GRAY CODES IN GRAPHS

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Abstract. This paper deals with special Gray codes associated with graphs. We examine labellings of a given graph where two labellings are considered successive whenever one can be obtained from the other by interchanging at most k edges.

Introduction

The codes which are now commonly known as Gray codes were invented and patented by F. Gray in 1953 [9]. For a given set X and a symmetric relation R of "small difference" on X, a Gray code is an ordering of all the elements of X such that every two immediately successive elements are in R.

Gray codes were examined for such sets as subsets of a given set ([7] and [12]), permutations ([11] and [19]), combinations ([4], [5], [13] and [17]), partitions of a natural number ([18]), binary trees ([10], [15] and [16]) etc. (See also [2], [3] and [6].)

The concept of Gray code is easily explained in graph-theoretical terms. Let $\Lambda(X)$ be a graph with the vertex set X, where two vertices x and y are joined by an edge whenever x and y are in the "small difference" relation. Then the problem of finding a Gray code on X is equivalent to the problem of finding a Hamiltonian path in $\Lambda(X)$, whereas the problem of finding a closed Gray code is equivalent to the problem of finding a Hamiltonian cycle in $\Lambda(X)$. This method was used in 1958 in the pioneering work of E. N. Gilbert, who examined the Hamiltonian paths on n-cube instead of finding the Gray codes on subsets of a given set.

In this paper we examine closed Gray codes on the set of all nonisomorphic labellings of vertices of a given graph Γ. If we remove an edge from Γ, we can have more possibilities for inserting a new edge such that the resulting graph is isomorphic to Γ. In this way from a labelling Γ_x of Γ we get a new labelling Γ_y and these labellings are in the relation "small difference" (see Definition 1.1). This relation has the following real-life motivation: Assume that we have n users in a network. By successive interchanges of just one line we want to generate all possible "realizations" of the given type of network in the way that no two configurations are repeated until the first is identical to the last. (In the case when Γ is a path or a cycle, we can regard our task as generating of Hamiltonian paths or Hamiltonian cycles, respectively, in a complete graph.)

Let Γ be an arbitrary graph; $V\Gamma$ and $E\Gamma$ are used for the vertex set and the edge set of Γ, respectively. The complement of Γ will be denoted by $\overline{\Gamma}$. By $G(\Gamma)$ we denote the automorphism group of Γ. Permutations of the set $\{a_1, a_2, \dots, a_n\}$ are given by the position of the elements $a_1, a_2, \dots, a_n$. So (a_1, a_3, a_2) means $a_1 \to a_1$, $a_2 \to a_3$ and $a_3 \to a_2$. Composition of mappings is always to be understood from right to left.

1. The k-Copylist of a graph

In this section we give precise definitions of basic notions and some elementary observations.

Let Γ be a graph with vertices $u_1, u_2, \dots, u_n$. In this way we ordered the vertex set of Γ. Let $x = (x_1, x_2, \dots, x_n)$ be any permutation of the set $\{1, 2, \dots, n\}$. Then the labelling of Γ by x, Γ_x, is the bijection

$$\Gamma_x : \{u_1, u_2, \dots, u_n\} \to \{1, 2, \dots, n\}$$

such that $\Gamma_x(u_i) = x_i$ for all i, for which $1 \le i \le n$. We remark that by Γ_x we denote also the graph Γ with vertices labelled by Γ_x; the meaning of Γ_x will always be clear from the context.

Let $G(\Gamma)$ be the automorphism group of Γ. Two labellings Γ_x and Γ_y of Γ are <u>Γ-equivalent</u> iff there is $g \in G(\Gamma)$ such that $\Gamma_x = g \circ \Gamma_y$.

Let us introduce the relation "small difference" on the labellings of Γ.

Definition 1.1. *Two labellings Γ_x and Γ_y are in the relation R_l^Γ iff there is a set A of l mutually different edges of Γ_x and a set B of l mutually different edges of $\overline{\Gamma_x}$ such that $(E\Gamma_x - A) \bigcup B = E\Gamma_y$, where x and y are the permutations of the set $\{1, 2, \ldots, n\}$, $n = |V\Gamma|$ and $l \geq 0$.*

Clearly, the relation R_l^Γ is symmetric.

Each class of Γ-equivalent labellings will be represented by a single labelling. Now we are able to introduce the basic concept of this work.

Definition 1.2. *Let $\mathcal{T}$ be the set of all classes of Γ-equivalent labellings of a graph Γ. The <u>k-Copylist of the graph Γ</u>, $B^k(\Gamma)$, is the graph for which*

$$VB^k(\Gamma) = \{\Gamma_x; \Gamma_x \in \mathcal{T}\} \qquad \textit{and}$$
$$EB^k(\Gamma) = \{[\Gamma_x, \Gamma_y];\ \Gamma_x, \Gamma_y \in \mathcal{T},\ \Gamma_x \neq \Gamma_y \textit{ and there is } l \leq k \textit{ such that } \Gamma_x\, R_l^\Gamma\, \Gamma_y\}, \blacksquare$$

where $k \geq 0$.

It is easy to see that this definition is correct for all $k \geq 0$. Note that $B^k(\Gamma) = B^l(\Gamma)$ if k and l are greater than or equal to $|E\Gamma|$.

