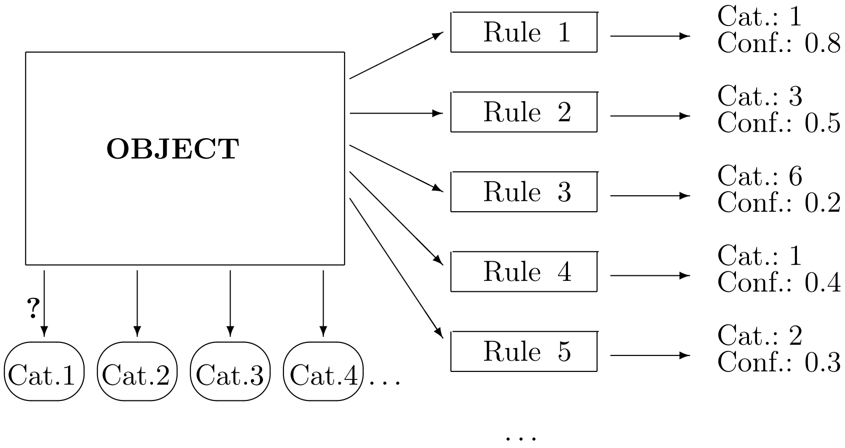
Multi-polarity in aggregation

Multi-cooperative games and multi-polar capacities

Andrea Mesiarová-Zemánková and Marek Hyčko

Classification

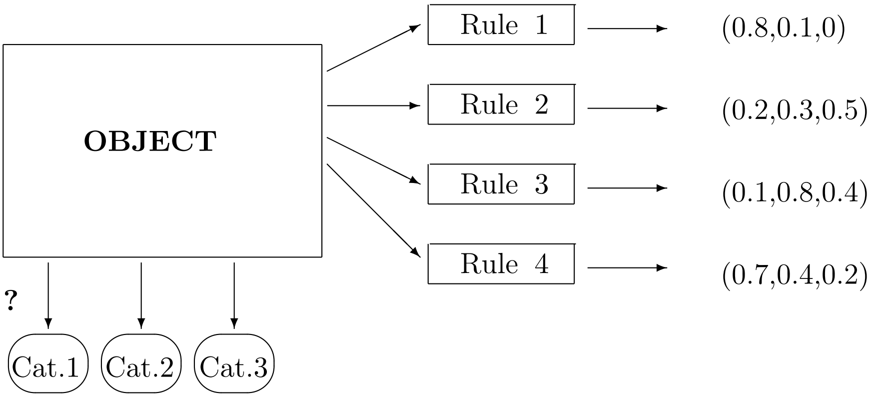


Classification

Object is described by the following rule outputs:

Rule	Category	Confidence
# 1	1	0.8
# 2	3	0.5
# 3	6	0.2
# 4	1	0.4
# 5	2	0.3
...	...	...

Classification for 3 categories and 4 rules:



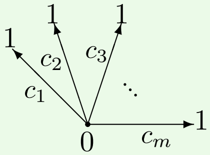
Classification for 3 categories and 4 rules:

Rule	Category	Confidence
# 1	1	0.8
	2	0.1
	3	0
# 2	1	0.2
	2	0.3
	3	0.5
# 3	1	0.1
	2	0.8
	3	0.4
# 4	1	0.7
	2	0.4
	3	0.2

Multi-polar input spaces

Multi-polar space

$K_m \times [0, 1]$, where $K_m = \{1, \ldots, m\}$ is the set of categories



input of the type (k, x), $k \in K_m$ and $x \in [0, 1]$

Extended multi-polar space

$EK_m = [0, 1]^m$

input of the type $(x_1, \ldots, x_m)$

Transition between multi-polar spaces

$K_m \times [0,1] \longrightarrow EK_m$

A multi-polar space $K_m \times [0,1]$ can be embedded into extended multi-polar space EK_m by injective function
$(k,x) \longrightarrow (0,\dots,0,\underbrace{x}_{k\text{-th}},0,\dots,0)$

$EK_m \longrightarrow K_m \times [0,1]$

- An input from an extended multi-polar space $(x_1,\dots,x_m)$ can be treated as m inputs from $K_m \times [0,1]$ of the form (i,x_i)
- Reduction function $EK_m \longrightarrow K_m \times [0,1]$ is such that $r((0,\dots,0,\underbrace{x}_{k\text{-th}},0,\dots,0)) = (k,x)$ for all $k \in K_m$ and $x \in [0,1]$ and have the same category monotonicity as multi-polar aggregation operators.

