Derivačné vzorce

1. $a' = 0\,, \;\; a \in \mathbb{R}$
2. $(x^a)' = a x^{a-1}, \;\; a \in \mathbb{R}$
3. $(\sin x)' = \cos x$
4. $(\cos x)' = -\sin x$
5. $(\operatorname{tg} x)' = \dfrac{1}{\cos^2 x}$
6. $(\operatorname{cotg} x)' = -\dfrac{1}{\sin^2 x}$
7. $(e^x)' = e^x$
8. $(a^x)' = a^x \ln a\,, \;\; a > 0, \; a \neq 1$
9. $(\ln x)' = \dfrac{1}{x}$
10. $(\log_a x)' = \dfrac{1}{x \ln a}\,, \; a > 0, \; a \neq 1$
11. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$
12. $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$

$x \in (-1, 1)$

13. $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$
14. $(\operatorname{arccotg} x)' = -\dfrac{1}{1+x^2}$

Derivačné pravidlá

a. $(a.f(x))' = a.f'(x)\,, \;\; a \in \mathbb{R}$

b. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

c. $(f(x).g(x))' = f'(x).g(x) + f(x).g'(x)$

d. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x).g(x) - f(x).g'(x)}{g^2(x)}$

e. $(f\,(g(x)))' = f'\,(g(x))\,.\,g'(x)$

f. $\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}$

1. Priamo z definície vypočítajte deriváciu funkcie $f(x) = \dfrac{1}{x}$ v bode $a = 2$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \lim_{x \to 2} \frac{\frac{2 - x}{2x}}{x - 2} = \lim_{x \to 2} \frac{\overset{-1}{\cancel{(2-x)}}}{\cancel{(x-2)} \cdot 2x} = \lim_{x \to 2} \frac{-1}{2x} =$$

$$= -\frac{1}{4}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f'(2) = -\frac{1}{4}$$

2. Priamo z definície vypočítajte deriváciu funkcie $f(x) = \operatorname{tg} x$ v bode $a = 0$.

$$\lim_{x \to 0} \frac{\operatorname{tg} x - 0}{x - 0} = \lim_{x \to 0} \frac{\operatorname{tg} x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{\sin x}{x \cdot \cos x} =$$

$$= \lim_{x \to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\frac{1}{\cos x}}_{\to 1} = \underline{\underline{1}}$$

$$f'(x) = \frac{1}{\cos^2 x}$$

$$f'(0) = \frac{1}{1} = 1$$

3. Priamo z definície vypočítajte deriváciu funkcie $f(x) = \ln x$.

$\boxed{a} \in D(f) \qquad D(f) = (0, \infty)$

$$\lim_{x \to a} \frac{\overbrace{\ln x - \ln a}^{= \ln\left(\frac{x}{a}\right)}}{x - a} = \lim_{t \to 0} \frac{\ln\left(\frac{t+a}{a}\right)}{t} = \lim_{t \to 0} \frac{\ln\left(1 + \frac{t}{a}\right)}{t} =$$

$t = x - a \Rightarrow x = t + a$

$t \to 0$

$u = \frac{t}{a} \Rightarrow t = a \cdot u$

$u \to 0$

$$= \lim_{u \to 0} \frac{\ln(u+1)}{a \cdot u} = \frac{1}{a}$$

$\to 1$

$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

4. Vypočítajte deriváciu funkcie $f(x) = (x^3 + 8)(x - 2)$.

$$f'(x) = \underbrace{3x^2(x-2)} + (x^3+8) \cdot 1 = 3x^3 - 6x^2 + x^3 + 8 = \underline{\underline{4x^3 - 6x^2 + 8}}$$

5. Vypočítajte deriváciu funkcie $f(x) = \dfrac{\ln x}{x}$.

$$f'(x) = \frac{\overbrace{\frac{1}{x}\cdot x} - \ln x \cdot 1}{x^2} = \underline{\underline{\frac{1-\ln x}{x^2}}}$$

