DISTANCES IN ITERATED LINE GRAPHS

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Abstract. For a connected graph G that is not a cycle, a path or a claw, let its k-iterated line graph have the diameter $diam_k$ and the radius r_k. Then $diam_{k+1} = diam_k + 1$ for sufficiently large k. Moreover, $\{r_k\}$ also tends to infinity and the sequence $\{diam_k - r_k - \sqrt{2\log_2 k}\}$ is bounded.

1. Introduction

In this paper we study distance properties of iterated line graphs. If G is a nontrivial graph then by its line graph $L(G)$ we mean such a graph whose nodes are the edges of G and two nodes in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. Further by L we mean the line graph function, i.e. the function which maps each nontrivial graph into its line graph. Later on L^0 is the identical graph function, while L^k is the composition $L^{k-1} \circ L$ for an integer $k \geq 1$.

Many papers have been written on line graphs [1] but only a few results are known about iterated line graphs [3]. The aim of this paper is to initiate the study of distance properties of iterated line graphs. Particularly, we focus our attention on sequences of graphs, in which any member but the first is the line graph of the preceding one. First we describe two types of them.

Note that the k-iterated line graph of a path on n nodes is the path on $n-k$ nodes for $k<n$ and such a graph does not exist if $k \geq n$. Moreover, each iterated line graph of a cycle is isomorphic to the original cycle and each iterated line graph of $K_{1,3}$ is a triangle. Hence it suffices to study connected graphs with at least four edges and the maximal degree at least three. Such graphs G will be called prolific, since each two members of the sequence $\{L^k(G)\}$ are distinct.

For a function f on graphs we create a sequence $f_k := f(L^k(G))$, or shortly an f-sequence and study its behavior. We do this for the order n, the diameter $diam$ and the radius r of a graph. Main results are presented in Chapter two and their proofs are postponed to Chapter three.

2. Results

Let G be a graph, then $n_0 = n(G)$ denotes the number of its nodes, $m(G)$ denotes the number of its edges and $\delta_0 = \delta(G)$ and $\Delta_0 = \Delta(G)$ correspond to its minimal and maximal degree. We will pay no attention to the m-sequence, since the equation $m_k = n_{k+1}$ relates it to the n-sequence. We will show that $\Delta_k{=}\Theta(2^k)$, $\delta_k{=}\Theta(2^k)$, $n_k{=}\Theta(2^{\frac{k^2-3k}{2}})$, $diam_k{=}\Theta(k)$, and $diam_k{-}r_k{=}\Theta(\sqrt{\ln k})$ are bounded for prolific graphs. The following Lemma gives bounds for these invariants in iterated line graphs.

Lemma 1. *For a prolific graph and $k{\geq}1$ we have*

$$2^k \cdot (\delta_0{-}2) + 2 \leq \delta_k \leq \Delta_k \leq 2^k \cdot (\Delta_0{-}2) + 2 \tag{1}$$

$$n_0 \cdot \prod_{i=0}^{k-1} [2^{i-1} \cdot (\delta_0{-}2) + 1] \leq n_k \leq n_0 \cdot \prod_{i=0}^{k-1} [2^{i-1} \cdot (\Delta_0{-}2) + 1] \tag{2}$$

Moreover, equalities hold for regular prolific graphs.

Better lower bounds for graphs with endnodes are given by the next theorem.

Theorem 2. *For a prolific graph G and $k{\geq}5$ we have*

$$n_k \geq 8 \cdot \prod_{i=0}^{k-5} (2^i + 1) \tag{3}$$

$$\Delta_k \geq 3 \cdot 2^{k-4} + 2 \tag{4}$$

Moreover, if $G = J$ (see Fig.1) then the equality in (4) holds.

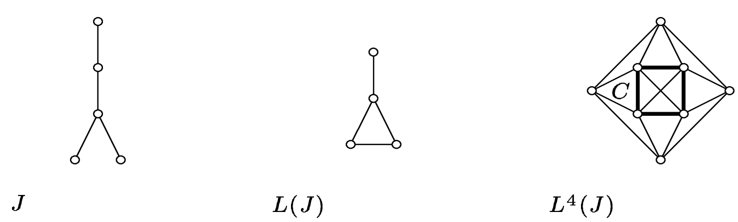

Fig.1.

It was shown in [2], that $|r_1{-}r_0| \leq 1$ and $|diam_1{-}diam_0| \leq 1$ for a nontrivial connected graph. Here we prove that the sequence $\{diam_k\}$ increases for sufficiently large k.