Multi-polar aggregation

(Extended) multi-polar aggregation operators

- For $n = 1$ aggregation is identity
- Zero and category maximal points are idempotent. In multi-polar case these are 0 and $(k, 1)$ for all $k \in K_m$. In extended multi-polar case these are $(0, \ldots, 0)$ and $(0, \ldots, 0, \underbrace{1}_{k\text{-th}}, 0, \ldots, 0)$ for all $k \in K_m$
- Category monotonicity: if input in i-th category is increased then output increase wrt category i, i.e., output in category i increases and in all other categories decreases.

(Extended) multi-polar aggregation operators

Example

- Basic extended multi-polar aggregation operators are those working coordinate-wisely, i.e., for each category a (unipolar) aggregation operator is used
- Basic multi-polar aggregation operators are those where inputs in separate categories are first aggregated by (unipolar) aggregation operators and then a reduction function is applied.
- omax – gives the input with the maximal absolute value, and gives 0 if two inputs with maximal absolute values have different classes
- The ordered category projection operator – gives the standard aggregation of values from the most important class that is present
- The union of the projections to a single coordinate

Addition on $K_m \times [0, \infty[$ $([0, \infty[^m)$

- Addition on $\mathbb{N}$

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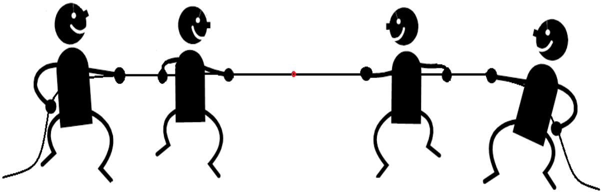

Addition

Bipolar addition

Sum up positive, sum up negative and make the difference, i.e., apply a reduction function – the difference.

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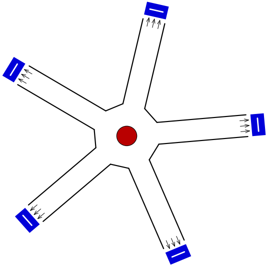

Summation reduction function

For $\mathbf{x} \in EK_m$ with $\mathbf{x} = (x_1, \ldots, x_m)$ let σ be a permutation such that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(m)}$. Then if $2 \leq q \leq m$ the function

$$U^q((k_1, x_1), \ldots, (k_n, x_n)) = (\mathrm{clx}(\mathbf{x}), \max(0, x_{\sigma(1)} - x_{\sigma(2)} - \cdots - x_{\sigma(q)})).$$

is called a q-summation reduction function. Additionally, for $q = 1$ the 1-summation reduction function is given by $U^1 = \mathrm{omax}$.

Aggregation on $[0, 1]$

Three steps of aggregation on $[0, 1]$

- (Monotone) Boolean functions $\{0, 1\}^n \longrightarrow \{0, 1\}$ – Boolean logic – decision making on $\{0, 1\}$ – voting games (simple cooperative games) – hypergraphs

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- Capacities (monotone) $\{0, 1\}^n \longrightarrow [0, 1]$, $\mu(0, \dots, 0) = 0$ – non-negative cooperative games – pseudo-Boolean functions
- Aggregation operators monotone $[0, 1]^n \longrightarrow [0, 1]$ with boundary conditions – decision making, optimization

Bipolar models according to Grabisch et al.

