FUZZY MODELS AND STOCHASTICS

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FUZZY INFORMATION

- Fuzzy Data
- Fuzzy a-priori Knowledge
- Fuzzy Probabilities
- Soft Computing ECSC

KINDS OF DATA UNCERTAINTY

Variability

Errors

Missing Values

Imprecision (Fuzzy Data)

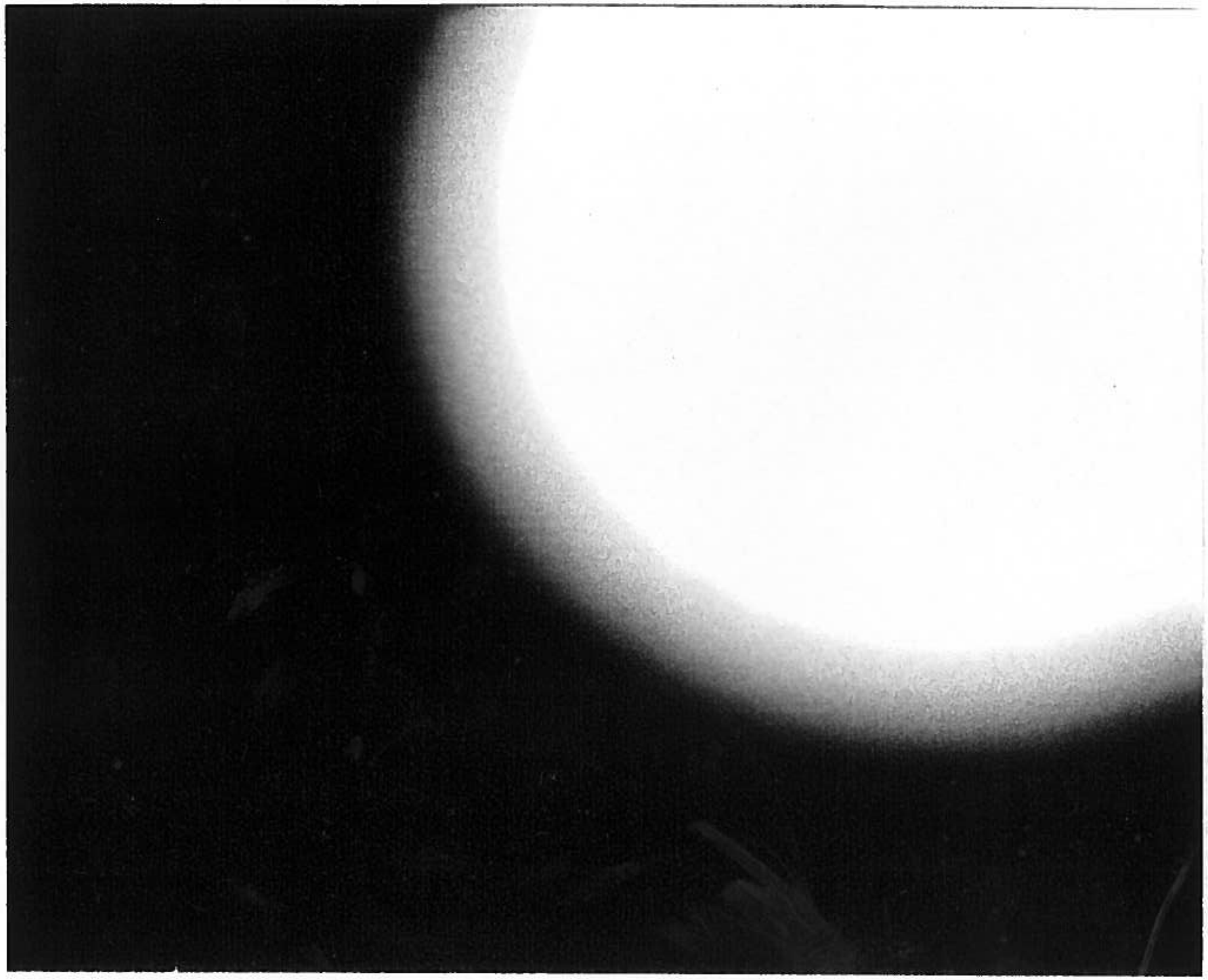
Description: Fuzzy Numbers
Fuzzy Vectors
Fuzzy Functions

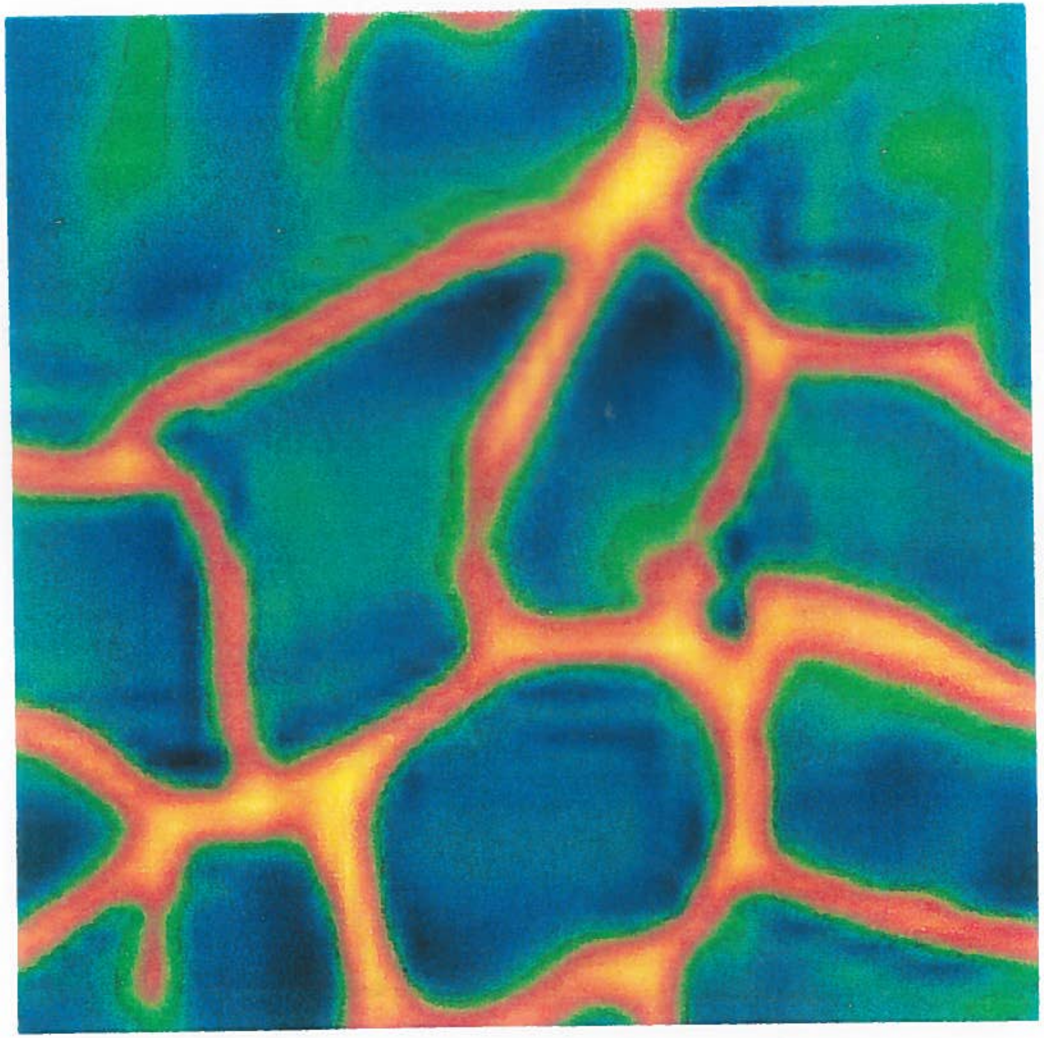
FUZZY DATA

- Environmental Loads
- Material Strength
- Dimensions
- Quality Characteristics
- Life Times