6. Vypočítajte deriváciu funkcie $f(x) = \dfrac{1}{x^2 - 1}$.

konštanta

$$\frac{1}{x^2-1} = (x^2-1)^{-1}$$

$$f'(x) = -1\,(x^2-1)^{-2} \cdot 2x = \frac{-2x}{(x^2-1)^2}$$

7. Vypočítajte deriváciu funkcie $f(x) = \ln\left(\sqrt{\frac{x-2}{x+2}}\right)$.

$$\left.\begin{aligned} a(x) &= \ln x \\ b(x) &= \sqrt{x} \\ c(x) &= \frac{x-2}{x+2} \end{aligned}\right\} f(x) = a(b(c(x)))$$

$$f'(x) = \frac{1}{\sqrt{\frac{x-2}{x+2}}} \cdot \frac{1}{2\sqrt{\frac{x-2}{x+2}}} \cdot \frac{\overbrace{1\cdot(x+2) - (x-2)\cdot 1}^{4}}{(x+2)^2} =$$

$$= \frac{1}{\cancel{2}\cdot\frac{x-2}{\cancel{x+2}}} \cdot \frac{\cancel{4}\,2}{(x+2)^{\cancel{2}}} = \frac{2}{(x-2)(x+2)} = \frac{2}{x^2-4}$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

8. Vypočítajte deriváciu funkcie $f(x) = e^{\sin^2 x^3} + \frac{x^7}{3} - \frac{1-x}{1+x}$.

$$\left(e^{\sin^2 x^3}\right)' = e^{\sin^2 x^3} \cdot \underbrace{2\sin x^3 \cdot \cos x^3}_{\sin 2x^3} \cdot 3x^2 = 3x^2 \cdot \sin 2x^3 \cdot e^{\sin^2 x^3}$$

$$\left(\frac{x^7}{3}\right)' = \frac{7x^6}{3}$$

$$\left(\frac{1-x}{1+x}\right)' = \frac{\overbrace{-1(1+x) - (1-x)\cdot 1}^{-2}}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$f'(x) = 3x^2 \cdot \sin 2x^3 \cdot e^{\sin^2 x^3} + \frac{7}{3}x^6 + \frac{2}{(1+x)^2}$$

9. Vypočítajte deriváciu funkcie $f(x) = x^x$.

$$\underline{x^x} = e^{\ln x^x} = \boxed{e^{\underline{x \cdot \ln x}}}$$

$$f'(x) = \underbrace{e^{x \cdot \ln x}}_{= x^x} \cdot \left(1 \cdot \ln x + x \frac{1}{x}\right) = \underline{x^x (\ln x + 1)}$$

10. Nájdite dotyčnicu a normálu ku grafu funkcie $f(x) = \underbrace{\dfrac{x+1}{x^2+1}}$ v bode $x_0 = 0$.

$k_t = f'(x_0)$ $\quad A = [x_0, y_0]$ $\quad \boxed{A = [0, 1]}$

$$f'(x) = \frac{1.(x^2+1) - (x+1).2x}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2}$$

$f'(0) = \underline{\underline{1}} \Rightarrow \boxed{k_t = 1}$ $\quad k_n = -\dfrac{1}{k_t}$ $\quad \boxed{k_n = -1}$

$\boxed{\text{Dotyčnica}}$ $\quad y = kx + q$ $\quad \boxed{\text{Normála}}$

$1 = 1.0 + q \Rightarrow \underline{\underline{q = 1}}$ $\quad 1 = -1.0 + q \Rightarrow q = 1$

$\boxed{t: \; y = x + 1}$ $\quad \boxed{n: \; y = -x + 1}$

11. Zistite pod akým uhlom pretína graf funkcie

$D(f) = R - \{-1, 0, 1\}$

$$f(x)=\frac{1}{x-1}+\frac{1}{x}+\frac{1}{x+1} \quad \text{os } o_x\,.$$

$$\frac{1}{x-1}+\frac{1}{x}+\frac{1}{x+1} = \frac{x^2+\cancel{x}+x^2-1+x^2-\cancel{x}}{(x-1)x(x+1)} = \frac{3x^2-1}{(x^2-1)x} = \boxed{\frac{3x^2-1}{x^3-x} = f(x)}$$

$$f(x)=0 \qquad 3x^2-1=0 \iff x^2=\frac{1}{3} \iff \boxed{x=\pm\frac{\sqrt{3}}{3}}$$