Theorem 3. *Let G be a prolific graph. Then*

$$diam_{k+1} = diam_k + 1 \qquad \textit{for} \quad k \geq 3{+}n_3. \tag{5}$$

Moreover, if G is a noncomplete graph with $\delta(G) \geq 3$, then

$$diam_0 + p - 2 \leq diam_p \leq diam_0 + p \quad \textit{for any} \quad p{\geq}1. \tag{6}$$

Note that the left equality in (6) holds if G is an N-dimensional cube for $N\geq 3$ and $p\geq 2$, the right equality holds for complete graphs with at least six nodes, and $diam_p = diam_0+p-1$ holds for K_n-e and $n\geq 6$. It could be of some interest, for a given graph G, to find a smallest number k_0 such that $diam_k$ increases for $k\geq k_0$. The bounds given in Theorem 3 are rather far from this value.

Theorem 4. *For a prolific graph there are numbers c_1 and c_2 such that $\sqrt{2\log_2 k}+c_1 < diam_k - r_k < \sqrt{2\log_2 k}+c_2$.*

We conclude this section with several open problems.

Conjecture 1. *There exists $k>0$ such that for any two prolific graphs G and H, $n(L^i(H)) = n(L^i(G))$ for $i=0,\dots,k$ implies $n(L^i(H)) = n(L^i(G))$ for all i.*

Conjecture 2. *There exists $k>0$ such that for any two prolific graphs G and H, $n(L^i(H)) \leq n(L^i(G))$ for $i=0,\dots,k$ implies $n(L^i(H)) \leq n(L^i(G))$ for all i.*

Conjecture 3. *Given a prolific graph, we have $\delta_{k+1} = 2\delta_k-2$ and $\Delta_{k+1} = 2\Delta_k-2$ for sufficiently large k.*

Note that Lemma 1 gives an exact value of n_k for a regular graph G. For a given graph G, $n(L(G))$ equals to the number of edges in G, while $n(L^2(G)) = \sum\binom{deg(v)}{2}$, where $deg(v)$ denote the degree of v in G and the summation goes over all nodes v of G. But to compute $n(L^k(G))$ in general seems to be an uneasy task unless G is regular.

The problems of the complexity of n_k, δ_k, Δ_k, $diam_k$, r_k for a prolific graph remain open.

3. Proofs

We will identify edges in a graph G with the corresponding nodes in $L(G)$. Hence if u and v are two adjacent nodes in G then by uv we mean an edge in G as well as the node in $L(G)$ corresponding to the edge uv. This notation enables us to consider a node in $L^k(G)$ ($k\geq 2$) as a pair of adjacent nodes in $L^{k-1}(G)$, either of these is a pair of adjacent nodes from $L^{k-2}(G)$, and so on. Furthermore we can define each node in $L^k(G)$ using only edges of G, and such a definition will be called the *recursive definition of v in G*.

Proof of Lemma 1. Let G be a prolific graph. Note that if $e{=}ab$ is an edge in G then $deg(e) = deg(a)+deg(b)-2$, hence $\Delta_k \leq 2\cdot\Delta_{k-1}-2$ and $\delta_k \geq 2\cdot\delta_{k-1}-2$ (for $k\geq 1$), which gives (1). Further as nodes in $L^k(G)$ are edges in $L^{k-1}(G)$, we have $n_k \leq \frac{\Delta_{k-1}\cdot n_{k-1}}{2} \leq [2^{k-2}\cdot(\Delta_0-2)+1]\cdot n_{k-1}$ from (1) and this recurrent relation gives the right inequality in (2). The proof of the left one is very similar. □

Let H be a nontrivial subgraph of G. By $L(H,G)$ we mean just the subgraph of $L(G)$ induced by nodes that are edges of H. Clearly $L(H,G)\cong L(H)$. Later on $L^1(H,G) = L(H,G)$ and $L^k(H,G)$ means $L^{k-1}(L(H,G),L(G))$.