Clearly, $B^k(\Gamma)$ is a regular graph. The classes of Γ-equivalent labellings Γ_z such that $\Gamma_z\, R_l^\Gamma\, \Gamma_{id}$ where $\Gamma_z \neq \Gamma_{id}$ and $l \leq k$ are called <u>generators</u> of $B^k(\Gamma)$. The generators can be determined by the sets A and B from Definition 1.1.

The elements of $VB^k(\Gamma)$ depend on the ordering of $V\Gamma$, but the structure of $B^k(\Gamma)$ does not.

Lemma 1.3. *Let Γ and Γ' be isomorphic graphs. Then $B^k(\Gamma)$ is isomorphic to $B^k(\Gamma')$ for all $k \geq 0$.*

Proof. Denote by φ the graph isomorphism between Γ and Γ'. Then φ maps labellings of Γ to labellings of Γ'. So φ induces an isomorphism between $B^k(\Gamma)$ and $B^k(\Gamma')$. $\square$

Now we introduce two basic lemmas.

Lemma 1.4. *The k-Copylist of a graph Γ is a vertex transitive graph.*

Proof. Let $\mathcal{T}$ be the set of all Γ-equivalent labellings. It is easy to see that $[\Gamma_u, \Gamma_v] \in EB^k(\Gamma)$ iff $[\Gamma_{u \circ z}, \Gamma_{v \circ z}] \in EB^k(\Gamma)$ for any permutation z of the set $\{1, 2, \ldots, n\}$.

Since $\Gamma_{x \circ x^{-1} \circ y} = \Gamma_y$, the mapping $\varphi : \mathcal{T} \to \mathcal{T}$ defined for all $\Gamma_z \in \mathcal{T}$ as $\varphi(\Gamma_z) = \Gamma_{z \circ x^{-1} \circ y}$ is automorphism of $B^k(\Gamma)$, which maps Γ_x to Γ_y. $\square$

Thus, the structure of $B^k(\Gamma)$ in any vertex is completely determined by the set of generators.

Lemma 1.5. *$B^k(\overline{\Gamma})$ is isomorphic to $B^k(\Gamma)$.*

Proof. Let $|V\Gamma| = n$. Denote by $u_1, u_2, \ldots, u_n$ the vertices of Γ and $\overline{\Gamma}$ such that $\Gamma \bigcup \overline{\Gamma} = K_n$, where K_n is the complete graph on n vertices.

Since $G(\Gamma) = G(\overline{\Gamma})$, we have $VB^k(\Gamma) = VB^k(\overline{\Gamma})$.

Let z be a generator of $B^k(\Gamma)$. Then there are l-element sets A and B such that $(E\Gamma_{id} - A) \bigcup B = E\Gamma_z$, where $l \leq k$. But since $A \bigcap B = \emptyset$, we have

$$E\overline{\Gamma_z} = \overline{(E\Gamma_{id} - A) \bigcup B} = \left(E\overline{\Gamma_{id}} \bigcup A\right) \bigcap \overline{B} = \left(E\overline{\Gamma_{id}} - B\right) \bigcup A$$

so z is also a generator for $\overline{\Gamma}$. Since the generators of $B^k(\Gamma)$ are just the generators of $B^k(\overline{\Gamma})$, we see that $EB^k(\Gamma) = EB^k(\overline{\Gamma})$. Thus, $B^k(\Gamma)$ is isomorphic to $B^k(\overline{\Gamma})$. $\square$

The following trivial assertions can be helpful in understanding the notion of k-Copylist.

Proposition 1.6. *For any graph Γ we have* $VB^k(\Gamma) = VB^l(\Gamma)$ *and* $EB^k(\Gamma) \supseteq EB^l(\Gamma)$ *if* $0 \leq l \leq k$.

Proposition 1.7. *Let Γ be a graph, $n{=}|V\Gamma|$, $m{=}|E\Gamma|$, $r{=}|G(\Gamma)|$ and $p = \frac{n!}{r}$. Then* $B^0(\Gamma) = D_p$ *and* $B^m(\Gamma) = K_p$, *where K_p and D_p are the complete and discrete graphs, respectively, on p vertices.*

Proposition 1.8. $B^k(K_n) = K_1$, *for all* $k \geq 0$

$B^1(K_n - e) = K_{\binom{n}{2}}$, *where* e *is an edge of* K_n

$B^{n-1}(K_{n,1}) = K_{n+1}$ *and* $B^{n-2}(K_{n,1}) = D_{n+1}$. $\blacksquare$

In the following sections we always choose a certain representation of Γ-equivalent classes. We thus consider only some simple labellings and not the classes of labellings. For brevity, the labelling Γ_x will be denoted just by x in what follows. So, the *labelling* x means Γ_x while the *permutation* x means just x.

2. Paths and Circuits

This section is devoted to finding Hamiltonian cycles in $B^1(P_n)$ and $B^2(C_{n+1})$, where $n \geq 3$, P_n is a path on n vertices and C_{n+1} is a circuit on $n+1$ vertices.

Let us denote the vertices of P_n as follows (see Fig. 2.1):

P_n :

Fig. 2.1

Then $G(P_n) = \{id, w\}$, where

$$id = (u_1, u_2, \ldots, u_n) \qquad \text{and}$$
$$w = (u_n, u_{n-1}, \ldots, u_1) .$$

So, $|VB^1(P_n)| = \frac{n!}{2}$.