  ⋮

- Precision Measurements





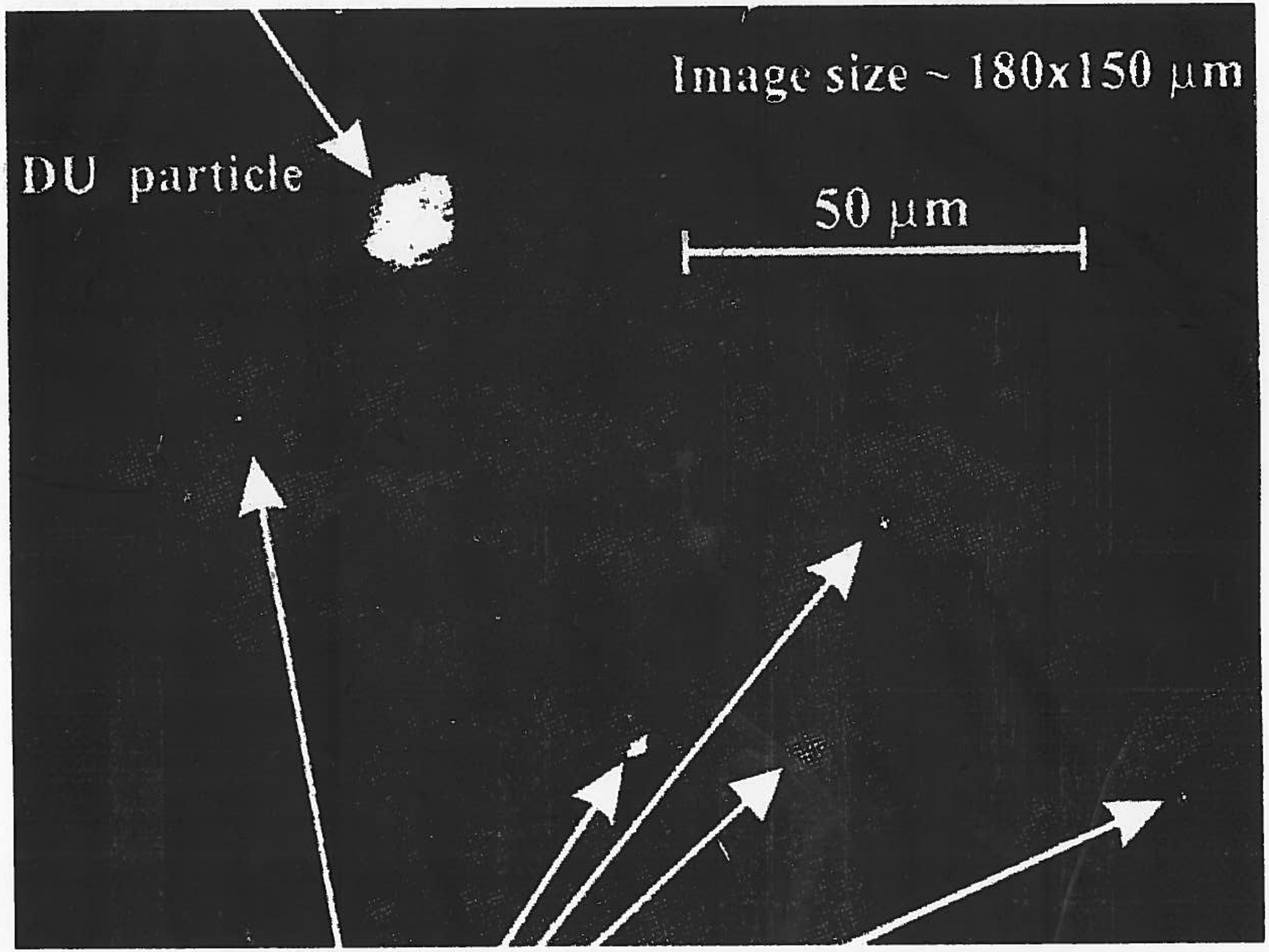
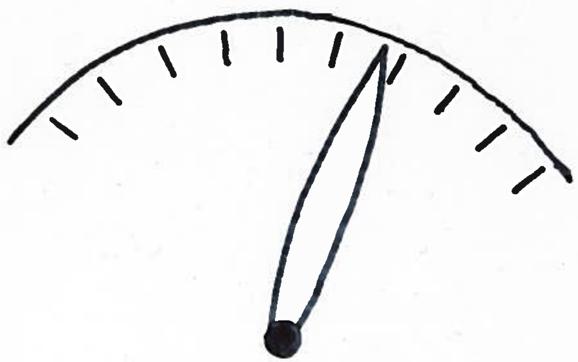


Image size ~ 180x150 µm
DU particle
50 µm

MEASUREMENTS



4.823

analog

digital

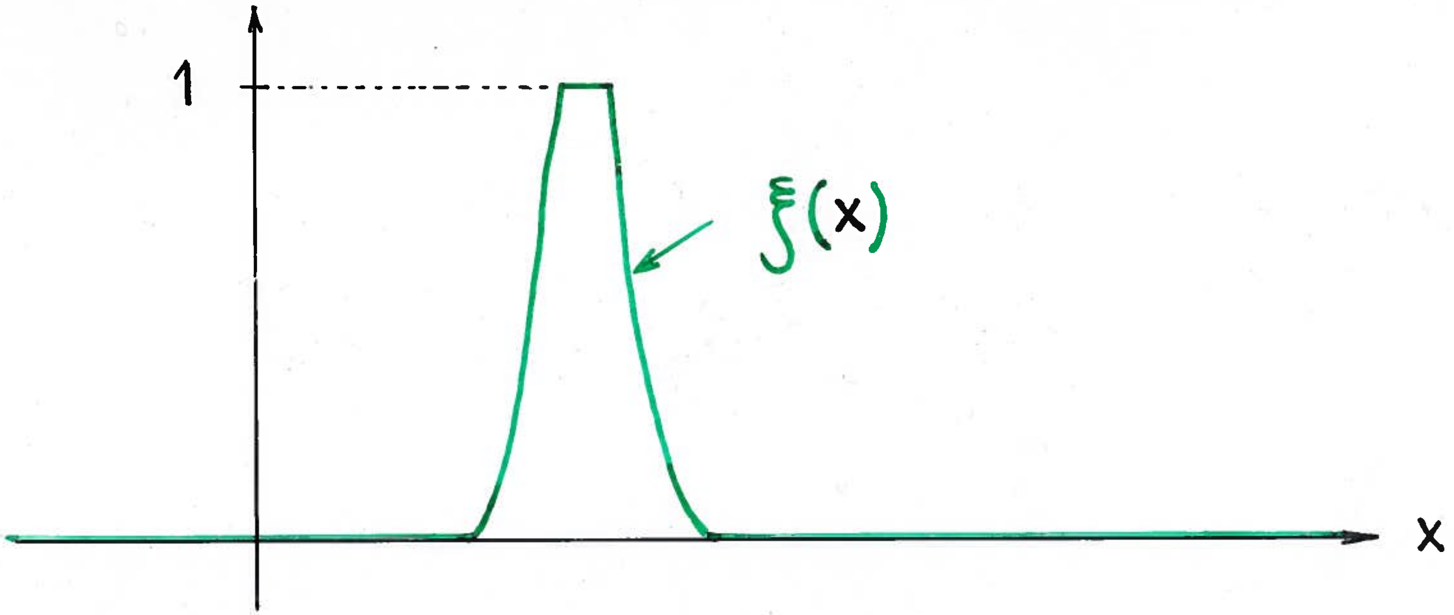
Results ?

MEASUREMENT RESULTS

Not precise numbers but more or less non-precise

Mathematical model : Fuzzy number x^*

Characterizing function $\xi(\cdot)$



Characterizing Function

(C1) $\xi : \mathbb{R} \to [0; 1]$

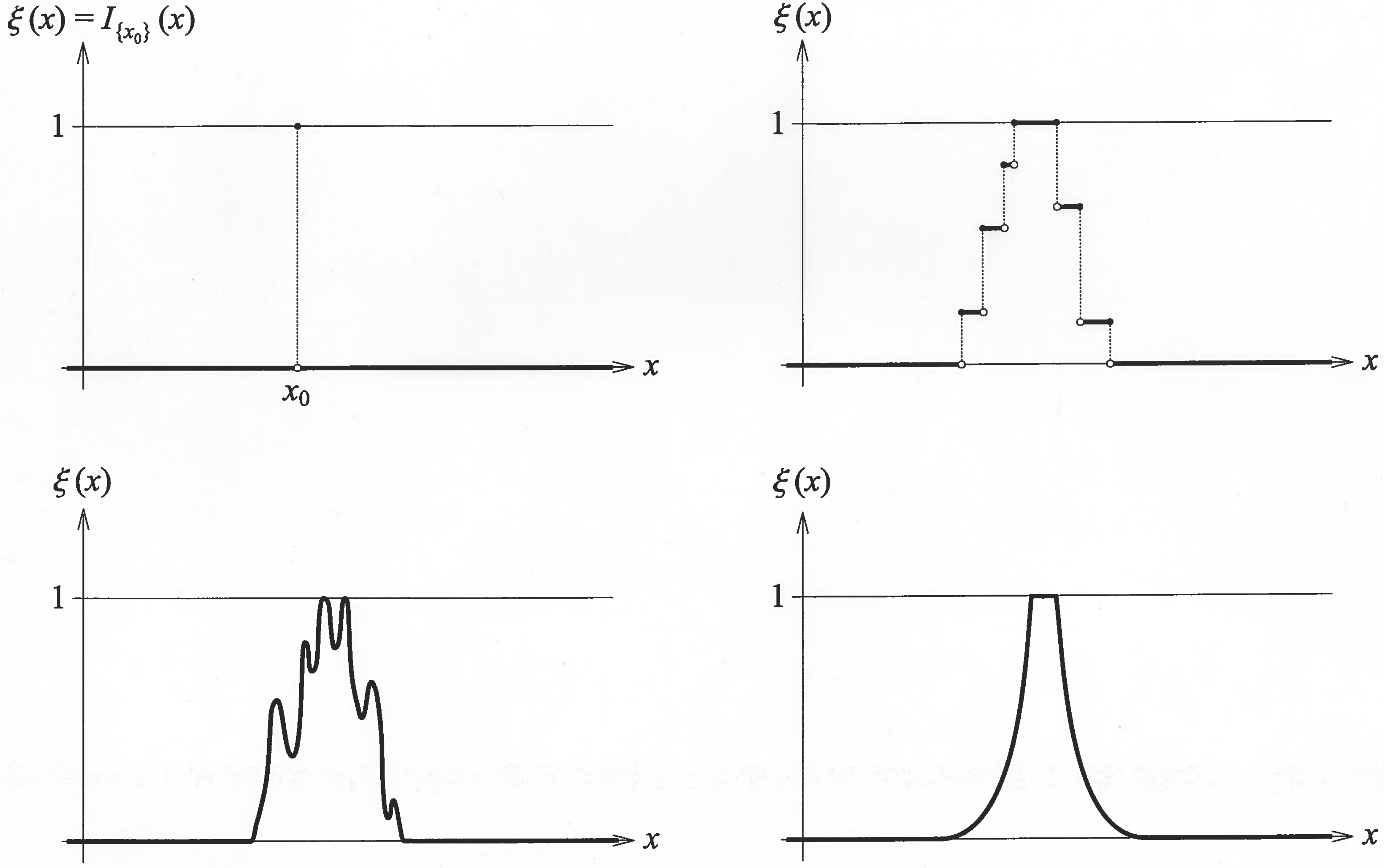
(C2) $\text{supp}[\xi(\cdot)]$ is bounded