Univariate bipolar model

corresponds to 2-polarity, inputs are from $[-1, 1]$

Bivariate unipolar model

corresponds to extended 2-polarity, inputs are from $[0, 1]^2$, or equivalently from $[0, 1] \times [-1, 0]$

Bipolar case

(monotone) bipolar Boolean functions $\longrightarrow$ bipolar crisp functions

$b\colon \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ – balanced three-valued logic – ternary voting games

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Bi- and bipolar capacities and generalized bipolar capacities

- Bi-capacity (Grabisch and Labreuche): monotone function $\nu\colon \{-1,0,1\}^n \longrightarrow [-1,1]$, $\nu(0,\ldots,0)=0$ – bi-cooperative games

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- Generalized bipolar capacity (Grabisch et al.): monotone function $\mu^*\colon (\{0,1\}^2)^n \longrightarrow [0,1]^2$, $\mu^*(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$

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- Bipolar capacity (Greco et al.): monotone function $\mu\colon \{-1,0,1\}^n \longrightarrow [0,1]^2$, $\mu(0,\dots,0)=\mathbf{0}$

Input spaces for capacities and bi- and bipolar capacities

Capacity: $\{0,1\}^n$ corresponds to $\mathcal{P}(X)$; inputs are $S \in \mathcal{P}(X)$

Bi-capacity: $\{-1,0,1\}^n$ corresponds to $\mathcal{Q}(X) = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \cap B = \emptyset\}$; inputs are $(A,B) \in \mathcal{Q}(X)$

Generalized bipolar capacity: $(\{0,1\}^2)^n$ corresponds to $\mathcal{Q}^*(X) = \{(C,D) \in \mathcal{P}(X) \times \mathcal{P}(X)\}$; inputs are $(C,D) \in \mathcal{Q}^*(X)$

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Theorem

Let $\mu \colon \{-1,0,1\}^n \longrightarrow [0,1]^2$ be a (normalized) bipolar capacity and let $r \colon EK_2 \longrightarrow K_2 \times [0,1]$ be a reduction function. Then $\nu \colon \{-1,0,1\}^n \longrightarrow [-1,1]$ given by $\nu = r \circ \mu$ is a (normalized) bi-capacity.

Bipolar aggregation

Aggregation on $[-1, 1]$

Several types of bipolar Choquet integral, symmetric Sugeno integral

Aggregation on $[0, 1]^2$, resp. $[0, 1] \times [-1, 0]$

YinYang bipolar fuzzy logic and YinYang bipolar t-norms and t-conorms $\longrightarrow$ different type of monotonicity – coordinatewise

Multi-polar case

Multi-polar crisp functions

$K_m \times \{0, 1\} = \{0, 1, \ldots, m\}$

$c\colon (K_m \times \{0, 1\})^n \longrightarrow K_m \times \{0, 1\}$ together with monotonicity and idempotency $\longrightarrow$ multi-polar crisp aggregation operators

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Commutative associative multi-polar crisp aggregation operators

Commutative, associative, idempotent operation implies order, i.e., it is isomorphic to meet (join) of some lower (upper) semi-lattice. Together with category monotonicity we obtain the following structure

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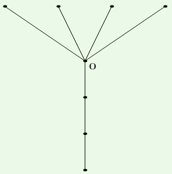

Multi-polar aggregation

Multi-polar t-norms, t-conorms and uninorms, multi-polar addition, multi-polar Choquet integral

Extended multi-polar aggregation

Extended multi-polar t-norms

Multi-polar $*$-ordinal sum of aggregation operators

Let A^k be an aggregation operator for all $k \in K_m$ and $*\colon EK_m \longrightarrow K_m \times [0,1]$ be an m-polar reduction function. Then

$$M^*((k_1, x_1), \ldots, (k_n, x_n)) = *(A^1(\mathbf{x}^1), \ldots, A^m(\mathbf{x}^m))$$

is an m-polar aggregation operator, which will be called an m-polar $*$-ordinal sum of aggregation operators.

Multi-polar aggregation

$\mathbf{x} = ((k_1, x_1), \ldots, (k_n, x_n))$

For $m = 3$ let $\mathbf{x} = ((1, 0.3), (2, 0.5), (1, 0.1), (3, 0.7), (2, 0.9))$ then
$\mathbf{x}^1 = (0.3, 0, 0.1, 0, 0)$
$\mathbf{x}^2 = (0, 0.5, 0, 0, 0.9)$
$\mathbf{x}^3 = (0, 0, 0, 0.7, 0)$

Multi-polar aggregation

Multi-polar t-norm

Commutative, associative, multi-polar aggregation operator, such that each its restriction to a single category is a (unipolar) t-norm.

$$T((k_1, x_1), \dots, (k_n, x_n)) = \begin{cases} T^i(x_1, \dots, x_n) & \text{if } k_1 = \dots = k_n = i \\ 0 & \text{else.} \end{cases}$$

Bipolar t-norm is a linear transformation of a nullnorm.

