Niektoré dôležité limity funkcií:

1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

2. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

3. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ln a\,,$ pre $a > 0$

4. $\lim\limits_{x\to\infty} \left(1 + \dfrac{a}{x}\right)^x = e^a,$ pre $a \in \mathbb{R}$

5. $\lim\limits_{x\to 0} \dfrac{(1+x)^p - 1}{x} = p\,,$ pre $p \in \mathbb{R}$

6. $\lim\limits_{x\to 0^+} x \log_a x = 0\,,$ pre $a > 0, \quad a \neq 1$

7. $\lim\limits_{x\to 0^+} x^n \ln x = 0\,,$ pre $n \in \mathbb{N}$

8. $\lim\limits_{x\to\infty} \dfrac{\log_a x}{x} = 0\,,$ pre $a > 0, \quad a \neq 1$

9. $\lim\limits_{x\to 0} \dfrac{\log_a(x+1)}{x} = \dfrac{1}{\ln a},$
   pre $a > 0, \quad a \neq 1$

10. $\lim\limits_{x\to 0} \dfrac{\arcsin x}{x} = 1$

11. $\lim\limits_{x\to 0} \dfrac{\operatorname{arctg} x}{x} = 1$

12. $\lim\limits_{x\to 1^-} \dfrac{\arccos x}{\sqrt{1-x}} = \sqrt{2}$

1. Vypočítajte limitu $\lim\limits_{x\to 1} \dfrac{x^m - 1}{x^n - 1}$, kde $m, n \in \mathbb{N}$.

$$\lim_{x\to 1} \frac{x^m - 1}{x^n - 1} = \left[\frac{0}{0}\right] = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{m-1} + x^{m-2} + \dots + x + 1)}{\cancel{(x-1)}(x^{n-1} + x^{m-2} + \dots + x + 1)} = *$$

$$\frac{0}{0}$$

$$(x^m - 1) : (x-1) = x^{m-1} + x^{m-2} + x^{m-3} + \dots + 1$$

$$-(x^m - x^{m-1})$$

$$0 \; (x^{m-1} - 1)$$

$$-(x^{m-1} - x^{m-2})$$

$$0 \; (x^{m-2} - 1)$$

$$\vdots$$

$$x^1 - 1$$

$$-(x - 1)$$

$$0 \quad 0$$

$$\cancel{0}$$

$$x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + \dots + x + 1)$$

$$* \lim_{x\to 1} \frac{\overbrace{x^{m-1} + x^{m-2} + \dots + x + \underset{}{1}}^{m}}{\underbrace{x^{n-1} + x^{n-2} + \dots + x + 1}_{n}} = \frac{m}{n}$$

$$1 = x^0$$

2. Vypočítajte limitu $\lim\limits_{x\to 0} \dfrac{\cos 3x - \cos 7x}{x^2}$.

$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos 3x - \cos 7x}{x^2} = \lim_{x\to 0} \frac{\cos(5x-2x) - \cos(5x+2x)}{x^2} =$$

$$= \lim_{x\to 0} \frac{\cancel{\cos 5x \cos 2x} + \sin 5x \sin 2x - (\cancel{\cos 5x \cos 2x} - \sin 5x \sin 2x)}{x^2} =$$

$$= \lim_{x\to 0} \frac{5 \cdot 2 \cdot 2 \sin 5x \sin 2x}{5x \cdot 2x} = \lim_{x\to 0} \left(20 \underbrace{\frac{\sin 5x}{5x}}_{t = 5x} \cdot \underbrace{\frac{\sin 2x}{2x}}_{u = 2x}\right) = 20 \cdot 1 \cdot 1 = 20$$

$\to 1$ $\quad \to 1$

$\boxed{\text{veta 190, strana 74 (Marko)}}$

3. Vypočítajte limitu $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^{x}$.

$$\boxed{\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}=e^{a} \qquad a\in\mathbb{R}}$$

1^{∞}

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}=\lim_{x\to\infty}\left(\frac{x-1+2}{x-1}\right)^{x-1+1}=\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{(x-1)+1}=$$

$$=\lim_{x\to\infty}\underbrace{\left(1+\frac{2}{x-1}\right)^{x-1}}_{t=x-1}\cdot\underbrace{\left(1+\frac{2}{x-1}\right)}_{\to 1}=e^{2}\cdot 1=e^{2}$$

$$\lim_{t\to\infty}\left(1+\frac{2}{t}\right)^{t}=e^{2}$$

$\boxed{\text{veta 190, strana 74 (Marko)}}$

4. Vypočítajte limitu $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{\sin^3 x}$.

$$\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{\sin^3 x} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin x \cdot \sin^2 x} = \lim_{x\to 0} \frac{\frac{\sin x - \sin x \cos x}{\cos x}}{\sin x\,(1-\cos^2 x)} =$$

$$\boxed{a^2 - b^2 = (a+b)(a-b)}$$

$$= \lim_{x\to 0} \frac{\cancel{\sin x}\,\overset{1}{\cancel{(1-\cos x)}}}{\cancel{\sin x}\cdot \cos x\,\cancel{(1-\cos x)}\,(1+\cos x)} =$$

$$= \lim_{x\to 0} \frac{1}{\underset{\substack{\shortparallel \\ 1}}{\cos x}\,\underbrace{(1+\cos x)}_{2}} = \frac{1}{2}$$