The classes of P_n-equivalent labellings will be represented by simple labellings x, where the elements $1, 2$ and 3 are in ordering $1, 2, 3$ or $2, 3, 1$ or $3, 1, 2$ in the permutation x. (There can be some other elements between $1, 2$ and 3.)

In $B^1(P_n)$ we have just three possibilities for choosing A and B to create the generators (see Fig. 2.2):

(**a**) $A = \{[u_i, u_{i+1}]\}$, $B = \{[u_1, u_{i+1}]\}$, where $2 \leq i \leq n-1$

(**b**) $A = \{[u_i, u_{i+1}]\}$, $B = \{[u_i, u_n]\}$, where $1 \leq i \leq n-2$

(**c**) $A = \{[u_i, u_{i+1}]\}$, $B = \{[u_1, u_n]\}$, where $2 \leq i \leq n-2$.

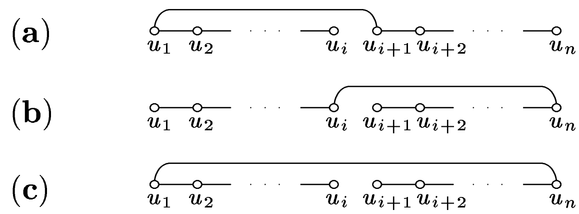

Fig. 2.2

We now present an algorithm for finding a Hamiltonian cycle in $B^1(P_n)$.

Algorithm 2.1.

STEP 1 $i := 0, \quad A_{i+1} := (1, 2, \dots, n)$.
STEP 2 $i := i + 1$. Now $A_i = (n, n-1, \dots, k+1, a_{i_1}, a_{i_2}, \dots, a_{i_k})$, where $a_{i_1} \neq k$ (but it is also possible that $k = n$, i.e., $A_i = (a_{i_1}, a_{i_2}, \dots, a_{i_n})$, where $a_{i_1} \neq n$).
STEP 3 If $k = 2$ then go to STEP 5.
STEP 4 $A_{i+1} := (a_{i_2}, a_{i_3}, \dots, a_{i_k}, a_{i_1}, k+1, k+2, \dots, n)$, go to STEP 2.
STEP 5 $A_{i+1} := (1, 2, \dots, n)$ end.

Here $a_{i_1}, a_{i_2}, \dots a_{i_n}$ stand for the elements of the set $\{1, 2, \dots, n\}$. The Algorithm 2.1 acts on permutations, but we can view these permutations as labellings (see section 1).

Proposition 2.2. *The sequence of labellings $A_1, A_2, \dots, A_{\frac{n!}{2}}, A_{\frac{n!}{2}+1}$ constructed by the Algorithm 2.1 is a Hamiltonian cycle in $B^1(P_n)$ for all $n \geq 3$.*

Proof. Clearly, $A_i \in VB^1(P_n)$ for all A_i constructed by the Algorithm 2.1. ∎

We divide the proof into three steps.

1. In each permutation A_i constructed by the Algorithm 2.1 the elements $1, 2$ and 3 are in ordering $1, 2, 3$ or $2, 3, 1$ or $3, 1, 2$.

This assertion is true if $i = 1$ and it is easy to see that this ordering can not be reversed neither in the STEP 4, nor in the STEP 5. So we have $A_i \neq w \circ A_j$ for all A_i and A_j constructed by the Algorithm 2.1, since the elements $1, 2$ and 3 are in ordering $3, 2, 1$ or $1, 3, 2$ or $2, 1, 3$ in $w \circ A_j$.

2. $[A_i, A_{i+1}] \in EB^1(P_n)$ for all i, for which A_i and A_{i+1} are constructed by the Algorithm 2.1.

   But A_{i+1} can be constructed only in STEP 4, or STEP 5. (In the second case $A_i = (n, n-1, \dots, 3, 1, 2)$.) In both these steps $[A_i, A_{i+1}]$ is an edge of $B^1(P_n)$ created by the generator of (**b**)-type (see above).

3. $A_1, A_2, \dots, A_{\frac{n!}{2}}, A_{\frac{n!}{2}+1}$ is a Hamiltonian cycle in $B^1(P_n)$.

   Let $A = (a_1, a_2, \dots, a_n)$ be a permutation constructed by the Algorithm 2.1 such that $a_k = n$, where $1 \le k \le n$. Then A was constructed from $B = (a_{k+1}, \dots, a_n, a_1, a_2, \dots, a_{k-1}, n)$ after $n-k$ (STEP 2-STEP 4)-cycles of the Algorithm 2.1.

   Let $B = (b_1, b_2, \dots, b_{n-1}, n)$ and $b_l = n-1$, where $1 \le l \le n-1$. Then B was constructed from $C = (b_{l+1}, \dots, b_{n-1}, b_1, \dots, b_{l-1}, n-1, n)$ on $(n-1-l) \cdot n$ cycles of the Algorithm 2.1. So, A was constructed from C on $((n-1)-l) \cdot n + n-k$ cycles of the Algorithm 2.1.