Multi-polar aggregation

Multi-polar t-conorm

Commutative, associative, multi-polar aggregation operator, such that each its restriction to a single category is a (unipolar) t-conorm.

Bipolar t-conorm is a linear transformation of a uninorm.

Minimal and maximal uninorm corresponds in multi-polar case to a OCP-ordinal sum of (unipolar) t-conorms: here we have a linear order on the set of categories.

Multi-polar pre-t-conorm

$$PC((k_1, x_1), \ldots, (k_n, x_n)) = f^{-1}(L^m(f((k_1, x_1)), \ldots, f((k_n, x_n))))$$

for a multi-polar function $f((k, x)) = (k, f_k(x))$ where f_k is an isomorphism on $[0, 1]$ with $f_k(0) = 0$ for all $k \in K_m$

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Multi-polar uninorm

Commutative, associative, multi-polar aggregation operator, such that each its restriction to a single category is either a (unipolar) t-norm or a (unipolar) t-conorm.
Bipolar continuous mixed uninorm is a linear transformation of an ordinal sum of two t-norms, or equivalently of two t-conorms.

Multi-polar Choquet integral

Multi-polar Choquet integral based on (unipolar) capacity

Multi-polar symmetric Choquet integral, multi-polar fusion Choquet integral, multi-polar balancing Choquet integral

Multi-polar OWA operators

Using the connection between the Choquet integral and OWAs, via a symmetric capacity we can define multi-polar OWAs: MSOWA, MFOWA, MBOWA with several interesting properties.
Using maximal and minimal (unipolar) capacity we can obtain several special operators as oriented maximum etc.

Extended multi-polar category t-norm

Commutative, associative extended multi-polar aggregation operators with category neutral elements $(0, \ldots, \underbrace{1}_{k\text{-th}}, \ldots, 0)$,

i.e., such an element is a neutral element on restriction of the t-norm onto elements of the form $(0, \ldots, \underbrace{x}_{k\text{-th}}, \ldots, 0)$

Extended multi-polar integrated t-norm

Commutative, associative extended multi-polar aggregation operators with neutral element $(1, \ldots, 1)$

Extended multi-polar integrated category t-norm

This extended multi-polar t-norm is given coordinate-wisely

Multi- and multi-polar capacities and generalized multi-polar capacities

- Multi-capacity: monotone function $\nu\colon (K_m \times \{0,1\})^n \longrightarrow K_m \times [0,1]$, $\nu(0,\ldots,0) = 0$ – multi-cooperative games

Capacities in multi-polar framework

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- Generalized multi-polar capacity: monotone function $\mu^*\colon (\{0,1\}^m)^n \longrightarrow [0,1]^m$, $\mu^*(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$

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Multi-polar case

Input spaces for multi- and multi-polar capacities

Multi-capacity: $(K_m \times \{0,1\})^n$ corresponds to $\mathcal{Q}_m(X) = \{(A_1, \ldots, A_m) \in (\mathcal{P}(X))^m \mid A_i \cap A_j = \emptyset \text{ for all } i \neq j, i, j \in \{1, \ldots, m\}\}$

Generalized multi-polar capacity: $(\{0,1\}^m)^n$ corresponds to $\mathcal{Q}_m^*(X) = \{(C_1, \ldots, C_m) \in (\mathcal{P}(X))^m\}$.