(C3) $\forall\, \delta \in (0; 1]$ the so-called δ-cut

$$C_\delta[\xi(\cdot)] := \{x \in \mathbb{R} : \xi(x) \geq \delta\} = [a_\delta ; b_\delta] \neq \emptyset$$

Remark: For a precise number $x_0 \in \mathbb{R}$:

$$\xi(\cdot) = \mathbb{1}_{\{x_0\}}(\cdot)$$



$\xi(x) = I_{\{x_0\}}(x)$
1
x
x_0
$\xi(x)$
1
x
$\xi(x)$
1
x
$\xi(x)$
1
x

NON-PRECISE NUMBERS

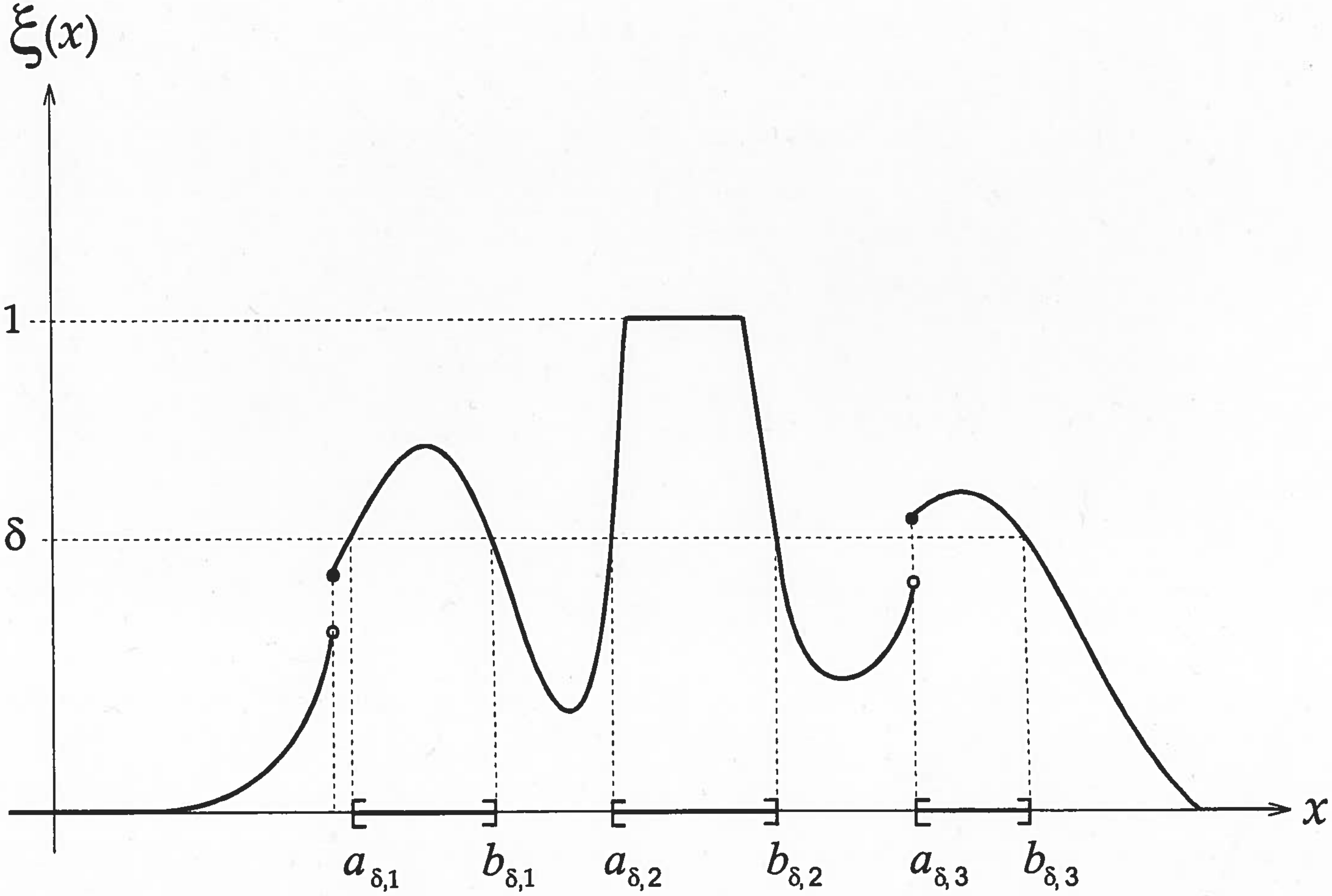
x^*, Characterizing Function $\xi(\cdot)$

(1) Support $[\xi(\cdot)] \subseteq [a; b]$ compact interval

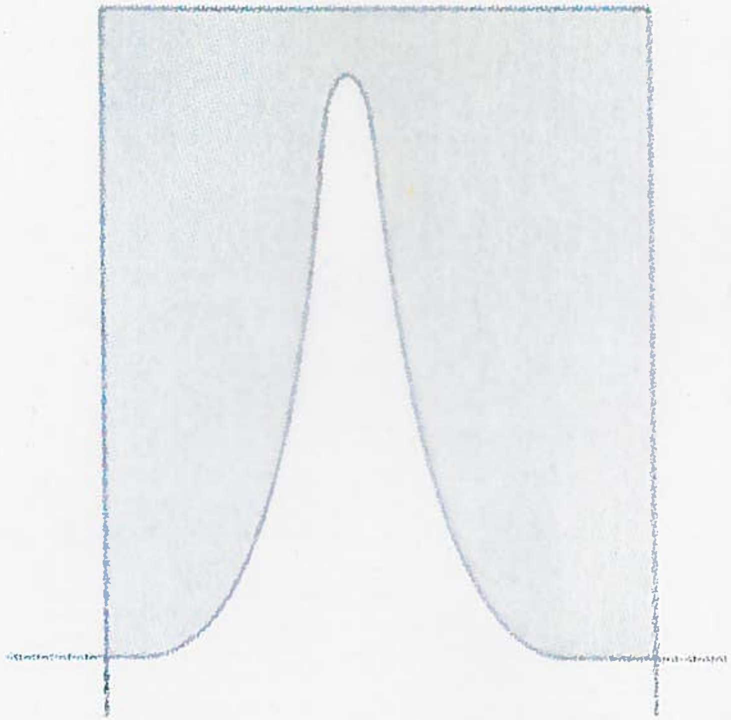
(2) All δ-Cuts $C_\delta := \{x \in \mathbb{R} : \xi(x) \geq \delta\}$

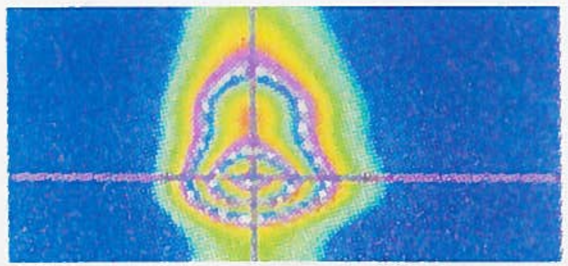
are non-empty with

$$C_\delta = \bigcup_{j=1}^{k_\delta} \left[a_{\delta,j} ; b_{\delta,j}\right] , \quad k_\delta \in \mathbb{N}$$