5. Vypočítajte limitu $\lim_{x\to\infty} 2^{\frac{3x}{x+2}}$.

$$\lim_{x\to\infty} 2^{\overbrace{\frac{3x}{x+2}}^{\to 3}} = 2^{\lim\limits_{x\to\infty}\frac{3x}{x+2}} = 2^{3} = \underline{\underline{8}}$$

$\boxed{\text{Marko: veta 197, strana 75}}$

$$2^{\frac{3x}{x+2}}$$

$$\left.\begin{aligned} f(x) &= 2^{x} \\ g(x) &= \frac{3x}{x+2} \end{aligned}\right\} f(g(x)) = 2^{\frac{3x}{x+2}}$$

6. Vypočítajte limitu $\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^{1+x}$.

$t = 2x$

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^{1+x} = \lim_{x\to 0} e^{\ln\left(\frac{\sin 2x}{x}\right)^{1+x}} = \lim_{x\to 0} e^{\overbrace{(1+x)}^{\to 1}\cdot\overbrace{\ln\left(\overbrace{\frac{\sin 2x}{2x}}^{\to 1}\cdot 2\right)}^{\to \ln 2}} = *$$

$$\boxed{f(x) = e^{\ln f(x)}}$$

$$\left.\begin{aligned} f(x) &= e^{x} \\ g(x) &= (1+x)\ln\left(\frac{\sin 2x}{x}\right)\end{aligned}\right\} f(g(x))$$

$$\boxed{\text{Matho! veta 197, strana 95} \leftarrow \text{zložená funkcia} \quad \text{veta 190, strana 94} \leftarrow \text{súčin funkcií}}$$

$$* = e^{1\cdot\ln 2} = e^{\ln 2} = \underline{\underline{2}}$$

7. Vypočítajte limitu $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin 2x}$.

$\frac{0}{0}$

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{\sin 2x} = \lim_{x\to 0}\frac{(e^x-1)-(e^{-x}-1)}{\sin 2x}\cdot\frac{1}{2x\frac{1}{2x}} =$$

$=1$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a \quad \text{pre } a>0$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

Marko: veta 190, strana 79

$$= \lim_{x\to 0}\left[\frac{e^x-1}{2x}\cdot\frac{1}{\frac{\sin 2x}{2x}} - \frac{e^{-x}-1}{2x}\cdot\frac{1}{\frac{\sin 2x}{2x}}\right] =$$

$$\lim_{x\to 0}\left[\frac{1}{2}\cdot\underbrace{\frac{e^x-1}{x}}_{\to 1}\cdot\frac{1}{\underbrace{\frac{\sin 2x}{2x}}_{t=2x\ \to 1}} + \frac{1}{2}\underbrace{\frac{e^{-x}-1}{(-x)}}_{u=-x\ \to 1}\cdot\frac{1}{\frac{\sin 2x}{2x}}\right] = \frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 1\cdot 1 =$$

$$= \underline{\underline{1}}$$

8. Vypočítajte limitu $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$.

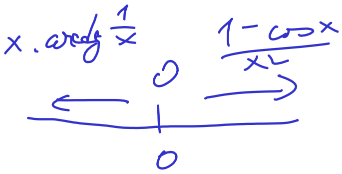
$$\lim_{x\to 0} \frac{\overbrace{\sqrt[3]{1+x}}^{=a} - \overbrace{\sqrt[3]{1-x}}^{b}}{x} \cdot \underbrace{\frac{\sqrt[3]{(1+x)^2}+\sqrt[3]{1-x^2}+\sqrt[3]{(1-x)^2}}{\sqrt[3]{(1+x)^2}+\sqrt[3]{1-x^2}+\sqrt[3]{(1-x)^2}}}_{=1} = *$$

$$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

$$* = \lim_{x\to 0} \frac{\overbrace{(1+x)-(1-x)}^{2x}}{x(\ldots\ldots\ldots)} = \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}\left(\sqrt[3]{(1+x)^2}+\sqrt[3]{1-x^2}+\sqrt[3]{(1-x)^2}\right)} =$$

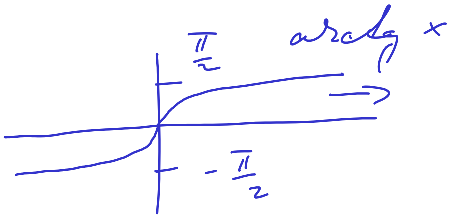
$$= \frac{2}{3}$$

9. Je funkcia $f(x)$ v bode $a = 0$ spojitá?



$$f(x) = \begin{cases} x.\mathrm{arctg}\,\frac{1}{x}\,, & x < 0 \\ 0\,, & x = 0 \\ \frac{1-\cos x}{x^2}\,, & x > 0 \end{cases}$$

$$\lim_{x\to 0^-} \overset{\nearrow 0}{x} . \mathrm{arctg} \underbrace{\frac{1}{x}}_{\to\infty} = 0 \cdot \frac{\pi}{2} = 0$$



$$\lim_{x\to 0^+} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

Ľ : $0 = \underset{f(0)}{\underset{\shortparallel}{0}} \neq \frac{1}{2}$: P $\Rightarrow$ Daná funkcia nie je spojitá.