   But since $(2, 1, 3, \dots, n)$ can not be constructed by the Algorithm 2.1 (see part **1** of this proof), the permutation A was constructed from $(1, 2, \dots, n)$ on m cycles of the Algorithm 2.1. Since $(n-1) + (n \cdot (n-2)) + \dots + n \cdot (n-1) \cdot \ldots \cdot 4 \cdot 2 =$

$$n \cdot ((n-1) \cdot (\ldots \cdot (4{\cdot}2 + 3) + \dots) + n-2) + n-1 =$$
$$n \cdot (\ldots k \cdot (\tfrac{(k-1)!}{2} - 1) + k-1 \ldots) + n-1 = \tfrac{n!}{2} - 1\,,$$

   m is at most $\frac{n!}{2} - 1$. So there is just one $i \le \frac{n!}{2}$ such that $A_i = A$, for any permutation A with 1, 2 and 3 in allowed ordering (i can be strictly computed).

   Since $A_{\frac{n!}{2}} = (n, n-1, \dots, 3, 1, 2)$, we have $A_{\frac{n!}{2}+1} = (1, 2, \dots, n)$ and $A_1, A_2, \dots, A_{\frac{n!}{2}+1}$ is a Hamiltonian cycle in $B^1(P_n)$. Clearly, the algorithm finishes in the STEP 5 with $i = \frac{n!}{2}$. □

Now we find a Hamiltonian cycle in $B^2(C_n)$, where $n \ge 4$. Let us denote the vertices of C_n as follows (see Fig. 2.3):

C_n :

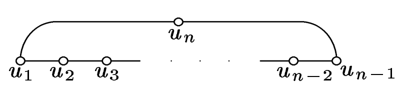

Fig. 2.3

Since $G(C_n)$ is the dihedral group, we have $|G(C_n)| = 2n$ and $|VB^2(C_n)| = \frac{(n-1)!}{2}$.

Note that $B^1(C_n) = D_m$, where $m = \frac{(n-1)!}{2}$. The classes of C_n-equivalent labellings will be represented by simple labellings x, where the element n is in the n-th position and the elements $1, 2$ and 3 are in ordering $1, 2, 3$ or $2, 3, 1$ or $3, 1, 2$ in the permutation x.

Then $\varphi : VB^2(C_{n+1}) \to VB^1(P_n)$, where $n \geq 3$, defined as

$$\varphi(a_1, a_2, \dots, a_n, n{+}1) = (a_1, a_2, \dots, a_n)$$

is a bijection between $VB^2(C_{n+1})$ and $VB^1(P_n)$.

Lemma 2.3. *Let $n \geq 3$ and $[c, d]$ be an edge of $B^1(P_n)$ created by a generator of (**b**)-type. Then $[\varphi^{-1}(c), \varphi^{-1}(d)]$ is an edge of $B^2(C_{n+1})$.*

Proof. Let $c = (c_1, c_2, \dots, c_n)$. Since the edge $[c, d]$ is created by the generator of (**b**)-type, we have

$$\begin{aligned} d =& (c_1, c_2, \dots, c_k, c_n, c_{n-1}, \dots, c_{k+1}) \qquad \text{or} \\ d =& (c_{k+1}, c_{k+2}, \dots, c_n, c_k, c_{k-1}, \dots, c_1) \end{aligned}$$

according to ordering the elements 1, 2 and 3, where $1 \leq k \leq n{-}2$.
In both these cases it is sufficient to choose $A = \{[u_n, u_{n+1}], [u_k, u_{k+1}]\}$ and $B = \{[u_k, u_n], [u_{k+1}, u_{n+1}]\}$ in Definition 1.1 and we see that $[\varphi^{-1}(c), \varphi^{-1}(d)]$ is an edge of $B^2(C_{n+1})$ (see Fig. 2.4). $\square$

Fig. 2.4

Let the Algorithm 2.4 be created from the Algorithm 2.1 by replacing of all the permutations $(x_1, x_2, \dots, x_n)$ by the permutations $(x_1, x_2, \dots, x_n, n{+}1)$. Then we have the following consequence of Proposition 2.2 and Lemma 2.3:

Proposition 2.5. *The Algorithm 2.4 finds a Hamiltonian cycle in $B^2(C_{n+1})$ for all $n \geq 3$.*

We remark that $B^2(C_{n+1})$ is not isomorphic to $B^1(P_n)$ if $n \geq 4$.

3. Bipartite graphs

This section is devoted to finding Hamiltonian cycles in $B^{m+n-2}(K_{m,n})$, where $m \geq n$ and $K_{m,n}$ is the complete bipartite graph.

Let us denote the vertices of $VK_{m,n}$ as shown in Fig. 3.1.

Then $|G(K_{m,n})| = m! \cdot n!$ if $m > n$, and $|G(K_{m,n})| = 2 \cdot (n!)^2$ if $m = n$.

The classes of $K_{m,n}$-equivalent labellings will be represented by simple labellings $x = (a_1, a_2, \ldots, a_{m+n})$, where $a_1 < a_2 < \ldots < a_m$ and $a_{m+1} < a_{m+2} < \ldots < a_{m+n}$. Moreover, we claim that $a_1 = 1$ if $m = n$.