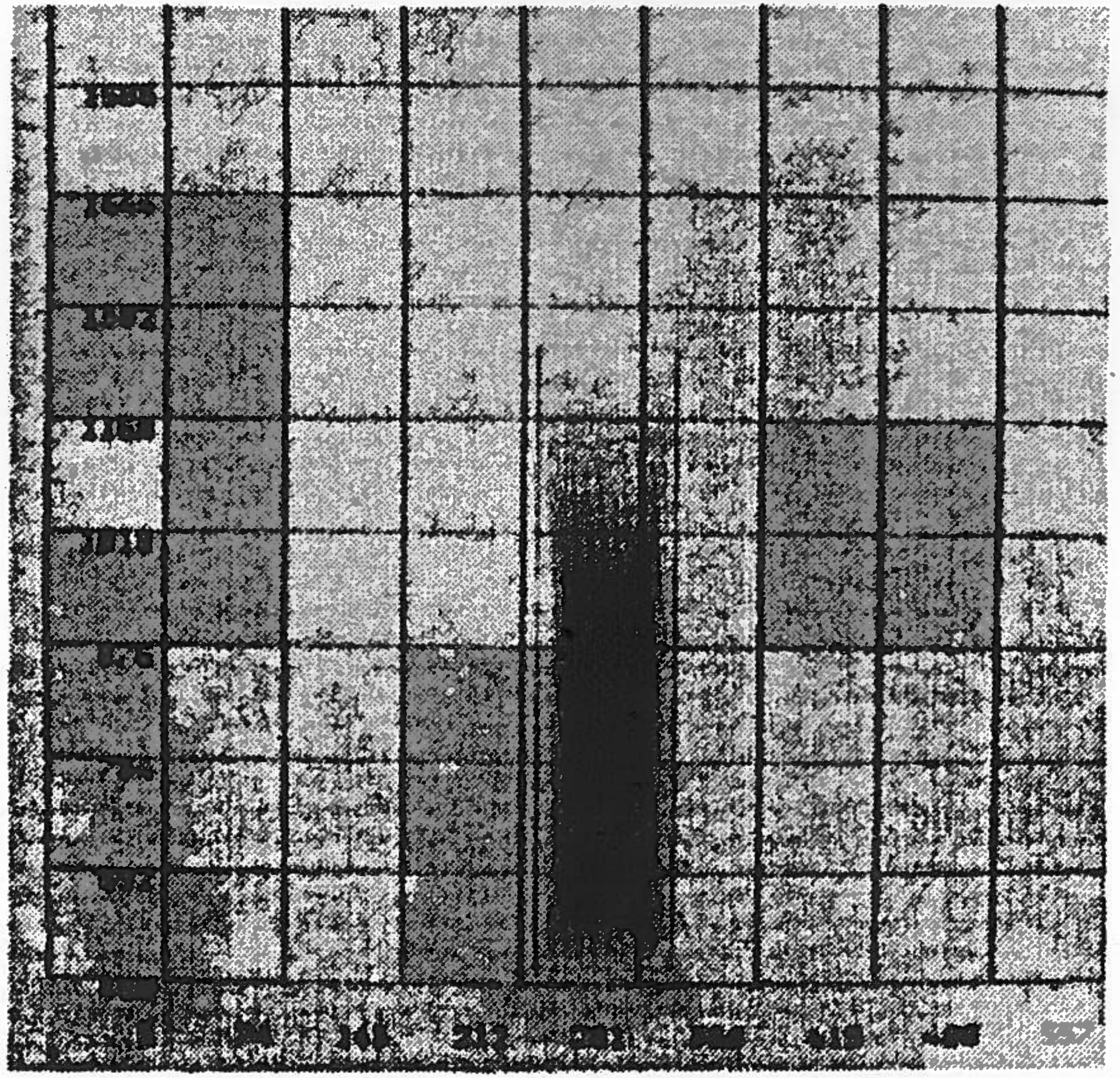


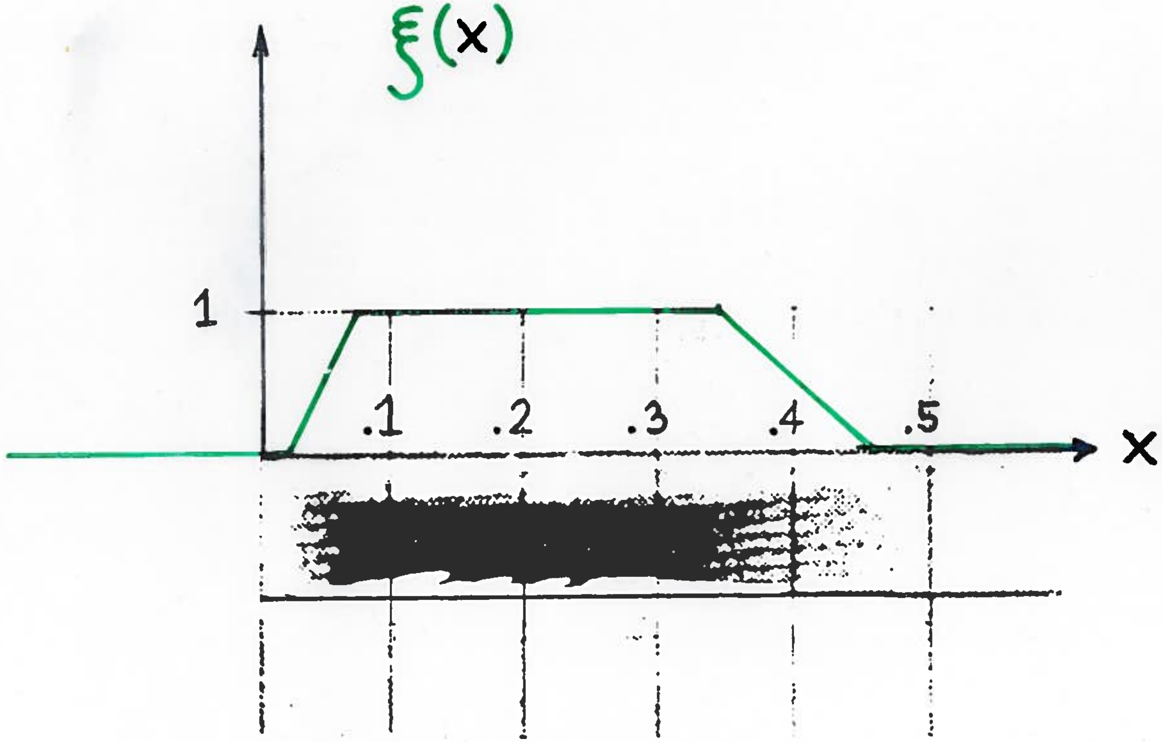
ξ(x)
1
δ
x
$a_{\delta,1}$
$b_{\delta,1}$
$a_{\delta,2}$
$b_{\delta,2}$
$a_{\delta,3}$
$b_{\delta,3}$