Proof of Theorem 2. It is easy to see that G contains either J or $L(J)$ (see Fig.1) or $K_{1,4}$ as a subgraph. Put $H = L^4(J)$ and note that $n(H)$=8 and $\delta(H)$=4. Then (2) gives

$$n(L^k(J)) = n(L^{k-4}(H)) \geq 8 \cdot \prod_{i=0}^{k-5}(2^i+1).$$

Moreover, $\Delta(H)$=5 and the four nodes of the degree five lie on a cycle C (see Fig.1). Then $L^p(C,H)$ is also a 4-cycle for any p. It is easy to check that for nodes v_p of $L^p(C,H)$ we have $deg(v_{p+1}, L^{p+1}(H)) = 2 \cdot deg(v_p, L^p(H)) - 2$, where $deg(v,G)$ means the degree of v in a graph G. This recurrent relation together with (1) gives

$$\Delta(L^k(J)) = \Delta(L^{k-4}(H)) = 3 \cdot 2^{k-4} + 2.$$

Now if G contains J then (3) and (4) follow as $L^k(G)$ contains $L^k(J)$. Further if G contains $L(J)$, then $L^k(G)$ contains $L^{k+1}(J)$ and (3) and (4) also follow. Finally, if G contains $K_{1,4}$, then as $L(J) \subseteq L(K_{1,4}) = K_4$, the graph $L(G)$ contains $L(J)$, the result follows. □

Let G be a graph and v be a node in $L^k(G)$, (k≥1). By the i-butt $B_i(v)$ of the node v in $L^k(G)$ we mean the subgraph of $L^{k-i}(G)$ induced by the edges involved into the recursive definition of the node v. The butt we will abbreviate to $B(v)$ when it is clear in which graph it is considered.

As $L^k(H,G)$ contains just those nodes of $L^k(G)$ whose recursive definition contains edges of H and only such edges, for any nontrivial graph $H \subseteq G$ we have

(7) A node v of $L^k(G)$ lies in $L^k(H,G)$ if and only if $B_k(v) \subseteq H$.

By P_k we mean a path on k edges. If H and J are two subgraphs of a graph G, then by $H \vee J$ we mean the edge-union of H and J. Symbols $V(G)$ and $E(G)$ denote the node set and the edge set of a graph G, respectively.

Lemma 5. *Let H be a subgraph of a graph G and $L^k(G)$ exist (k≥1). Then H is a butt for some node in $L^k(G)$ if and only if H is a connected graph with at most k edges, distinct from any path with less than k edges.*

Proof. First assume $H = B_k(v)$ for some node v of $L^k(G)$. We prove that $B_k(v)$ is a connected graph with at most k edges distinct from paths on at most k nodes by induction on k. This clearly holds for k=1, 2, as each 1-butt is an edge and each 2-butt is a path P_2. For k≥2 suppose that the inductive hypothesis holds for values less than k. First we prove that $B_k(v)$ is connected. Note that v=$(xy)(yz)$ for some nodes x, y, z of $L^{k-2}(G)$. Moreover, $B_k(v)$ is the edge-union $B_{k-1}(xy) \vee B_{k-1}(yz)$ and both these butts are connected and contain a common subgraph $B_{k-2}(y)$, hence the butt $B_k(v)$ is connected.

Further as $B_{k-1}(v)$ is connected and has at most $k-1$ edges, it contains at most k nodes. Since $V(B_{k-1}(v)) = E(B_k(v))$, $B_k(v)$ contains at most k edges.

Finally, for $H = B_k(v)$, (7) gives that v lies in $L^k(B_k(v), G)$, hence $B_k(v)$ cannot be a path P_t with t<k, as the graph $L^k(P_t, G)$ does not exist.

Conversely, let H be a graph that satisfies the hypothesis of Lemma 5. We prove by the induction on k that H is a butt for some node in $L^k(G)$. Clearly this holds for $k{=}1$. Assume $k{\geq}2$ and let the statement hold for numbers less than k. Distinguish three cases.

(*i*) Let H be a path on k edges. Let v be a node of $L^k(H,G)$. (Remember, that $L^k(H,G) \cong L^k(H)$.) Then $B_k(v) = H$ follows from (7) and the first part of this proof.

(*ii*) Let H be a claw or a cycle of length $l \leq k$. Since the i-iterated line graph of a claw is isomorphic to the i-iterated line graph of a triangle for $i \geq 1$, it suffices to study the case when H is a cycle. It is clear that $L^i(H,G)$ is a cycle with l nodes for any $i \geq 0$. Put $L^{k-l}(H,G) = H'$ and represent this cycle as an edge-union $P_{l-1} \vee P'_{l-1}$ with $P_{l-1} \cap P'_{l-1} = P_{l-2}$. Denote $u = L^{l-1}(P_{l-1}, L^{k-l}(G))$ and $u' = L^{l-1}(P'_{l-1}, L^{k-l}(G))$. Then u and u' are adjacent nodes in $L^{k-1}(G)$ and form a 1-butt of a node $v \in L^k(G)$. Thus, $H = B_k(v)$.

