z \) in \( G/z_1z_2 \), \( a \geq 2 \). Then \( z \) is placed in \( \varphi^c \) in a self-singular point \( p \) of \( S_{ic}^* \) that is joined to \( S_{ic} \) by at least \( 2 \) edges in \( B_S \).

Now we introduce a lexicographical ordering of pseudosurfaces \( S_i^* \) for which \( S_i \cong S_i \), according the multiplicities of edges in \( B_{S_i} \). Let \( S_{i_1} \cong S_{i_2} \). Let \( S_{i_k}^* \) contain \( b_k \) self-singular points with multiplicities (i.e. the multiplicities of edges in \( B_{S_{i_k}} \) ) \( a_k^1 \geq \cdots \geq a_k^k \), \( 1 \leq k \leq 2 \). We write \( S_{i_1}^* \leq S_{i_2}^* \) if and only if from \( a_j^1 > a_j^2 \), \( 1 \leq j \leq b_1 \) it follows that there is \( j' \), \( 1 \leq j' < j \), with \( a_j^1 < a_j^2 \). If \( S_{i_1}^* \leq S_{i_2}^* \) and \( S_{i_1}^* \neq S_{i_2}^* \), we write \( S_{i_1}^* < S_{i_2}^* \).

Let \( z \) be a vertex of \( G/z_1z_2 \) that has been obtained by the identification of a vertex from \( V \), and possibly some other vertices. Denote by \( P_z \) the collection of the paths joining \( z \) to \( H''_i \) that contain no vertex from \( \{ x_1, \ldots, x_m \} \) (except possibly \( z \)). Clearly, for each such \( z \) there is at least one path in \( P_z \) with this property. Denote by \( G_z^* \) the subgraph of \( G/z_1z_2 \) induced by \( H''_i \) and the paths \( P_z \), where \( z \) is obtained by the identification of a vertex from \( V \), see Fig. 3 (the vertices \( z_{i,j} \) in Fig. 3, \( 1 \leq j \leq 3 \), need not necessarily be distinct). Since \( H''_i \) is embedded in \( S_{ic} \) in \( \varphi^c \), the graph \( G_z^* \) is embedded in \( S_{ic}^* \), by (ii).

Suppose that \( S_i \) is not a sphere. Then \( S_{ic} \cong S_i \), by (i). Moreover,
we have $S^*_{i_c} \succeq S^*_1$, by (ii) and (iii). Note that $S^*_{i_c} \succeq S^*_1$ also if $z_1$ and $z_2$ are self-singular points of $S^*_1$. (We remark that $S^*_{i_c} \succeq S^*_1$ is only a necessary but not a sufficient condition for embeddability of $G^*_i$ in $S^*_{i_c}$.) Let $J = \{j : S^*_j \succ S^*_i\}$. By (i) (the second part), if $k_1 \neq k_2$, and $k_1, k_2 \in J$, then $S_{i_{k_1}} \neq S_{i_{k_2}}$. Since $J$ is a finite set, $S^*_{i_c} \succeq S^*_1$ contradicts $S^*_{j_c} \succeq S^*_j$, $j \in J$. Hence, $S^*_1 \simeq S^*_1$.

Now suppose that $S_i$ is a sphere, but $S^*_1$ is not. Moreover, suppose that $S^*_{i_c}$ is not a sphere, either. Then there is a $j$ such that $S_j$ is not a sphere and $\varphi^c$ induces an embedding of $G^*_j$ in $S^*_{j_c}$ with $S^*_{j_c} = S^*_{i_c}$, by (i) (the second part). As shown above, we have $S^*_j \simeq S^*_j$. Since $S^*_j$ is not a sphere, there is a self-singular point in $S^*_j$. Since $z_1 \neq z_2$ in $G$, there are at least two vertices, say $u, v \in V_i$ that have been identified into a single vertex in $G^*_i$. However, $\varphi^c$ induces an embedding of $H''_i$ in $S^*_i$ that arises from the embedding of $H_i$ in $S_i$ (see the note below (ii)). Thus, at least one of the vertices $u$ and $v$, say $u$, is separated from each vertex from $V_j$ (and also from $V_i - u$) by a cycle in $S^*_i$ in $\varphi^c$. Since $G^*_j$ is embedded in $S^*_{j_c}$ in $\varphi^c$, we have $S^*_{i_c} \succeq S^*_j$, as shown above. Since $u$ is separated from each vertex from $V_j \cup (V_i - u)$ by a cycle in $S^*_i$ in $\varphi^c$, we have $S^*_{i_c} \succeq S^*_j$, which contradicts $S^*_{i_c} \simeq S^*_j$. Hence, if $S^*_1$ is not a sphere but $S_i$ is, then $S^*_{i_c}$ is a sphere, too.