$K_{m,n}$:

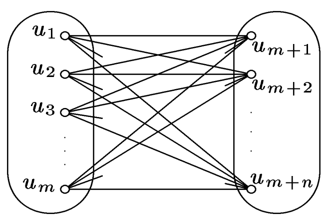

Fig. 3.1

We have only one type of generator in $B^{m+n-2}(K_{m,n})$ if $m > n+2$ or $m = n$ (see Fig. 3.2 (**a**) - reversing an edge of $K_{m,n}$). We call it a generator of (**a**)-type. Certainly the generator of (**a**)-type is also a generator for $B^{m+n-2}(K_{m,n})$, where $m = n+2$ or $m = n+1$. However, we have still one more type of generator in $B^{m+n-2}(K_{m,n})$ if $m = n+2$ (see Fig. 3.2 (**b**)). We call it a generator of (**b**)-type.

It is easy to check that $B^{m+n-3}(K_{m,n})$ is a discrete graph whenever $m \neq n+1$ (use Lemma 1.5). But $B^n(K_{m,n})$ is not discrete if $m = n+1$, since in $B^n(K_{n+1,n})$ we have a generator of (**c**)-type (see Fig. 3.2 (**c**)).

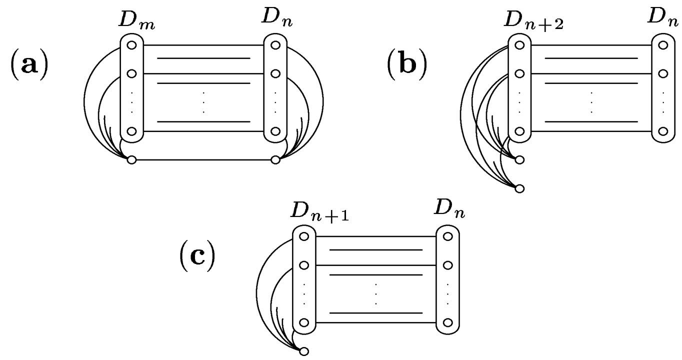

Fig. 3.2

In the following we use only the generators of (**a**)-type.

Denote by $C_{k,l}$ the graph whose vertex set is the set of all l-element combinations of k-element set, where two vertices are joined by an edge whenever they differ as sets in just one element. Then we have:

Lemma 3.1. *There is a graph homomorphism from $C_{k,l}$ into $B^{m+n-2}(K_{m,n})$ for some k and l depending on m and n.*

Proof. We distinguish two cases.

1. $m > n$.

Let $\varphi : C_{m+n,m} \to B^{m+n-2}(K_{m,n})$ be a mapping defined as

$$\varphi\{a_1, a_2, \dots, a_m\} = (b_1, b_2, \dots, b_m, b_{m+1}, \dots, b_{m+n})$$

where $\{a_1, a_2, \dots, a_m\} = \{b_1, b_2, \dots, b_m\}$, $b_1<b_2<\dots<b_m$, $b_{m+1}<b_{m+2}<\dots<b_{m+n}$ and $\{b_1, \dots, b_{m+n}\} = \{1, \dots, m+n\}$. Then φ is a bijection from $VC_{m+n,m}$ to $VB^{m+n-2}(K_{m,n})$.

Two vertices A and A' are joined by an edge in $C_{m+n,m}$ whenever they differ in just one element. But then $\varphi(A)$ and $\varphi(A')$ are joined by an edge created by the generator of (**a**)-type in $B^{m+n-2}(K_{m,n})$. So φ is a graph homomorphism.

2. $m = n$.

Let $\varphi : C_{2n-1,n-1} \to B^{2n-2}(K_{n,n})$ be a mapping defined

$$\varphi\{a_1, a_2, \dots, a_{n-1}\} = (1, b_2, b_3, \dots, b_n, b_{n+1}, \dots, b_{2n})$$

where $\{a_1, a_2, \dots, a_{n-1}\} = \{b_2-1, b_3-1, \dots, b_n-1\}$, $b_2<\dots<b_n$, $b_{n+1}<\dots<b_{2n}$ and $\{b_2, \dots, b_{2n}\} = \{2, \dots, 2n\}$.

Then it can be shown that φ is a bijection from $VC_{2n-1,n-1}$ to $VB^{2n-2}(K_{n,n})$ which is a graph homomorphism by arguments similar to the previous ones. $\square$

In [4] P. J. Chase gives an algorithm finding a Hamiltonian cycle in $C_{k,l}$ for all k and l such that $k>l>0$ (see also [5]). G. Ehrlich gives another algorithm in [6]. Thus, Lemma 3.1 can be used for finding Hamiltonian cycles in $B^{m+n-2}(K_{m,n})$ from those in $C_{k,l}$. However, since $C_{k,l}$ can be decomposed into two graphs Γ and Γ' isomorphic to $C_{k-1,l}$ and $C_{k-1,l-1}$, respectively, and $C_{l+1,l}$ is isomorphic to K_{l+1} and $C_{k,1}$ is isomorphic to K_k, it can be proved that $C_{k,l}$ is Hamiltonian-connected by induction (see section 4, part 3 of proof of Lemma 4.1). Thus, $B^{m+n-2}(K_{m,n})$ is Hamiltonian-connected graph as well (see section 4 for the notion of the Hamiltonian-connectivity).

As we mentioned above, $B^{m+n-3}(K_{m,n})$ is a discrete graph for $m \neq n+1$, while $B^{m+n-2}(K_{m,n})$ has a Hamiltonian cycle. But if $m = n+1$, even the graph $B^n(K_{n+1,n})$ is not discrete. In $B^n(K_{n+1,n})$, edges are created by the generators of (**c**)-type.