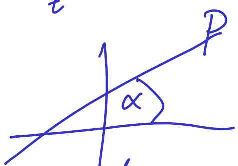
$$f'(x) = \frac{6x(x^3-x)-(3x^2-1)(3x^2-1)}{(x^3-x)^2} = \frac{6x^4-6x^2-(3x^2-1)^2}{(x^3-x)^2} =$$

$$= \frac{6x^4-\cancel{6x^2}-9x^4+\cancel{6x^2}-1}{(x^3-x)^2} = \frac{-3x^4-1}{(x^3-x)^2}$$

$$\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \qquad \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$\left.\begin{aligned} f'\left(-\frac{\sqrt{3}}{3}\right) &= \dots = -9 \\ f'\left(\frac{\sqrt{3}}{3}\right) &= \dots = -9 \end{aligned}\right\} \quad k_t = -9$$

$$\boxed{\alpha = \operatorname{arctg}(-9)}$$



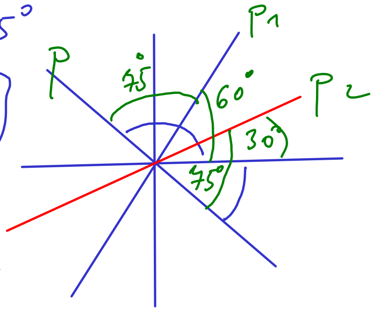
$k_p = \operatorname{tg}\alpha$

12. Nájdite dotyčnicu a normálu ku grafu funkcie $f(x) = \sqrt{2x}$ ak dotyčnica zviera s priamkou $x + y + 7 = 0$ uhol $\frac{5}{12}\pi$.

$\frac{5}{12}\pi \leftrightarrow 75^\circ$

$\boxed{\alpha_1 = 60^\circ = \frac{\pi}{3}}$

$\alpha_2 = 30^\circ = \frac{\pi}{6}$



$y = -x - 7$

$k = -1 \implies \alpha = 135^\circ$

-45°

$\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 60^\circ = \frac{1}{2}$

$\operatorname{tg} 60^\circ = \sqrt{3} = k_t$

$\boxed{k_t = \sqrt{3}, \quad k_n = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}}$

$f'(x) = \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}}$

$f'(x_0) = k_t$

$y = kx + q$

$\frac{1}{\sqrt{2x}} = \sqrt{3} \quad / \cdot \sqrt{2x}$

$1 = \sqrt{6x} \quad /^2$

$1 = 6x$

$x_0 = \frac{1}{6} \qquad y_0 = \frac{\sqrt{3}}{3}$

$\boxed{A = \left[\frac{1}{6}, \frac{\sqrt{3}}{3}\right]}$

Dotyčnica

$\frac{\sqrt{3}}{3} = \sqrt{3} \cdot \frac{1}{6} + q \implies q = \frac{\sqrt{3}}{6}$

$\boxed{t: \ y = x\sqrt{3} + \frac{\sqrt{3}}{6}}$

Normála

$\frac{\sqrt{3}}{3} = -\frac{\sqrt{3}}{3} \cdot \frac{1}{6} + q$

$q = \frac{7}{18}\sqrt{3}$

$\boxed{n: \ y = \frac{-x\sqrt{3}}{3} + \frac{7}{18}\sqrt{3}}$

13. Zistite, či je funkcia f spojitá a či je diferencovateľná v bode $a = 0$.

$$f(x) = \begin{cases} x.\sin\frac{1}{x} & , \text{ pre } x \neq 0 \\ 0 & , \text{ pre } x = 0 \end{cases}$$