Now analogously as above, if $S^*_1$ is not a sphere but $S_i$ is, we have $S^*_{i_c} \succeq S^*_1$, by (ii) and (iii). Moreover, if $S^*_{i_k}$ is not a sphere but $S_{i_k}$ is, $1 \leq k \leq 2$, we have $S^*_{i_k} \neq S^*_{i_2}$. Hence, we have $S^*_{i_c} \simeq S^*_1$, since the set of those $j$ for which $S^*_j \succeq S^*_1$ is finite. Thus:

(iv) For each $i$, $1 \leq i \leq l$, $\varphi^c$ induces an embedding of $G^*_i$ in $S^*_i$. If $S^*_i$ is not a sphere, we have $S^*_{i_c} \simeq S^*_1$. Moreover, if $i_1 \neq i_2$, and $S^*_{i_1}$ and $S^*_{i_2}$ are not spheres, then $S^*_{i_1} \neq S^*_{i_2}$.

**Necessary conditions for $S$**

To obtain the necessary conditions in Theorem 1, we now need to utilize the "finer structure" of $G$, that is, the way how $G$ depends on the labelling of the surfaces and the singular points. In fact, we only need to consider the vertices $z_1$ and $z_2$ in $G$.

Suppose that $z_1$ and $z_2$ are placed in two self-singular points of $S^*_1$ in $\varphi$. Let $z_j$ be obtained by the identification of $t_j$ vertices of $G_1$, $1 \leq j \leq 2$. Suppose that $G^*_i$ is embedded in $S^*_{i_c}$ in $\varphi^c$. Since $t_1 + t_2 - 1 > \max\{t_1, t_2\}$, we have $S^*_{i_c} \succeq S^*_1$, by (i) and (iii). By (iv) we have:

1. No pseudosurface $S^*_i$ contains more than one self-singular point, $1 \leq i \leq l$.

Suppose that $B_S$ contains a cycle of length at least four. Let $p_1, S_1, p_2, \ldots, p_t, S_t, p_1$ be a shortest cycle in $B_S$ such that $t \geq 2$. Let $z_1$ be placed in $p_1$ and $z_2$ be placed in $p_2$ in $\varphi$. Let $G$ be the subgraph of $G/z_1z_2$ induced by $G^*_2, G^*_3, \ldots, G^*_t$.
By (iv) each $G_i^*$, $2 \leq i \leq t$, is embedded in one pseudosurface $S_i^*$ in $\varphi^c$. Hence, $\overline{G}$ is embedded in one pseudosurface, say $S_k^*$, in $\varphi^c$ if $t = 2$. Now suppose that $t \geq 3$. Then $G_2^*, \ldots, G_t^*$ are joined to a $(t-1)$-cycle, by (ii). (More precisely, if we replace each $G_i^*$ by a single vertex $g_i$, and join $g_i$ by an edge whenever $G_i^*$ and $G_j^*$ have some common vertices, then $\overline{G}$ will result to a cycle on $t-1$ vertices.) Since $2t$ is the length of a shortest cycle of length at least four in $B_S$, the graph $\overline{G}$ is embedded in one pseudosurface, say $S_k^*$, if $t \geq 3$.

Since $\overline{G}$ is not a planar graph, $S_k^*$ is not a sphere. By (iv) (the second part), at most one pseudosurface from $S_2^*, \ldots, S_t^*$ is not a sphere. If $S_k^* \not\equiv S_i^*$ for each $i$, $2 \leq i \leq t$, then the finiteness of the set of those $j$ for which $S_j^* \equiv S_k^*$ contradicts (iv). Hence, $S_k^* \equiv S_j^*$ for some $j$, $2 \leq j \leq t$, and $S_j^*$ is the unique pseudosurface from $S_2^*, \ldots, S_t^*$ which is not a sphere.

Suppose that $t = 2$. Then $\overline{G} = G_j^* = G_2^*$ and the vertex $z_1 z_2$ is embedded in a self-singular point of $S_k^*$ in $\varphi^c$. Hence we have $S_k^* \not\equiv S_2^*$, by (iii), which contradicts $S_k^* \equiv S_j^*$.