Two vertices $(a_1, a_2, \ldots, a_{2n+1})$ and $(b_1, b_2, \ldots, b_{2n+1})$ are joined by a generator of (**c**)-type in $B^n(K_{n+1,n})$ whenever

$$\left|\{a_1, a_2, \ldots, a_{n+1}\} \bigcap \{b_1, b_2, \ldots, b_{n+1})\}\right| = 1\,.$$

It means that

$$\left|\{a_{n+2}, a_{n+3}, \ldots, a_{2n+1}\} \bigcap \{b_{n+2}, b_{n+3}, \ldots, b_{2n+1}\}\right| = 0\,.$$

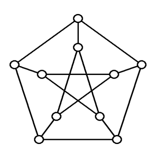

Fig. 3.3

Thus, $B^n(K_{n+1,n}) = O_{n+1}$, where O_{n+1} are the odd graphs (see [1] and [14]). The odd graphs have been studied intensively. It is known that O_n has a Hamiltonian cycle for $n \in \{4,5,6,7\}$ ([14]), but for $n>7$ it is still an open problem. However, $B^2(K_{3,2}) = O_3$ has no Hamiltonian cycle, because O_3 is the well-known Petersen graph (see Fig. 3.3).

4. Forks

This section is devoted to finding Hamiltonian cycles in 1-Copylist of the fork F_n, where $n \geq 5$.

Fork F_n is a tree consisting of a path on $n-2$ vertices, ($n-2 \geq 3$), with two new vertices adjoined to one end of the path. Let us denote the vertices of F_n as shown in Fig. 4.1.

Then $G(F_n) = \{id, w\}$, where

$$\begin{aligned} id =& (u_1, u_2, \ldots, u_n) \qquad \text{and} \\ w =& (u_1, u_2, \ldots, u_{n-2}, u_n, u_{n-1}) . \end{aligned}$$

So, $|VB^1(F_n)| = \frac{n!}{2}$.

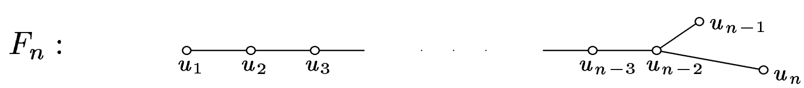

Fig. 4.1

The classes of F_n-equivalent labellings will be represented by simple labellings $x = \{x_1, \ldots x_{n-1}, x_n\}$, where $x_{n-1} < x_n$.

In $B^1(F_n)$ we have three types of generators:

(a) $A = \{[u_{i-1}, u_i]\}$, $B = \{[u_1, u_i]\}$, where $3 \leq i \leq n-2$

(b) $A = \{[u_{n-4}, u_{n-3}]\}$, $B = \{[u_{n-4}, u_{n-1}]\}$ or
$A = \{[u_{n-4}, u_{n-3}]\}$, $B = \{[u_{n-4}, u_n]\}$

(c) $A = \{[u_{n-2}, u_{n-1}]\}$, $B = \{[u_2, u_{n-1}]\}$ or
$A = \{[u_{n-2}, u_n]\}$, $B = \{[u_2, u_n]\}$,

where A and B are the sets from Definition 1.1 (see Fig. 4.2).

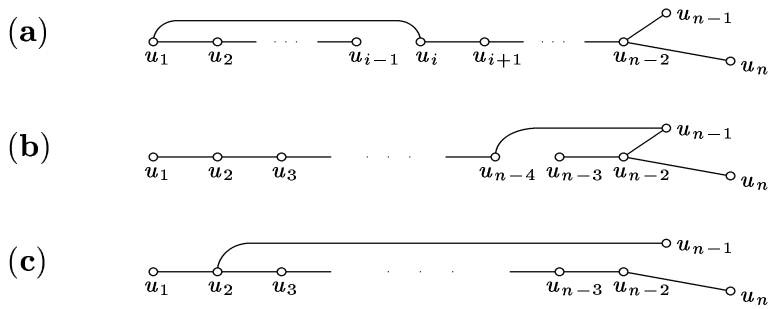

Fig. 4.2

We recall that a graph Γ is Hamiltonian-connected iff there is a Hamiltonian path between any two distinct vertices of Γ. It is easy to see that there is a Hamiltonian cycle in Γ if Γ is Hamiltonian-connected and $|V\Gamma| > 2$.

Lemma 4.1. *$B^1(F_n)$ is Hamiltonian-connected if $n \geq 7$.*

Proof. We divide the proof into five steps.

1. The maximal connected subgraphs S_7 of $B^1(F_7)$ with edges created only by generators of (**a**)-type are Hamiltonian-connected.

We remark that all such graphs S_7 are mutually isomorphic and have $(7-3)! = 24$ vertices. One of the graphs S_7 is in Fig. 4.3. The vertices $A, \dots, Z$ are labellings of F_7 and below we give the first four members of these labellings, since the last three are always 5,6,7 in this ordering. So, instead of $Z = (1, 2, 3, 4, 5, 6, 7)$ we simply write $Z = 1234$.