Spojitosť

$$\lim_{x\to 0^-} \underbrace{x}_{0} \cdot \underbrace{\sin\frac{1}{x}}_{\text{ohraničená}} = 0$$

$$\lim_{x\to 0^+} x \cdot \sin\frac{1}{x} = 0$$

$\Rightarrow$ $f(x)$ je spojitá v $a = 0$

Diferencovateľnosť

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$f(0) = 0$

$x - 0 = x$

$$\lim_{x\to 0^-} \frac{\cancel{x} . \sin\frac{1}{x} - 0}{\cancel{x} - 0} = \lim_{x\to 0^-} \sin\frac{1}{x} \quad \nexists$$

$$\lim_{x\to 0^+} \ldots = \lim_{x\to 0^+} \sin\frac{1}{x} \quad \nexists$$

$\Rightarrow$ $f(x)$ nie je difer. v $a = 0$

14. Nájdite diferenciál funkcie $f(x) = x^2 + 2x - 3$ v bode $x_0 = 1$.

$$df_{x_0}(x) = f'(x_0)(x - x_0)$$

$$\left.\begin{aligned} f'(x) &= 2x+2 \\ f'(x_0) &= 4 \end{aligned}\right\} \quad df_1(x) = 4(x-1) = \underline{\underline{4x-4}}$$

15. Nájdite diferenciál funkcie $f(x) = \dfrac{1}{x}$.

$$df_{x_0}(x) = f'(x_0)(x - x_0)$$

$$f'(x) = -\frac{1}{x^2}$$

$$\boxed{df_{x_0}(x) = -\frac{1}{x_0^2}(x - x_0)}$$

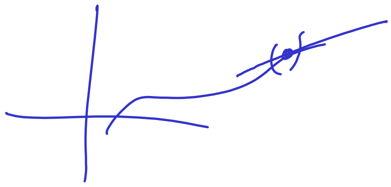
$$x_0 \in D(f)$$

16. Pomocou diferenciálu funkcie vypočítajte približnú hodnotu arctg 1,05.

$a = 1{,}05$

$x_0 = 1$

$\operatorname{arctg} 1 = \frac{\pi}{4}$



$$f(a) \approx f(x_0) + df_{x_0}(a)$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(1) = \frac{1}{2} \Rightarrow df_1(x) = \frac{1}{2}(x-1) = \cancel{\frac{x}{2} - \frac{1}{2}}$$

$$\operatorname{arctg} 1{,}05 \approx \operatorname{arctg} 1 + df_1(1{,}05) = \frac{\pi}{4} + \frac{1}{2}(1{,}05 - 1) = \frac{\pi}{4} + \frac{1}{2} \cdot 0{,}05 =$$

$$\pi = 3{,}1415\ldots$$

$$\operatorname{arctg} 1{,}05 \approx 0{,}8104 \doteq 0{,}81$$

kalkulačka $0{,}8098 \doteq 0{,}81$

17. Raketa vystrelená zo Zeme letí kolmo nahor a jej vzdialenosť (km) od Zeme v čase t (min) vyjadruje funkcia:

$$v(t) = 110t - 18t^2\,.$$

Vypočítajte rýchlosť rakety v čase $t = 3\,\mathrm{min}$, čas, v ktorom sa pohyb rakety smerom nahor zastaví a najväčšiu výšku, ktorú raketa dosiahne. Vypočítaný čas zokrúhlite na celé sekundy a výšku v kilometroch zaokrúhlite na dve desatinné miesta.

$v'(t) = 110 - 36t$ ← rýchlosť rakety v čase t

$v'(3) = 110 - 3.36 = \underline{\underline{2\ \mathrm{km/min}}}$ ←

$v'(t) = 0 \qquad 110 - 36t = 0$

$\boxed{t = \frac{110}{36}} = 3\frac{2}{36} \dots 3\,\mathrm{min}\ \frac{10}{3}\,\mathrm{s} \dots \underline{\underline{3\ \mathrm{min}\ 3\ \mathrm{sek.}}}$

$v\left(\frac{110}{36}\right) = \dots \doteq \underline{\underline{168{,}06\ \mathrm{km}}}$