Example

Assume a reduction function based on m-summation, $X = \{1, \dots, 100\}$ and let v be a multi-polar capacity given by $v(A_1, \dots, A_m) = (\frac{\max(0, \mathrm{Card}(A_1) - 2\cdot\mathrm{Card}(A_3))}{100}, \frac{\mathrm{Card}(A_2)}{100}, \dots, \frac{\mathrm{Card}(A_m)}{100})$.
Assume any A_1, A_2, A_3, B_3, B such that $\mathrm{Card}(A_1) = 40$, $\mathrm{Card}(A_2) = 33$, $\mathrm{Card}(A_3) = 2$, and $B_3 = A_3 \cup B$ with $\mathrm{Card}(B_3) = 10$.
Then v is a multi-polar capacity, however,

$$r(v(A_1, A_2, A_3, \emptyset, \dots, \emptyset)) = r((\frac{36}{100}, \frac{33}{100}, \frac{2}{100}, 0, \dots, 0)) = (1, \frac{1}{100})$$

and

$$r(v(A_1, A_2, B_3, \emptyset, \dots, \emptyset)) = r((\frac{20}{100}, \frac{33}{100}, \frac{10}{100}, 0, \dots, 0)) = (2, \frac{3}{100}).$$

Thus although the set corresponding to the third category was increased, the output changed from the first to the second category, i.e., $r \circ v$ is not a multi-capacity.

Theorem

Let $\mu\colon (K_m \times \{0,1\})^n \longrightarrow [0,1]^m$ be a (normalized) multi-polar capacity given coordinate-wisely and let $r\colon EK_m \longrightarrow K_m \times [0,1]$ be a reduction function. Then $\nu\colon (K_m \times \{0,1\})^n \longrightarrow K_m \times [0,1]$ given by $\nu = r \circ \mu$ is a (normalized) multi-capacity.

Examples of multi-(polar) capacities

- Basic multi-(polar) capacities are additive, symmetric, category symmetric and decomposable.

- For $p < m$ an m-capacity restricted to inputs from p categories is a p-capacity.

- For n inputs if $n < m$ an m-capacity is given by a collection of $\binom{m}{n}$ compatible partial n-capacities

Bolger: games with n players and m alternatives

- X - set of all players, $C(j)$ set of players who choose alternative j

- $(C(1), \ldots, C(m))$ is called an arrangement of the players among the m alternatives

- if for all arrangements Γ and all $S \in \Gamma$ we set $\nu(S, \Gamma) \in \mathbb{R}$ it is represented as the 'worth' of S with respect to the arrangement Γ

- (X, m, ν) is a game on X with m alternatives provided $\nu(T, \Gamma) = 0$ whenever $T = \emptyset$ (ν is an (X, m) game).

Bolger: games with n players and m alternatives

Input space for (X, m) games

$$\mathcal{S}_m(X) = \{(A_1, ..., A_m) \in \mathcal{Q}_m(X) | \bigcup_{i=1}^{m} A_i = X\}$$

$$\mathcal{Q}_{m-1}(X) = \mathcal{S}_m(X) \subseteq \mathcal{Q}_m(X)$$

(X, m) game

$\nu : \mathcal{S}_m(X) \longrightarrow \mathbb{R}^m \quad (\nu : \mathcal{S}_m(X) \longrightarrow [0,1]^m)$

Thus an m-polar capacity is a monotone $(X, m+1)$ game such that there is a 0 gain if all players abstain.

Bolger gives Shapley value and Banzhaf index for (X, m) games

- bi-cooperative simple (voting) game - $(X, 3)$ game fulfilling nil condition with embedded range $\{-1, 0, 1\}$
- bi-cooperative game - $(X, 3)$ game fulfilling nil condition with embedded range $[-1, 1]$
- m-cooperative simple (voting) game - $(X, m+1)$ game fulfilling nil condition with embedded range $K_m \times \{0, 1\}$
- m-cooperative game - $(X, m+1)$ game fulfilling nil condition with embedded range $K_m \times [0, 1]$
- bi-capacity - monotone $(X, 3)$ game fulfilling nil condition, with embedded range $[-1, 1]$
- bipolar capacity - monotone $(X, 3)$ game fulfilling nil condition
- m-capacity - monotone $(X, m+1)$ game fulfilling nil condition with embedded range $K_m \times [0, 1]$
- m-polar capacity - monotone $(X, m+1)$ game fulfilling nil condition

References – Multi-polarity

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Thank you for your attention