(*iii*) Let T denote a spanning tree in a connected graph $L(H,G)$ and T do not be a path. Such a tree exists, because $L(H,G)$ has a node of degree greater than two. Then T has at most $k{-}1$ edges, as $n(T){=}m(H){\leq}k$. Now according to the induction hypothesis, T is a $(k{-}1)$-butt for some node u of $L^{k-1}(L(G))$. Since $E(B_k(u)){=}V(B_{k-1}(u)){=}V(T){=}E(H)$, the k-butt $B_k(u)$ is edge-induced by $E(H)$ and $H = B_k(u)$. $\square$

The distance $d(H,J)$ between two subgraphs H and J of a graph G equals to the length of a shortest path in G joining a node from H to a node from J. The following lemma enables us to compute distances between nodes in iterated line graphs.

Lemma 6. *Let G be a connected graph, $L^k(G)$ exist for an integer $k{\geq}1$, and let u and v be distinct nodes in $L^k(G)$. Then*

(S1) $d(u,v) = k + d(B_k(u), B_k(v))$ *if the k-butts of v and u are edge-disjoint.*

(S2) $d(u,v) = \max\{t;\, t\text{-butts of } u \text{ and } v \text{ are edge-disjoint}\}$ *if k-butts of u and v have a common edge.*

Proof. First we prove (S1) by the induction on k. It is not difficult to check up that if u and v are two distinct edges in G (i.e. the nodes in $L(G)$), then

$$d_{L(G)}(u,v) = 1 + d\left(B_1(u), B_1(v)\right). \tag{8}$$

This verifies (S1) for $k{=}1$. Now let (S1) hold for integers less than k and assume that $u{=}u_1u_2$ and $v{=}v_1v_2$ are distinct nodes in $L^k(G)$ with edge-disjoint butts in G. Then (8) gives

$$d_{L^k(G)}(u_1u_2, v_1v_2) = 1 + \min\left(d_{L^{k-1}(G)}(u_i, v_j)\middle|\ i,j \in \{1,2\}\right) \tag{9}$$

as u_1u_2 is the only edge in the butt $B_1(u)$ and v_1v_2 is the only edge in $B_1(v)$. Since $B_k(u)$ and $B_k(v)$ are edge-disjoint and $B_{k-1}(u_i) \subseteq B_k(u)$ and $B_{k-1}(v_i) \subseteq B_k(v)$ for $i \in \{1,2\}$, also $B_{k-1}(u_i)$ and $B_{k-1}(v_j)$ are edge-disjoint for $i,j \in \{1,2\}$. Hence the induction hypothesis gives

$$d_{L^{k-1}(G)}(u_i, v_j) = (k{-}1) + d\left(B_{k-1}(u_i), B_{k-1}(v_j)\right)$$

and after substituting to (9) we have $d_{L^k(G)}(u,v) = 1 + (k{-}1) + d\left(B_{k-1}(u_1) \vee B_{k-1}(u_2), B_{k-1}(v_1) \vee B_{k-1}(v_2)\right)$. Now the observations $B_k(u) = B_{k-1}(u_1) \vee B_{k-1}(u_2)$ and $B_k(v) = B_{k-1}(v_1) \vee B_{k-1}(v_2)$ complete the proof of (S1).

In order to prove (S2) assume that t is maximal integer such that the t-butts of u and v in $L^{k-t}(G)$ are edge-disjoint ($t<k$ as $B_k(u)$ and $B_k(v)$ have a common edge). Then the (t+1)-butts of u and v in $L^{k-t-1}(G)$ have a common edge e, which is also a node in $L^{k-t}(G)$. Finally, statement (S1) reduces the problem on computing distances in $L^{k-t}(G)$, $d(u,v) = k - (k-t) + d\left(B_t(u), B_t(v)\right) = t$ as both butts in question contain the node e. □

Proof of Theorem 3. By $\lambda_p(G)$ we mean the maximal distance between any two p-butts in a prolific graph G.

(10) If $p \geq 1$ and G contains two edge-disjoint p-butts, then $diam(L^p(G)) = p + \lambda_p(G)$.

Proof of (10): Lemma 6 implies that for any $p \geq 1$, two nodes in $L^p(G)$ with edge-disjoint p-butts have the distance at least p, while two nodes whose p-butts possess a common edge have the distance less than p. Hence in computing the diameter of $L^p(G)$ we can restrict ourselves to pairs of nodes with edge-disjoint p-butts, and (S1) gives (10).