The assertion **1** will be proved by simple enumeration of Hamiltonian paths. Since S_7 is vertex-transitive, it is enough to find Hamiltonian paths from all the vertices of S_7 to the vertex Z (see Fig. 4.3):

$ABCDEFGHIJKLMNOPRSTUVXYZ$ $\quad A = 2134$
$BGHIVXYPRJKLMNOCDEFSTUAZ$ $\quad B = 4312$
$CBGFEDKLMHIJRSTNOPYXVUAZ$ $\quad C = 3412$
$DCOPYXEFGBAUVIHMNTSRJKLZ$ $\quad D = 1432$
$EFGBCDKLMHIJRSTNOPYXVUAZ$ $\quad E = 4132$
$FGBCDEXYPONTSRJKLMHIVUAZ$ $\quad F = 3142$
$GBCONTSFEDKLMHIJRPYXVUAZ$ $\quad G = 1342$
$HIJKLMNTSRPOCDEFGBAUVXYZ$ $\quad H = 2431$

$IVXYPONMHGFEDCBAUTSRJKLZ$	$I = 4231$
$JIHMLKDEFGBCONTSRPYXVUAZ$	$J = 3241$
$KJRPYXVIHGBAUTSFEDCONMLZ$	$K = 2341$
$LMHIJKDEFGBCONTSRPYXVUAZ$	$L = 4321$
$MNOCDEFSTUABGHIVXYPRJKLZ$	$M = 3421$
$NTSRJKDCOPYXEFGBAUVIHMLZ$	$N = 1243$
$OPRSTNMLKJIHGFEDCBAUVXYZ$	$O = 2143$
$PRJKLMNOCDEFSTUABGHIVXYZ$	$P = 4123$
$RPONTSFGHMLKJIVUABCDEXYZ$	$R = 1423$
$STNOCBGFEDKLMHIJRPYXVUAZ$	$S = 2413$
$TSRPONMLKJIHGFEDCBAUVXYZ$	$T = 4213$
$UTSFEDCONMLKJRPYXVIHGBAZ$	$U = 3124$
$VUTSFGHIJRPYXEDKLMNOCBAZ$	$V = 1324$
$XVIHGBAUTSFEDCONMLKJRPYZ$	$X = 2314$
$YXVUTSRPONMLKJIHGFEDCBAZ$	$Y = 3214$

2. The maximal connected subgraphs S_n of $B^1(F_n)$ with edges created only by generators of (**a**)-type are Hamiltonian-connected if $n \geq 7$.

We prove this assertion by induction.

If $n > 7$, the graph S_n consists of $n-3$ copies of S_{n-1} joined by edges created by the generator z with $A = \{[u_{n-3}, u_{n-2}]\}$ and $B = \{[u_1, u_{n-2}]\}$ (see Definition 1.1). The edges created by the generator z form a linear factor in S_n and between any two distinct copies of S_{n-1} in S_n there are exactly $(n-5)!$ edges created by the generator z. (We fix the elements in the first, $n-3$-rd, …, n-th positions in labellings.) In this way we obtain K_{n-3} from S_n by contraction of all the copies of S_{n-1} into single points.

For any $A, B \in VS_n$ we find a Hamiltonian path from A to B in S_n. We distinguish two cases:

a. A and B are in the same copy of S_{n-1} (see Fig. 4.4).

We can find a Hamiltonian path $\mathcal{H}$ from A to B in S_{n-1} by induction. Since the edges created by z form a linear factor, there are two successive vertices on $\mathcal{H}$, say X and Y, such that $z \circ X$ and $z \circ Y$ are in distinct copies of S_{n-1} in S_n. Let us order the remaining copies of S_{n-1} arbitrarily. Since $n>7$, we have $(n-5)! > 2$. Thus, we can choose nonadjacent edges between the copies of S_{n-1} which join them in the required order (see

Fig. 4.4). Then we can complete $\mathcal{H} - [X, Y]$ to a Hamiltonian path in S_n using induction.

b. A and B are in distinct copies of S_{n-1}.

Let us order the copies of S_{n-1} such that the one containing A will be the first and that containing B will be the last. Then we can find a Hamiltonian path in S_n as in the previous case.

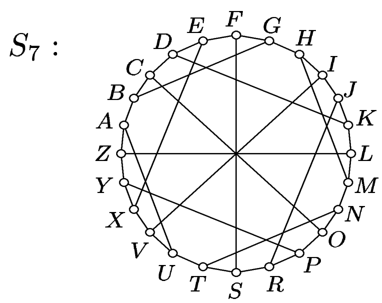

Fig. 4.3

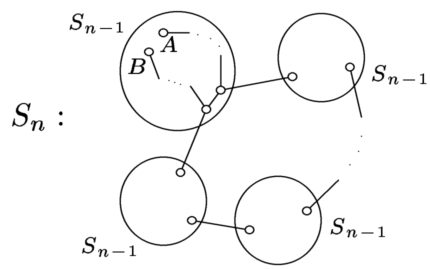

Fig. 4.4

3. There is a path from B to C in $C_{k,2}$ traversing all the vertices of $C_{k,2}$ just once and missing the vertex A for all mutually different A, B, C, where $A, B, C \in VC_{k,2}$ and $k \geq 3$.

Here $C_{k,2}$ is the vertex-transitive graph defined in the section 3. Again, we prove this assertion using induction.

If $k = 3$, then $C_{k,2} = K_3$ and the assertion trivially holds.

