MV-pairs and state operators

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Introduction: MV-pairs

- A key relationship between Boolean algebras and MV-algebras lies in the fact that the set of all idempotents of an MV-algebra M is a Boolean algebra, in fact the greatest Boolean subalgebra of M. The Boolean algebra of idempotents can be considered as a system of classical propositions, while the surrounding algebra M can be considered as an extension of the classical logic by fuzzy or unsharp propositions.

Introduction: MV-pairs

- Another relation between MV-algebras and Boolean algebras was shown by Jenča in 2007 - a representation theorem for MV-algebras is given in terms of Boolean algebras and their automorphism groups. Actually, Jenča showed that given a Boolean algebra B and a subgroup G of its automorphism group satisfying certain conditions, the pair (B, G) can be canonically associated with an MV-algebra. Such pairs (B, G) are called *MV-pairs*.

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- Conversely, given an MV-algebra M, if $B(M)$ denotes its R-generated Boolean algebra and $G(M)$ is a special subgroup of the automorphism group of $B(M)$, it turns out that $(B(M), G(M))$ forms an MV-pair. Independently, a similar study of certain type of (B, G)-pairs which yield an MV-algebra, so called ambiguity algebras, were studied by Vetterlein (2008). A comparison of these two approaches were made by De la Vega last year. In 2009, Di Nola, Holčapek and Jenča came up with a categorical development of the results concerning MV-pairs.

Introduction: State Operators

- Recently (2009) the notion of a state on an MV-algebra was generalized by Flaminio and Montagna to an algebraically defined notion for MV-algebras. The language of MV-algebras has been enlarged by a unary operation σ, called an *internal state* or a *state operator*. Such MV-algebras are called *state-MV-algebras*. These algebras are now intensively studied by Di Nola, Dvurečenskij, Lettieri and many others.

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- In this talk, internal states in connection with MV-pairs are discussed; namely, a relations between state MV-algebras and state Boolean algebras, which are connected by an MV-pair.

Definition [Chang, 1958]

An algebra $(A, \boxplus, ', 0)$ with a binary operation $\boxplus$, a unary operation $'$ and a special element 0 is called an *MV-algebra* if it satisfies the following conditions for all $x, y, z \in A$:

- $x \boxplus y = y \boxplus x$.
- $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$.
- $x \boxplus 0 = 0$.
- $x'' = x$.
- $x \boxplus 0' = 0'$.
- $(x' \boxplus y)' \boxplus y = (y' \boxplus x)' \boxplus x$.

- ordering: $x \leq y$ iff $x' \boxplus y = 1$
- distributive lattice: $x \vee y = (x' \boxplus y)' \boxplus y = (y' \boxplus x)' \boxplus x$, $x \wedge y = (x' \vee y')'$
- another operations: $a \boxminus b := (a' \boxplus b)'$, $a \boxdot b := (a' \boxplus b')'$

Basic Notions: Effect algebra

Definition [Foulis and Bennett, 1994]

An *effect algebra* (EA) is a partial algebra $(E, \oplus, 0, 1)$ where E is a nonempty set, $0, 1$ are special elements and the partial operation $\oplus$ is such that $\forall a, b \in E$:

- if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$
- if $a \oplus b$ and $(a \oplus b) \oplus c$ are defined then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- for every $a \in E$ there exists exactly one $a' \in E$ such that $a \oplus a' = 1$
- if $a \oplus 1$ does exist then $a = 0$

- orthogonality: $a \perp b$ iff $a \oplus b$ exists
- ordering: $a \leq b$ iff $\exists\, c \in E : a \oplus c = b$
- minus operation: $b \ominus a = c$ iff $a \oplus c = b$

MV-effect algebra

An *MV-effect algebra* is a lattice ordered effect algebra E with the property:

$$\forall\, a, b \in E : (a \vee b) \ominus a = b \ominus (a \wedge b)$$

or equivalently with the Riesz decomposition property

$$(RDP) : \forall a, b, c \in E : a \leq b \oplus c \ \Rightarrow \exists\, b_1, c_1 \in E : b_1 \leq b, c_1 \leq c, a = b_1 \oplus c_1$$

By [Kôpka and Chovanec, 1997], MV-effect algebras and MV-algebras are in one-to-one correspondence:

- $(M, \boxplus, ', 0) \rightarrow (M, \oplus, 0, 1)$: $a \oplus b := a \boxplus b$ only for $a \leq b'$
- $(M, \oplus, 0, 1) \rightarrow (M, \boxplus, ', 0)$: $a \boxplus b := a \oplus (a' \wedge b)$

BG pairs

Let B be a Boolean algebra, $Aut(B)$ the group of automorphisms of B and G a subgroup of $Aut(B)$. Then (B, G) is called a BG-pair. We will use the following notation:

$$L(a, b) := \{a \wedge f(b) : \ f \in G\},$$
$$L^{+}(a, b) := \{g(a) \wedge f(b) : \ f, g \in G\}$$

Let M be a bounded distributive lattice. Up to isomorphism, there exists a unique boolean algebra $B(M)$ such that M is a 0,1-sublattice of $B(M)$ and M generates $B(M)$ as a Boolean ring. This Boolean algebra is called R-generated by M. For every element $x \in B(M)$ there exists a finite chain $x_1 \leq \ldots \leq x_n$ in M such that $x = x_1 + \ldots + x_n$, where $+$ denotes the symmetric difference. We call this chain an *M-chain representation* of the element x.

BG pairs

Theorem [Jenča, 2004]

Let M be an MV-effect algebra. The mapping $\phi_M : B(M) \to M$ given by

$$\phi_M(x) = \bigoplus_{i=1}^{n} (x_{2i} \ominus x_{2i-1})$$

where $\{x_i\}_{i=1}^{2n}$ is an M-chain representation of x, is a surjective morphism of effect algebras.

MV-pairs

Definition [Jenča, 2007]

A BG-pair (B, G) is called an *MV-pair* iff the following conditions are satisfied:

1. For all $a, b \in B, f \in G$ such that $a \leq b$ and $f(a) \leq b$, there is $h \in G$ such that $h(a) = f(a)$ and $h(b) = b$.
2. For all $a, b \in B$ and $x \in L(a, b)$, there exists $m \in \max(L(a, b))$ with $m \geq x$.

Definition [Pulmannová, 2009]

A BG-pair is an *MV*-pair* if the following conditions are satisfied for any $a, b \in B$:

1. For all $a, b \in M$, $f \in G$ such that $a \perp b$ and $a \perp f(b)$, there is $h \in G$ with $h(a \vee b) = a \vee f(b)$
2. For all $a, b \in B$ and $x \in L^+(a, b)$, there exists an element $m \in \max(L^+(a, b))$ such that $x \leq m$.

MV-pairs

Let us denote

$$a, b \in B : a \sim_G b \Leftrightarrow \exists f \in G : b = f(a)$$

Theorem [Jenča, 2007]

Let (B, G) be an MV-pair. Then

- $\sim_G$ is an effect algebra congruence
- B/G is an MV-effect algebra
- for all $a, b \in B$

$$[a]_G \wedge [b]_G = \max(L^+(a, b))$$

  where the "$=$" is the set equality
- $\max(L(a, b)) \subseteq \max(L^+(a, b))$

MV-pairs

Theorem [Jenča, 2007]

Let M be an MV-algebra. We denote

$$G(M) := \{f \in Aut(B(M)) : \text{ for all } x \in B(M), \phi_M(x) = \phi_M(f(x))\}.$$

The following hold:

- $(B(M), G(M))$ is an MV-pair;
- for all $x, y \in B(M)$, $x \sim_{G(M)} y$ iff $\phi_M(x) = \phi_M(y)$;
- $B(M)/G(M)$ is isomorphic to M, where the isomorphism is given by

$$\beta_M([x]_{G(M)}) = \phi_M(x).$$

State Operators

Definition [Flaminio and Montagna, 2009]

Let $(M, \boxplus, ', 0)$ be an MV-algebra. We say that a mapping $\sigma : M \to M$ which satisfies:

1. $\sigma(0) = 0$
2. $\sigma(x') = \sigma(x)'$
3. $\sigma(x \boxplus y) = \sigma(x) \boxplus \sigma(y \boxminus (x \boxdot y))$
4. $\sigma(\sigma(x) \boxplus \sigma(y)) = \sigma(x) \boxplus \sigma(y)$

is a *state operator* on M.