Now we prove (6): As $\delta(G) \geq 3$ and G is noncomplete, any claws W_1 and W_2 in G, whose central nodes have the distance $diam(G)$, are edge-disjoint p-butts (for any $p \geq 3$) according to Lemma 5. Evidently, $diam(G) - 2 \leq d(W_1, W_2) \leq \lambda_p(G) \leq diam(G)$. If p=1 or 2, we take appropriate subgraphs of these claws.

Now we show that $H = L^3(G)$ contains two edge-disjoint p-butts for any $p \geq 1$ and any prolific G. Since G contains either J (see Fig.1) or $L(J)$ or $K_{1,4}$ as a subgraph, the graph $L^3(G)$ contains $L^3(J)$ or $L^4(J)$ or $L^3(K_{1,4}) = L^2(K_4)$ as a subgraph and one can directly verify that each of them contains two edge-disjoint claws. Clearly a claw is a p-butt for $p \geq 3$. If p=1 or 2, we take appropriate subgraphs of these claws and obtain edge-disjoint p-butts.

Now, according to Lemma 6, $\lambda_p(H) = diam_p - p$ is a constant function for all p, $p \geq n(H)$, as the sets of minimal p-butts remain the same. Since $H = L^3(G)$, (10) completes the proof. □

Lemma 7. *For a prolific graph and $t \geq 2r_0$, we have $\delta_t \geq 3$.*

Proof. Let $H(G)$ be the subgraph of a prolific graph G induced by the nodes with the degree at most two. Then $H(G)$ consist of paths with at most $2r_0$ nodes. Since $H(L(G)) = L(H(G))$, $H(L^{2r_0}(G))$ is empty and $\delta_{2r_0} \geq 3$. □

Proof of Theorem 4. Due to Lemma 7 and Lemma 1 we can suppose that $\delta(G) \geq 4$. In this case for the sequences $\{n_i\}$, $\{m_i\}$ holds the next inequality which is sharp for $i > 0$

$$n_{i+1} = m_i \geq 2n_i \ , \qquad i = 0, 1, \dots \ . \tag{11}$$

For the given graph G we shall consider the sequence

$$\omega_i = diam_i - r_i$$

and try to estimate it for sufficiently large n.

Let s^0 be the minimal number with the property that there exists $v_0 \in V(L^{s^0}(G))$ such that $E(B_{s^0}(v_0)) \cap E(B_{s^0}(v)) \neq \emptyset$ for all $v \in V(L^{s^0}(G))$. The existence of s^0 follows from Lemma 5. The complement of $B_{s^0}(v_0)$ in G does not contain s^0-butt of any node $v \in V(L^{s^0}(G))$ so it should be a forest with path components of length less then s^0. Thus, we obtain

$$m_0 \geq s^0 \geq |E(B_{s^0}(v_0))| > m_0 - n_0 \ . \tag{12}$$

Now we show that $|E(B_{s^0}(v_0))| = s^0$. According the inequalities (11) and (12) the complement of $B_{s^0}(v_0)$ in G has less then $s^0{-}1$ edges and it cannot contain $(s^0{-}1)$-butt of any node. Now suppose that $|E(B_{s^0}(v_0))| < s^0$. In this case $B_{s^0}(v_0)$ is also an $(s^0{-}1)$-butt of a node $v_0' \in V(L_{s^0-1}(G))$ with the property $E(B_{s^0-1}(v_0')) \cap E(B_{s^0-1}(v)) \neq \emptyset$ for any $v \in V(L^{s^0-1}(G))$ what is a contradiction with the minimality of s^0.

We show that v_0 is a central node of $L^{s^0}(G)$. Suppose that w is a node from $L^{s^0}(G)$ with eccentricity less then $s^0{-}1$. Then $E(B_{s^0-1}(w)) \cap E(B_{s^0-1}(v)) \neq \emptyset$ for all $v \in V(L^{s^0}(G))$ from (S2) and $m_1 \geq s^0{-}1 > m_1 - n_1 \geq m_0$ from (11) which is a contradiction with (12). Since the eccentricity of v_0 is $s^0{-}1$, we have $r_{s^0} = s^0{-}1$.