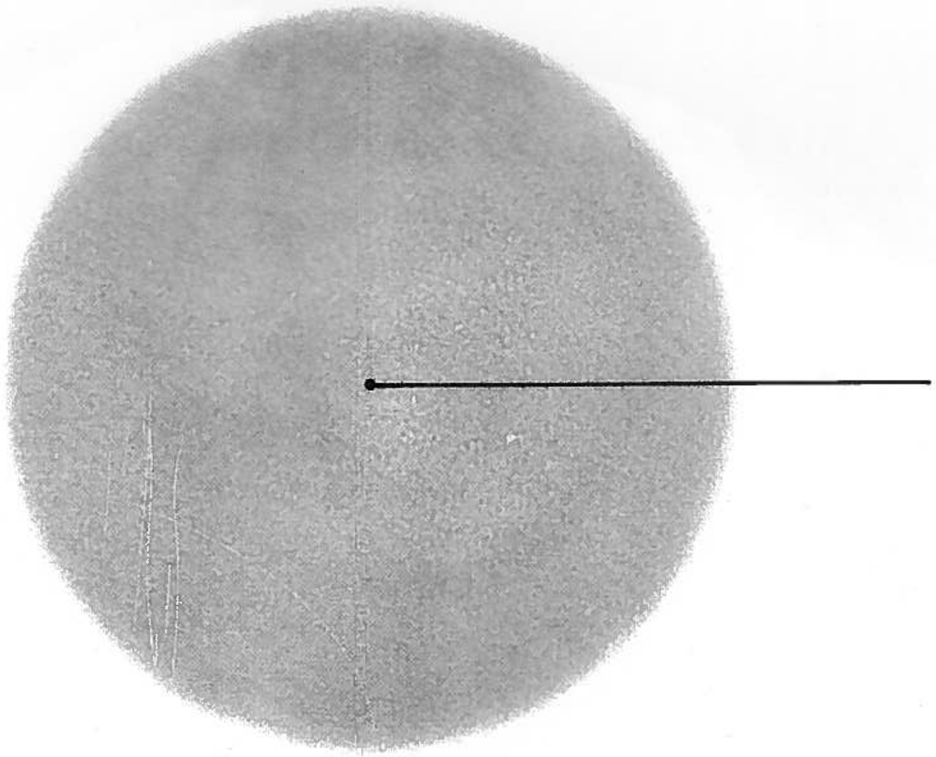




one observation as presented by the screen




ξ(x)
1
.1
.2
.3
.4
.5
x



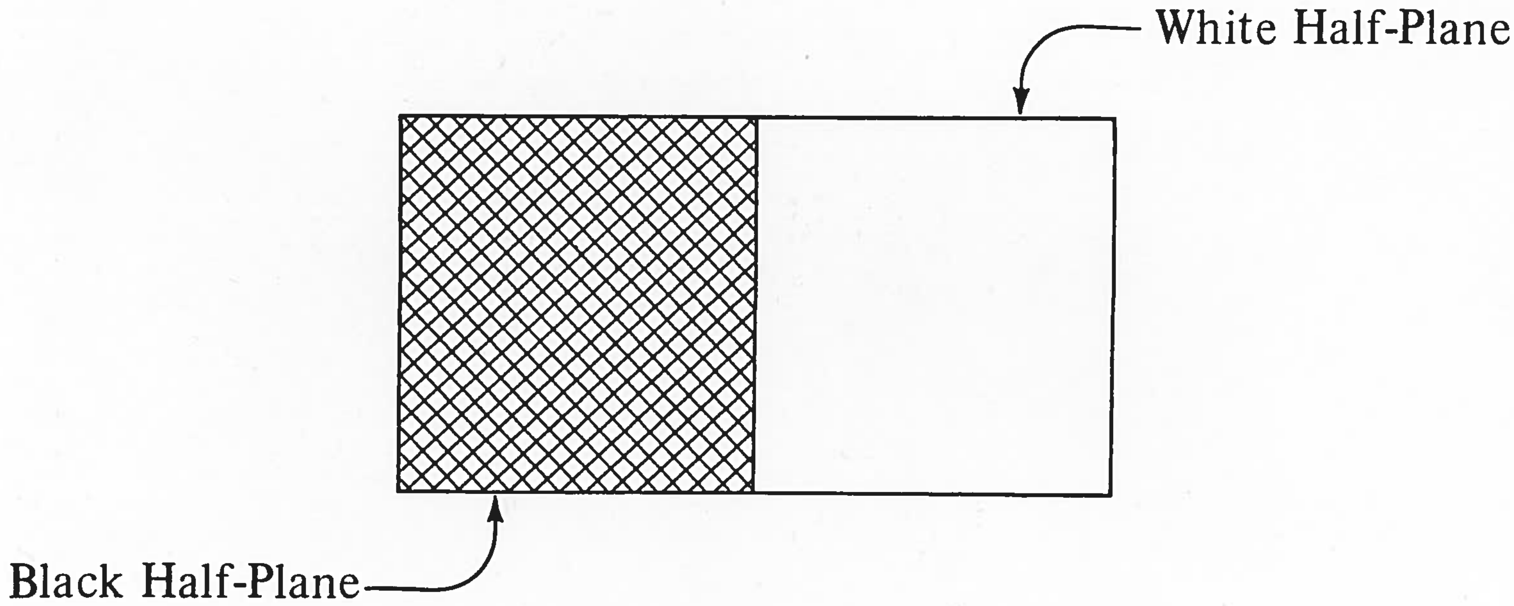


Figure Black and white half-planes

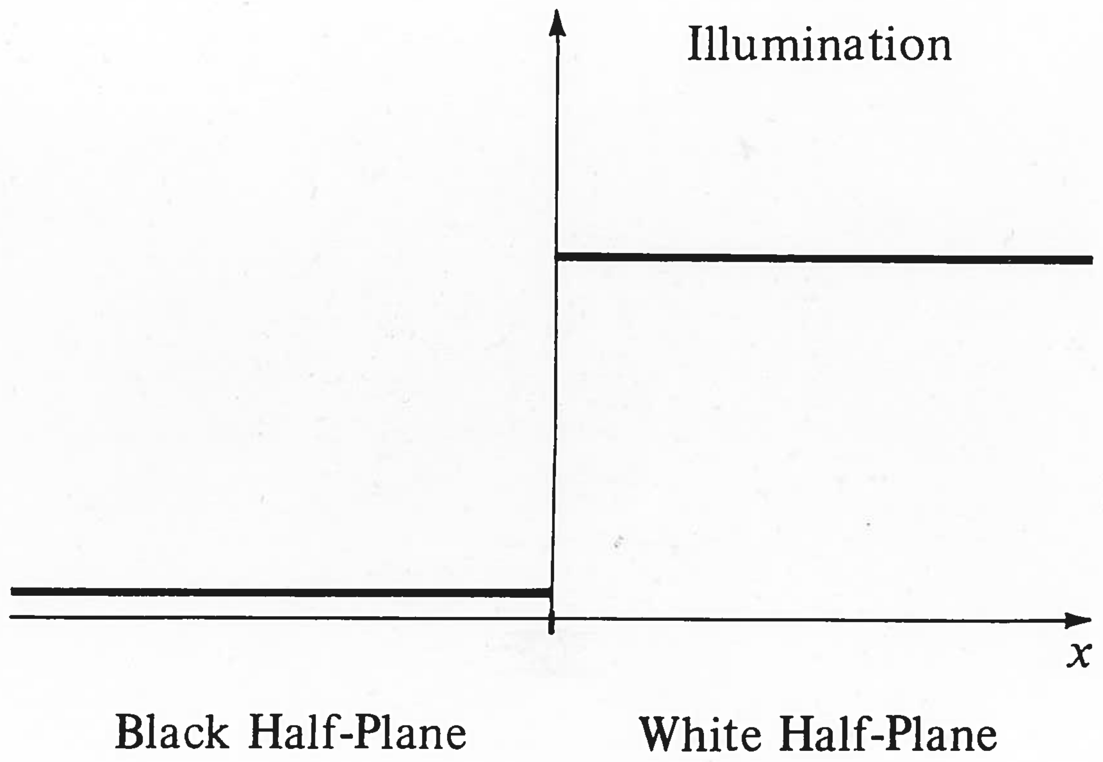


Figure Ideal illumination on a line

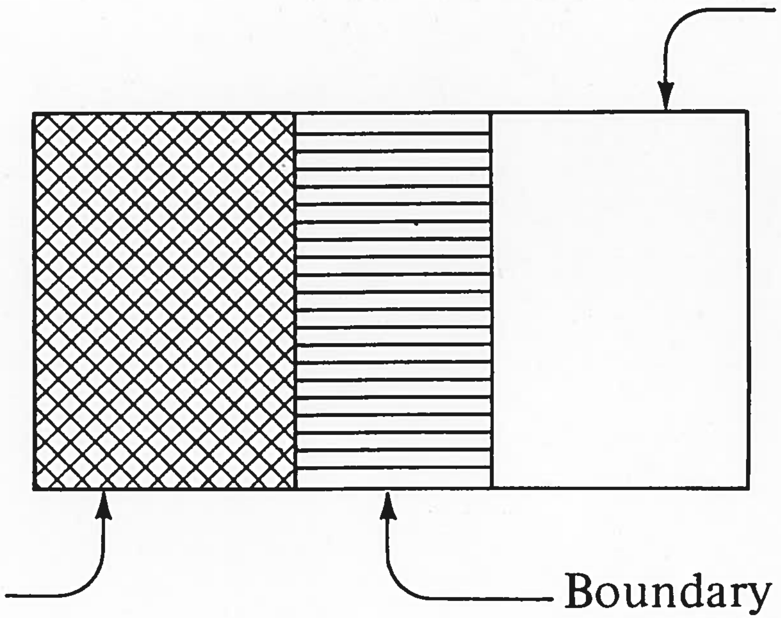
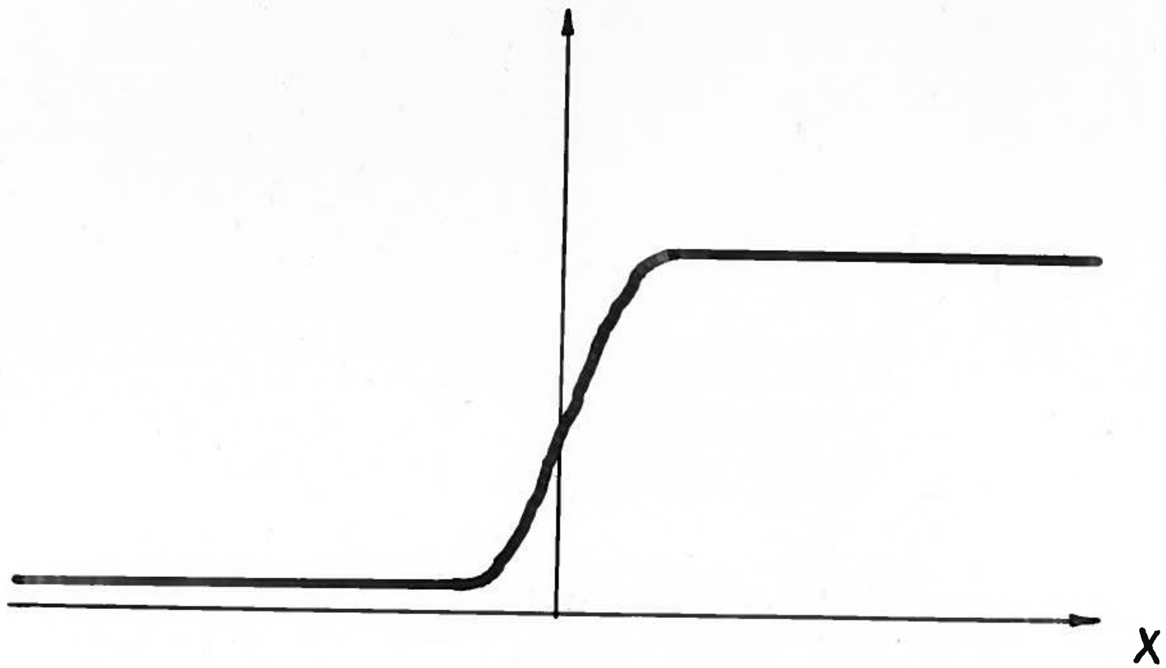
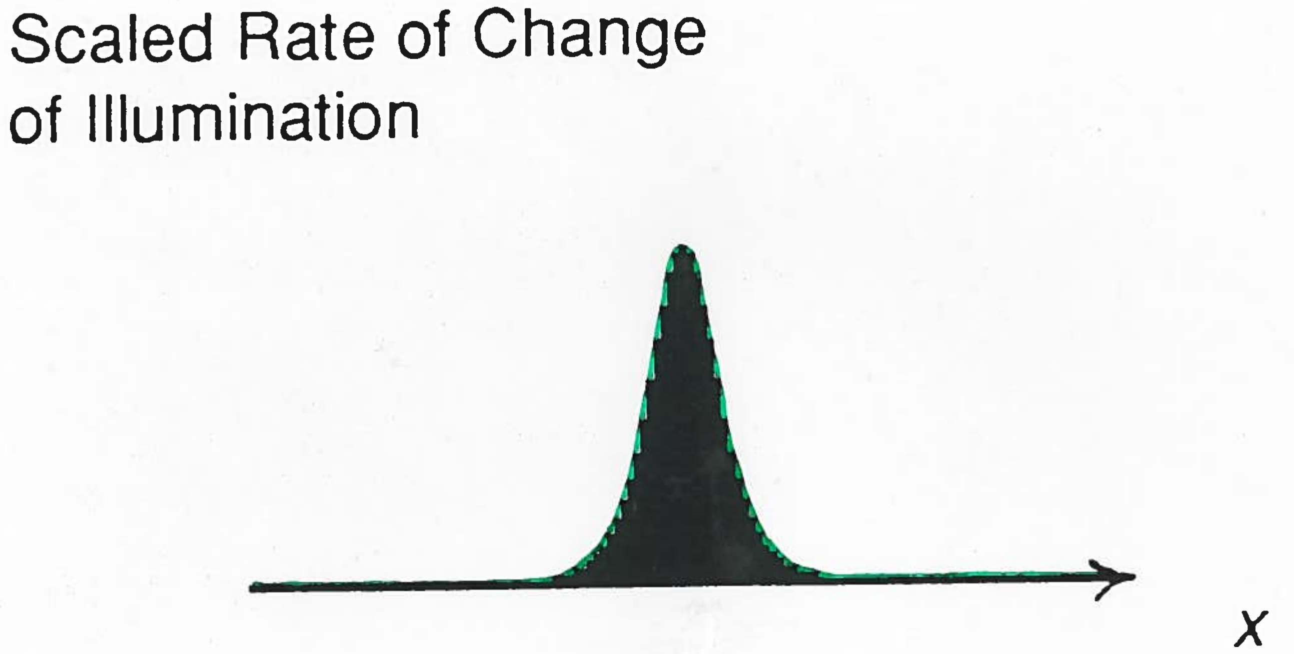