Definition

A *state morphism* of an MV-algebra M is a state operator which is also an MV-algebra morphism.

Definition [Buhagiar, Chetcuti and Dvurečenskij, 2011]

Let $(E, \oplus, ', 0, 1)$ be an effect algebra. We say that a mapping $\sigma : E \to E$ which satisfies:

1. $\sigma(1) = 1$
2. $\sigma(a \oplus b) = \sigma(a) \oplus \sigma(b)$ whenever $a \oplus b$ is defined
3. $\sigma(\sigma(a)) = \sigma(a)$

   is a *state operator* on E.

   A state operator σ on E is *strong* iff, in addition,
4. $\sigma(\sigma(a) \wedge \sigma(b)) = \sigma(a) \wedge \sigma(b)$ whenever $\sigma(a) \wedge \sigma(b)$ exists in E.

- Remark: If E is an MV-effect algebra, then an effect algebra state operator σ is also an MV-algebra state operator iff σ is strong.

state MV-algebra $\to$ state MV-pair

Theorem

Let M be an MV-algebra, $B(M)$ the R-generated Boolean algebra and $\sigma : M \to M$ a (strong) state operator on M. The mapping

$$\sigma^*(a) := \sigma(\phi_M(a))$$

on the Boolean algebra $B(M)$ is a (strong) state operator on $B(M)$.

- by a *state operator* we mean in all following slides an *effect algebra state operator*

state MV-pair $\to$ state MV-algebra

Theorem

Let (B, G) be an MV-pair and $\sigma_B : B \to B$ be a state operator on the Boolean algebra B. The mapping

$$\sigma_*([a]_G) = [\sigma_B(a)]_G$$

is a state operator on the MV-effect algebra $M = B/G$ if and only if the following condition holds:

$$\sigma_B(O(a) \subseteq O(\sigma_B(a)), \ a \in B. \qquad \text{(O)}$$

In addition, if σ_B is strong and the equality holds in (O), then σ_* is strong as well.

- orbit: $O(x) := \{y \in B : \exists f \in G; y = f(x)\}$

State MV-pair

Definition

A *state-MV-pair* is a triple (B, G, σ), where (B, G) is an MV-pair and $\sigma : B \to B$ is a state operator satisfying condition (O).

Corollary

If (B, G, σ) is a (strong) state-MV-pair, then $M = B/G$ is a (strong) state-MV-effect algebra with the state operator $\sigma_*([a]_G) = [\sigma(a)]_G$. Conversely, if (M, σ) is a (strong) state-MV-effect algebra, then $(B(M), G(M), \sigma^*)$, where $\sigma^*(a) = \sigma(\phi_M(a))$, is a state-MV-pair and $(\sigma^*)_* = \sigma$.

State morphisms

Theorem

Let (B, G) be an MV-pair and σ_B be a state morphism on the Boolean algebra B such that the equality in (O) is satisfied. Then the state operator σ_* is a state morphism on B/G.

Theorem

If $\sigma : M \to M$ is a state morphism on an MV-algebra M, then there exists a morphism $\nabla(\sigma)$ on Boolean algebra $B(M)$ such that $\phi_M(\nabla(\sigma)) = \phi_M(\nabla(\sigma))^2$.

Subdirectly irreducible MV-algebras

- we say that an MV-algebra is subdirectly irreducible, if it has a smallest nontrivial ideal

Theorem

Let (B, G) be an MV-pair and let $M := B/G$ be the corresponding MV-algebra. Then M is subdirectly irreducible if and only if B has a smallest nontrivial G-invariant ideal.

Subdirectly irreducible MV-algebras

- any ideal I in $M = B(M)/G(M)$ uniquely extends to an ideal I^* in $B(M)$ such that
$$a \in I^* \;\Leftrightarrow\; \phi_M(a) \in I$$
- an ideal J in $B(M)$ is *G-invariant* iff $a \in J$ implies $f(a) \in J$ for all $f \in G$

Corollary

Let M be an MV-algebra, and $(B(M), G(M))$ the corresponding MV-pair. Then M is subdirectly irreducible with a smallest ideal I_s if and only if $B(M)$ has a smallest nontrivial $G(M)$-invariant ideal I_s^* that extends I_s.

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