Now let $k > s^0$. For the radius r_k of $L^k(G)$ we have $r_{k-1} - 1 \leq r_k \leq r_{k-1} + 1$, Theorem 3 from [2]. Let s^1 be the minimal number $s^1 > s^0$ with $r_{s^1} \leq r_{s^1-1}$. Then $r_{s^1} \leq s^1{-}2$ and there exists a node $v_1 \in V(L^{s^1}(G))$ with property $E(B_{s^1-1}(v_1)) \cap E(B_{s^1-1}(v)) \neq \emptyset$ for any $v \in V(L^{s^1}(G))$. From the minimality of s^1 we obtain

$$\begin{aligned} m_1 \geq s^1 - 1 > m_1 - n_1 \\ |E(B_{s^1-1}(v_1))| = s^1 - 1 \end{aligned}$$

in the same way as above. Since $m_2 - n_2 > m_1$ from (11), we have $r_{s^1} = r_{s^1-1}$ and $r_{s^1} = s^1{-}2$. Thus,

$$\begin{aligned} r_{s^0+i} &= s^0 + i \\ \omega_{s^0+i} &= \omega_{s^0} \end{aligned}$$

for all nonnegative $i < s^1 - s^0$.

Analogously, let s^j be the least number with a property that there exists $v_j \in V(L^{s^j}(G))$ such that $E(B_{s^j-j}(v_j)) \cap E(B_{s^j-j}(v)) \neq \emptyset$ for any $v \in V(L^{s^j}(G))$. By the same method as above we have

$$\begin{aligned} m_j \geq & s^j - j > m_j - n_j \\ r_{s^j} = & s^j - (j{+}1) \end{aligned}$$

and for nonnegative $i < s^j - s^{j-1}$

$$r_{s^{j-1}+i} = s^{j-1} - j + i \ .$$

From Theorem 3 it follows that there exists $k_0 = s^{j_0}$ with the property

$$diam_k = diam_{k-1} + 1$$

for any $k \geq k_0$. On the other hand

$$r_k = r_{k-1} + 1$$

for all $k > s^0$ except for $k = s^j$. Thus, $\{\omega_k\}$ is a constant function on all intervals $[s^j, s^{j+1} - 1]$ and in s^j it increases by one for $j \geq j_0$.

In the following we estimate the values of ω in $k = s^j$, $j \geq j_0$. If $\omega_{s^{j_0}} = c + j_0$ then $\omega_{s^j} = c + j$ for $j \geq j_0$.

Now we bound the value of s^j. Using (11) and Lemma 1 we have

$$m_j - n_j \geq n_j > 2^{\binom{j}{2}} \cdot n_0 \ .$$

Let $l = \lceil \log_2(\Delta_0 - 1) \rceil$. Then we have

$$m_j = n_{j+1} \leq n_0 \cdot \prod_{i=0}^{j} \left[2^{i-1}(\Delta_0 - 2) + 1 \right] < 2^{\binom{j+l}{2}} \cdot n_0 \ .$$

Putting together the above inequalities we obtain

$$2^{\binom{j+l}{2}} \cdot n_0 > m_j \geq s^j - j > m_j - n_j > 2^{\binom{j}{2}} \cdot n_0 \ .$$

Since $2^{\binom{j+l+1}{2}} \cdot n_0 > 2^{\binom{j+l}{2}} \cdot n_0 + j$ and $\omega_{s^j} = c + j$, for $k = s^j$ and $j \geq j_0$, we have

$$2^{\binom{\omega(k)-c+l+1}{2}} \cdot n_0 > k > 2^{\binom{\omega(k)-c}{2}} \cdot n_0$$

for $k = s^j$ and $j \geq j_0$. Using $y^2 > y(y-1) > (y-1)^2$ after a short computing we obtain

$$\sqrt{2\log_2 k - 2\log_2 n_0} + c - l - 1 < \omega_k < \sqrt{2\log_2 k - 2\log_2 n_0} + c + 1$$

thus,

$$\sqrt{2\log_2 k} + c_1 < \omega_k < \sqrt{2\log_2 k} + c_2$$

for some constants c_1 and c_2 that does not depend on k. Now let k be an arbitrary integer greater than or equal to s^{j_0}. Since $\{\omega_k\}$ is a constant function on all intervals $[s^j, s^{j+1} - 1]$ and in s^j it increases by one for $j \geq j_0$,

$$\sqrt{2\log_2 k} + c_1 - 1 < \omega_k < \sqrt{2\log_2 k} + c_2$$

for all $k \geq s^{j_0}$. Now changing the constants completes the proof. □

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