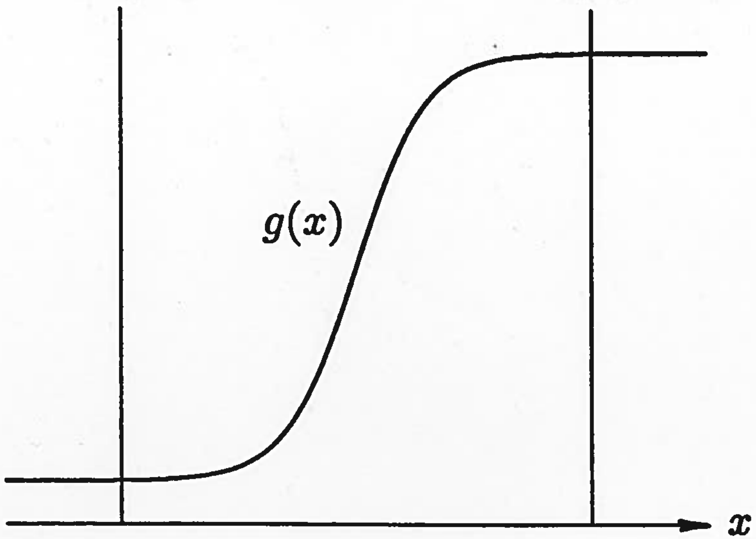
Figure Boundary between half-planes

Realistic illumination

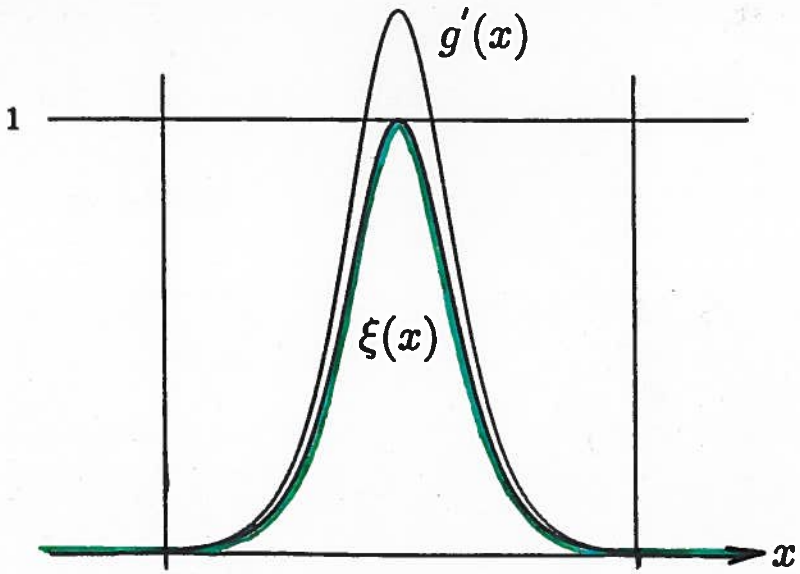




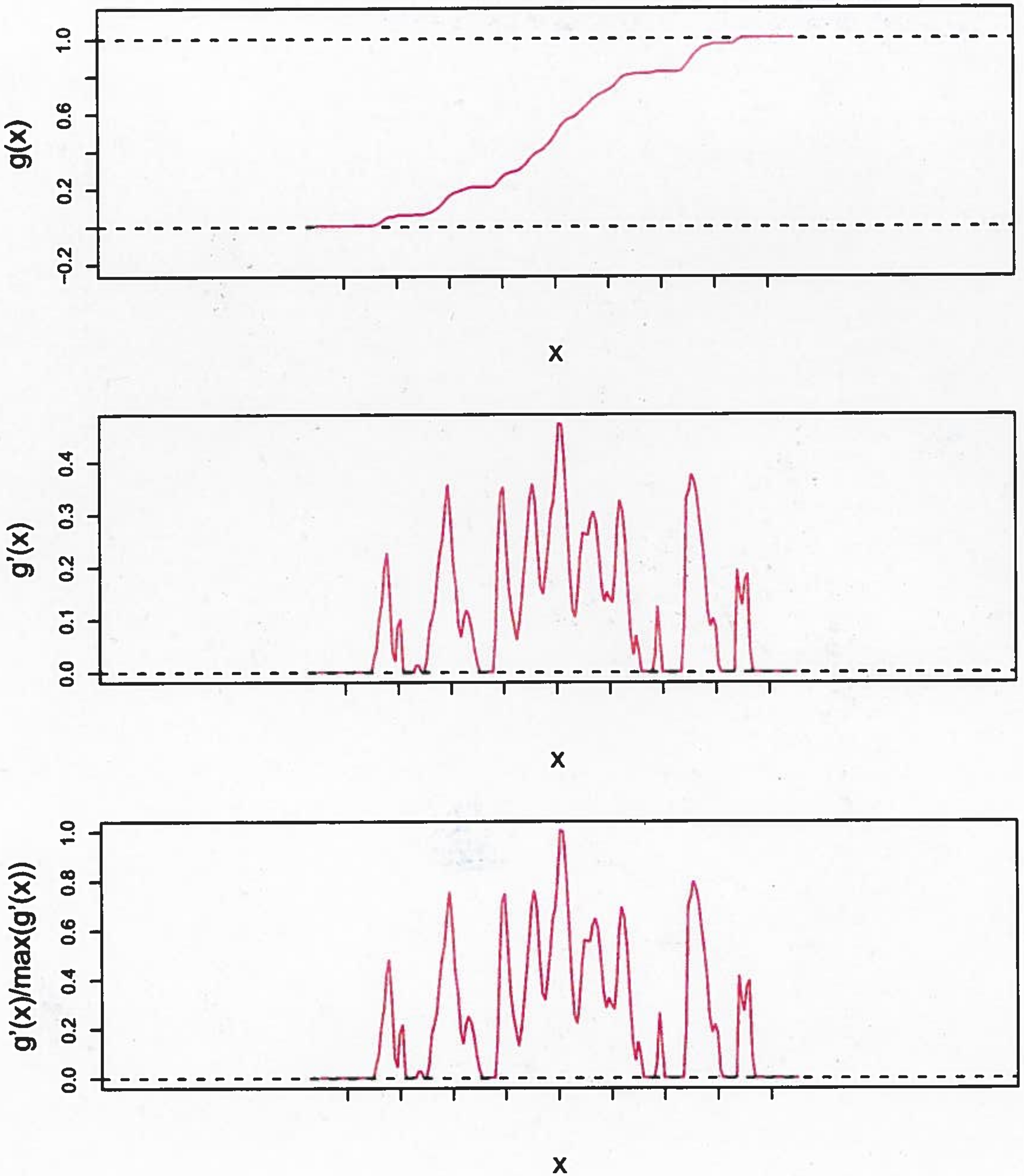
Derivative of illumination function displayed