Let $k > 3$. Then $C_{k,2}$ can be decomposed into two graphs Γ and Γ' (all combinations in Γ' contain the element k, but those of Γ do not), where Γ is isomorphic to $C_{k-1,2}$ and Γ' is isomorphic to K_{k-1} (see Fig. 4.5). Since $C_{k,2}$ is a vertex-transitive graph, we can suppose that $A \in \Gamma$. We distinguish three cases:

a) $B, C \in V\Gamma$

b) $B \in V\Gamma$, $C \in V\Gamma'$

c) $B, C \in V\Gamma'$

a) There is a path $\mathcal{H}$ in Γ traversing all the vertices of Γ except of A (by induction). Let X and Y be two successive vertices on $\mathcal{H}$. Then there are $X', Y' \in V\Gamma'$, such that $X' \neq Y'$ and X is joined to X' and Y is joined to Y'. (Each vertex from Γ is joined to exactly two vertices in Γ'.) Since Γ' is isomorphic to K_{k-1}, we can complete $\mathcal{H} - [X, Y]$ to the required path in $C_{k,2}$.

The remaining cases **b**) and **c**) can be proved similarly using the fact that each vertex of Γ is joined to exactly two vertices of Γ' and each vertex of Γ' is joined to exactly $k-2$ vertices of Γ.

We remark that the assertion **3** implies that $C_{k,2}$ is Hamiltonian-connected.

$C_{k,2}$:

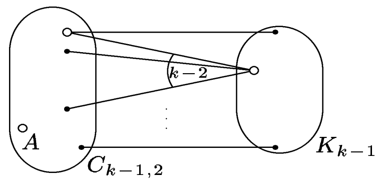

Fig. 4.5

4. Each maximal connected subgraph S_n' of $B^1(F_n)$ created only by generators of (**a**) and (**b**)-types is Hamiltonian-connected.

Again, such subgraphs are mutually isomorphic, so the definition of S_n' is correct. Let all S_n-subgraphs of S_n' be contracted into single points. Then the resulting graph is isomorphic to $C_{n-1,2}$.

Now we can prove the assertion **4** by arguments similar to those used in the proof of the assertion **2**. If the vertices A, B of S_n' are in the same copy of S_n we use the assertion **3**, and if the vertices A and B are in distinct copies of S_n we use the Hamiltonian-connectivity of $C_{n-1,2}$.

5. $B^1(F_n)$ is Hamiltonian-connected.

Let all subgraphs S_n' of $B^1(F_n)$ be contracted into single points. Then the resulting graph is isomorphic to K_n and so the assertion **5** can be proved by arguments similar to those used in the proof of the assertion **2**. $\square$

The following lemma completes the previous one.

Lemma 4.2. *$B^1(F_n)$ is Hamiltonian-connected if $n \in \{5,6\}$.*

Proof. Let S_n' be the maximal connected subgraph of $B^1(F_n)$ created only by generators of (**a**) and (**b**)-types, where $n \in \{5,6\}$.

The graph S'_5 containing the labelling $(1,2,3,4,5)$ is in Fig. 4.6, where

$$\begin{array}{lll} A=(1,2,3,4,5), & B=(1,4,3,2,5), & C=(4,1,3,2,5), \\ D=(4,5,3,1,2), & E=(4,2,3,1,5), & F=(2,4,3,1,5), \\ G=(2,1,3,4,5), & H=(2,5,3,1,4), & I=(5,2,3,1,4), \\ J=(5,4,3,1,2), & K=(5,1,3,2,4), & L=(1,5,3,2,4). \end{array}$$

It is easy to verify that S'_5 is Hamiltonian-connected.

S'_5 :

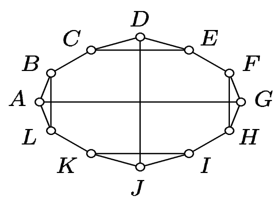

Fig. 4.6

Denote by Γ the maximal connected subgraph of $B^1(F_6)$ created only by generators of (**b**)-types and one generator of (**a**)-type with $A = \{[u_3, u_4]\}$ and $B = \{[u_1, u_4]\}$ (see Definition 1.1). Then Γ is isomorphic to S'_5. So, S'_6 consists of four copies of S'_5 connected by a generator with $A = \{[u_2, u_3]\}$ and $B = \{[u_1, u_3]\}$. Moreover, there are exactly three edges between any two copies of S'_5 in S'_6. Since we get K_4 by contraction of all the copies of S'_5 in S'_6, using the arguments from the assertion **2** of the previous proof we obtain that S'_6 is Hamiltonian-connected.

So the graphs S'_n are Hamiltonian-connected if $n \in \{5,6\}$. But then the assertion **5** of the previous proof completes the proof. $\square$

We conclude this section with the following proposition:

Proposition 4.3. *$B^1(F_n)$ is Hamiltonian-connected.*

We remark that the proofs of Lemma 4.1 and Lemma 4.2 can be used for finding Hamiltonian paths between any two given vertices of $B^1(F_n)$.

Concluding Remarks

It would be interesting to characterize those graphs Γ for which $B^k(\Gamma)$

is connected for "small" k. Such k express some sort of stability property of Γ. (The concept of semi-stable graph (e.g. [8]) is in close relation to such idea of stability.)

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