Now suppose that $t \geq 3$. Let $G'$ be the subgraph of $G$ induced by $G_i^*$, $2 \leq i \leq t$ and $i \neq j$. Then $G'$ is a connected graph containing two distinct vertices, say $z_1$ and $z_2$, of $G_j^*$. Let $z'$ be obtained by the identification of a set $V_j^1$ vertices from $V_j$, and possibly some vertices outside $V_j$, $1 \leq i \leq 2$. Since $S_k^* \equiv S_j$, the graph $H''_j$ is uniquely and faithfully embeddable in $S_k$. Hence, each pair of vertices from $V_j$ is separated by at least two vertex-disjoint cycles in $S_k$ in $\varphi^c$ (see the note before (iii)). Hence, also the sets of vertices $V_j^1$ and $V_j^2$ are separated by two vertex-disjoint collections of cycles in $S_k$ in $\varphi^c$. Since $G'$ joins $z^1$ with $z^2$ in $S_k^*$ in $\varphi^c$, there is a self-singular point of $S_k^*$ that allows this connection. Hence, we have $S_k^* \not\equiv S_j^*$ by (iii), which contradicts $S_k^* \equiv S_j^*$. Thus:

(2) There is no cycle of length greater than two in $B_S$.

Thus, $S$ has a “tree structure”. Suppose that at least two pseudosurfaces from $S_1^*, \ldots, S_t^*$ are not spheres. Let $S_2, p_1, S_1, p_2, S_3, p_3, \ldots, p_{t-1}, S_t$ be a longest path in $B_S$ such that both $S_2^*$ and $S_t^*$ are not spheres. Suppose that $t \geq 3$. Let $z_1$ be placed in $p_1$ and $z_2$ be placed in $p_2$ in $\varphi$.

Let $\overline{G}$ be the subgraph of $G/z_1 z_2$ induced by $G_2^*, G_3^*, \ldots, G_t^*$. By (iv) each $G_i^*$, $2 \leq i \leq t$, is embedded in one pseudosurface $S_i^*$ in $\varphi^c$. Moreover, $G_2^*, \ldots, G_t^*$ are joined to a $(t-1)$-path, by (ii). (More precisely, if we replace each $G_i^*$ by a single vertex $g_i$, and join $g_i$ with $g_j$ by an edge whenever $G_i^*$ and $G_j^*$ have some common vertices, then $\overline{G}$ will result to a path on $t-1$ vertices.) Thus, there are surfaces, say $S_{i_1}$ and $S_{i_2}$, at distance $2t$ in $B_S$, such that $S_{i_1}^*$ and $S_{i_2}^*$ are not spheres, and either $S_{i_1}^*$ or $S_{i_2}^*$ are covered by planar graphs, possibly empty, in $\varphi^c$, which contradicts (iv). Hence:

(3) If $S_i^*$ and $S_j^*$ are not spheres, then $S_i$ and $S_j$ are at distance two in $B_S$. 

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Let $S_1^*$ and $S_2^*$ be not spheres. Let $pS_1$ and $pS_2$ be edges of $B_S$, and let $z_1$ be placed in $p$ in $\varphi$. Suppose that $S_1^*$ contains a self-singular point different from $p$ that is occupied by $z_2$ in $\varphi$. Since $p$ is the unique singular point lying in at least two pseudosurfaces that are not spheres, by (2) and (3), the vertex $z_1z_2$ is placed in $p$ in $\varphi^c$, by (ii).

Let $p$ be a self-singular point of $S_i^*$, $2 \leq i \leq l$. Then $p$ is a self-singular point of $S_i^*$, by (ii) and (3). However, $p$ is a self-singular point of $S_1^*$, while $p$ is not a self-singular point of $S_1^*$, by (1). Since there is just a finite set of $S_j^*$ that contains $p$ as a self-singular point, by (iv) (the second part) we have:

(4) If $S_i^*$ and $S_j^*$ are not spheres and $p$ is a self-singular point of $S_i^*$, then $p \in S_j^*$.

Hence, if there are three pseudosurfaces, say $S_1^*$, $S_2^*$, and $S_3^*$, that are not spheres, then they are glued in a unique singular point $p$, by (2) and (3). Moreover, if one of them, say $S_1^*$, contains a self-singular point $p'$, then $p' = p$, by (4). Thus, the spherically-irreducible subspace of $S$ contains at most one singular point. This completes the proof. \( \square \)

We remark that if a pseudosurface $S$ is not minor-closed, then there are infinitely many graphs $G$ embeddable in $S$ such that $G/xy$ is not embeddable in $S$ for some $xy \in E(G)$, by Lemma 3. The problem of determining whether or not the embeddability in a given non-minor-closed pseudosurface can be characterized by a finite set of forbidden subgraphs remains open.

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References