$g(x)$
x

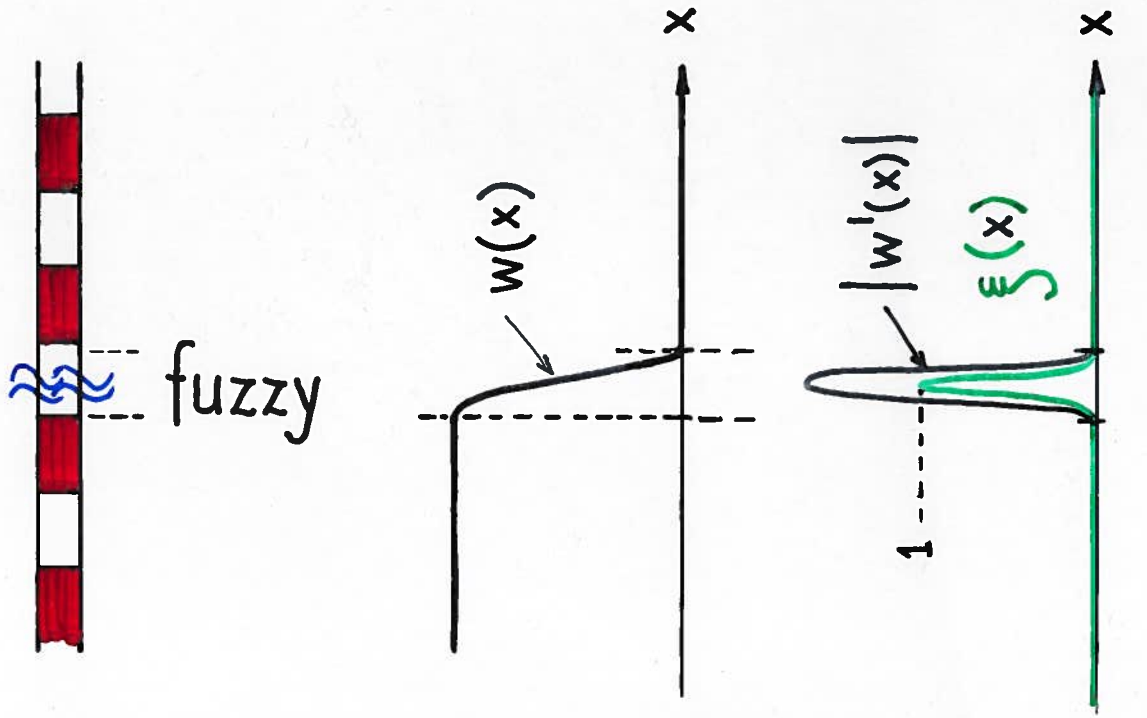


$g'(x)$
1
$\xi(x)$
x



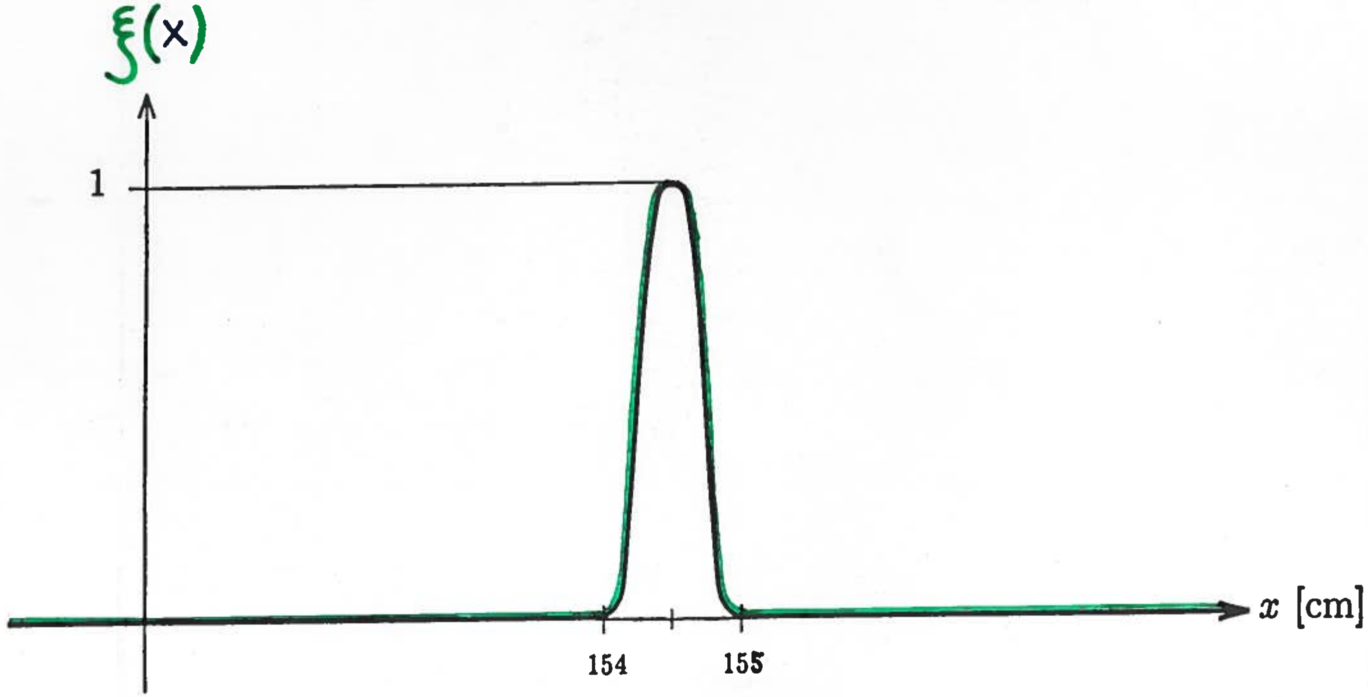
g(x)
1.0
0.6
0.2
-0.2
x
g'(x)
0.4
0.3
0.2
0.1
0.0
x
g'(x)/max(g'(x))
1.0
0.8
0.6
0.4
0.2
0.0
x

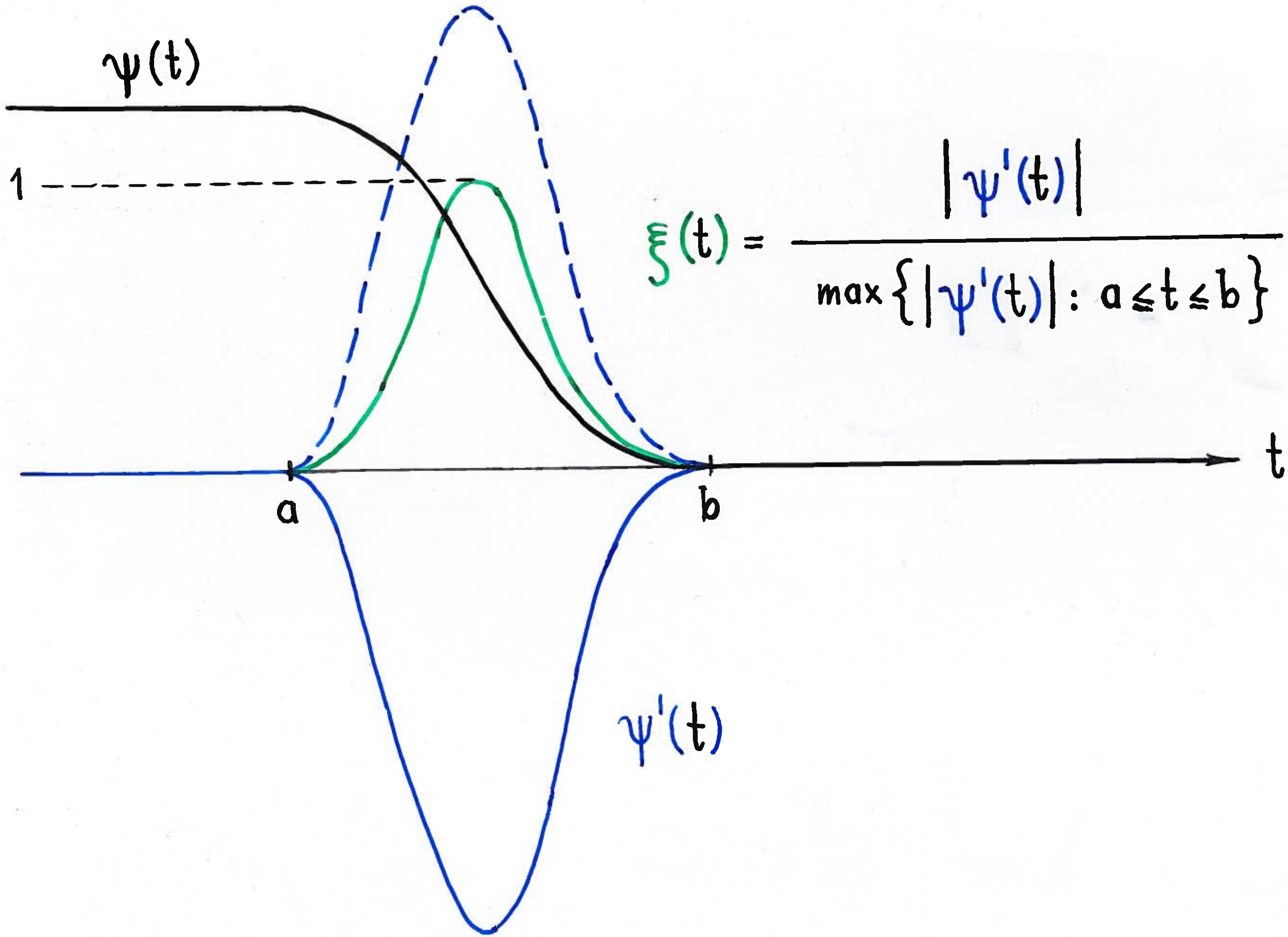
WATER LEVEL

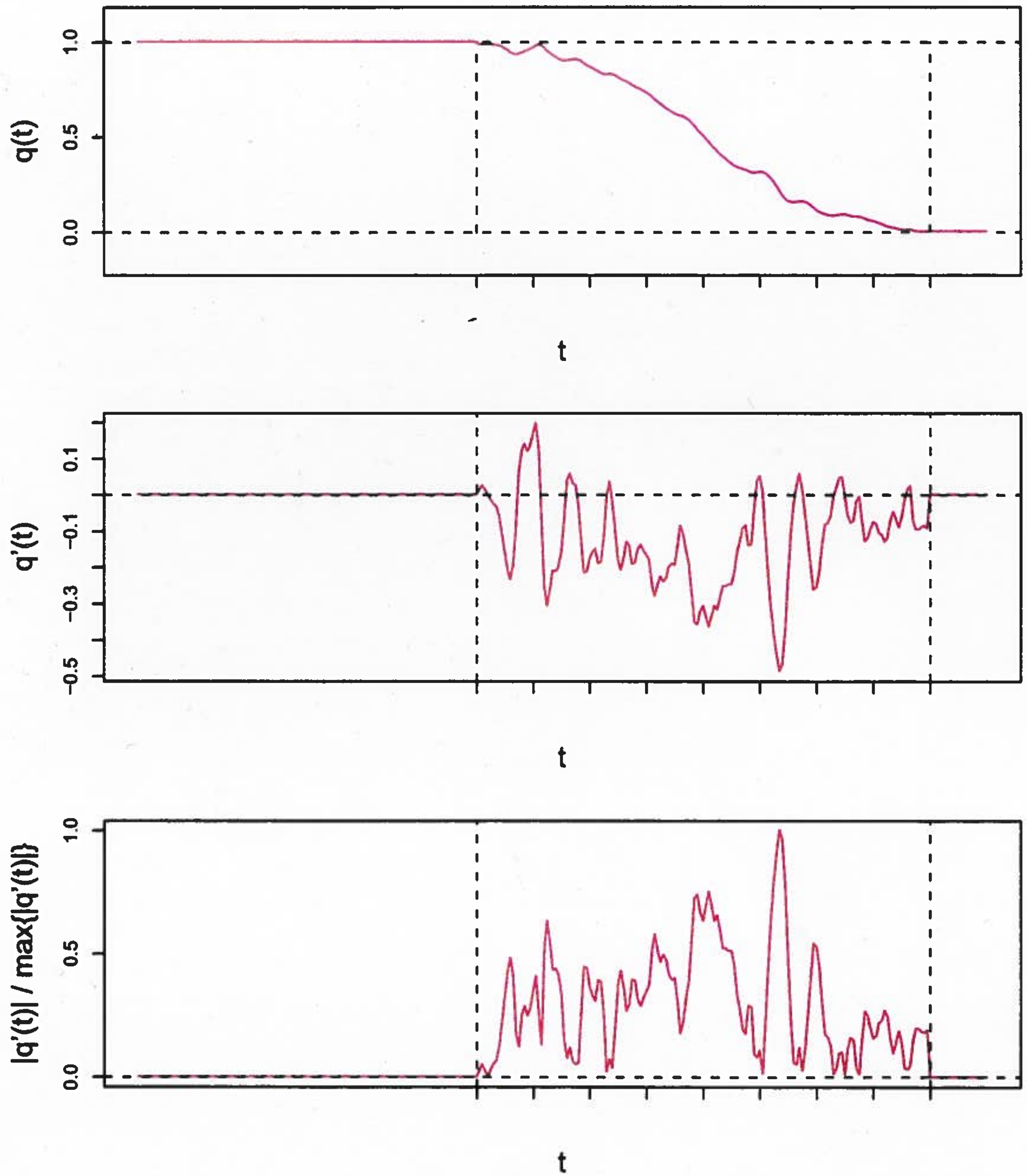


$$\xi(x) = \frac{|w'(x)|}{\max\{|w'(x)| : x \in \mathbb{R}\}} \qquad \forall x \in \mathbb{R}$$

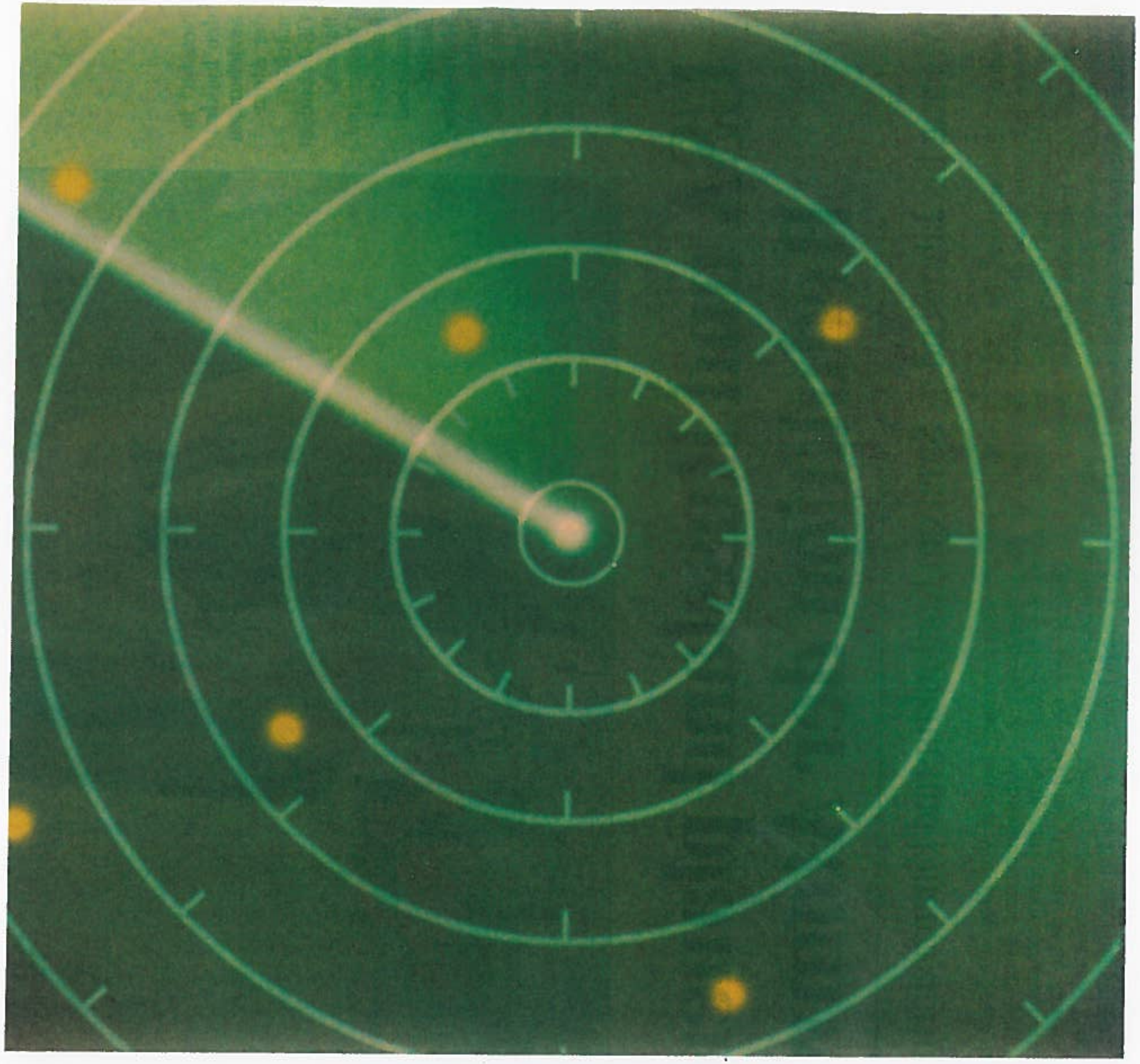
Figure : *Water level*

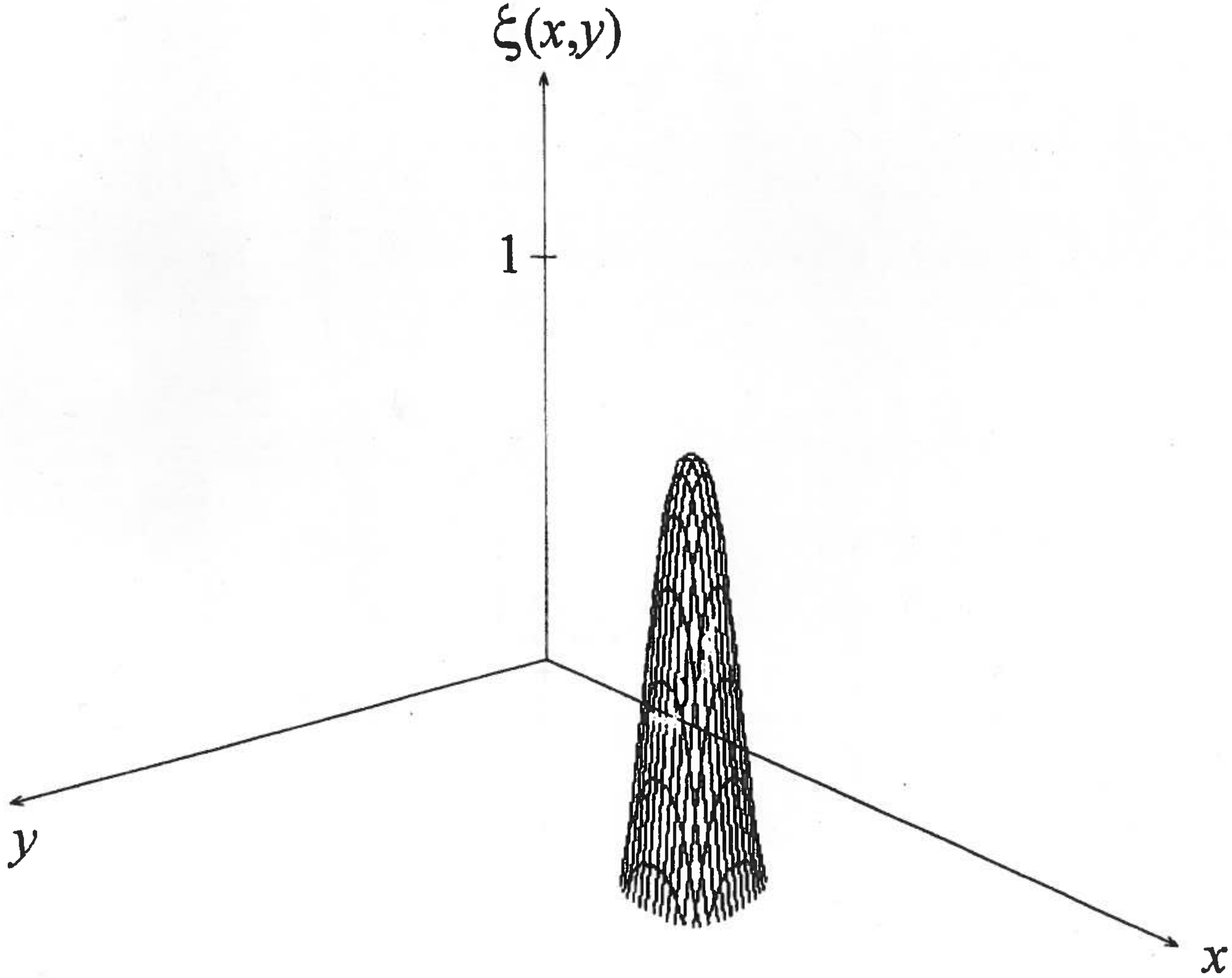



ψ(t)
1
ξ(t) = |ψ'(t)| / max{|ψ'(t)| : a ≤ t ≤ b}
t
a
b
ψ'(t)



q(t)
1.0
0.5
0.0
t
q'(t)
0.1
-0.1
-0.3
-0.5
t
|q'(t)| / max{|q'(t)|}
1.0
0.5
0.0
t




ξ(x,y)
1
y
x

FUZZY VECTOR

vector-characterizing function $\xi(\cdot)$

$\xi : \mathbb{R}^k \longrightarrow [0;1]$

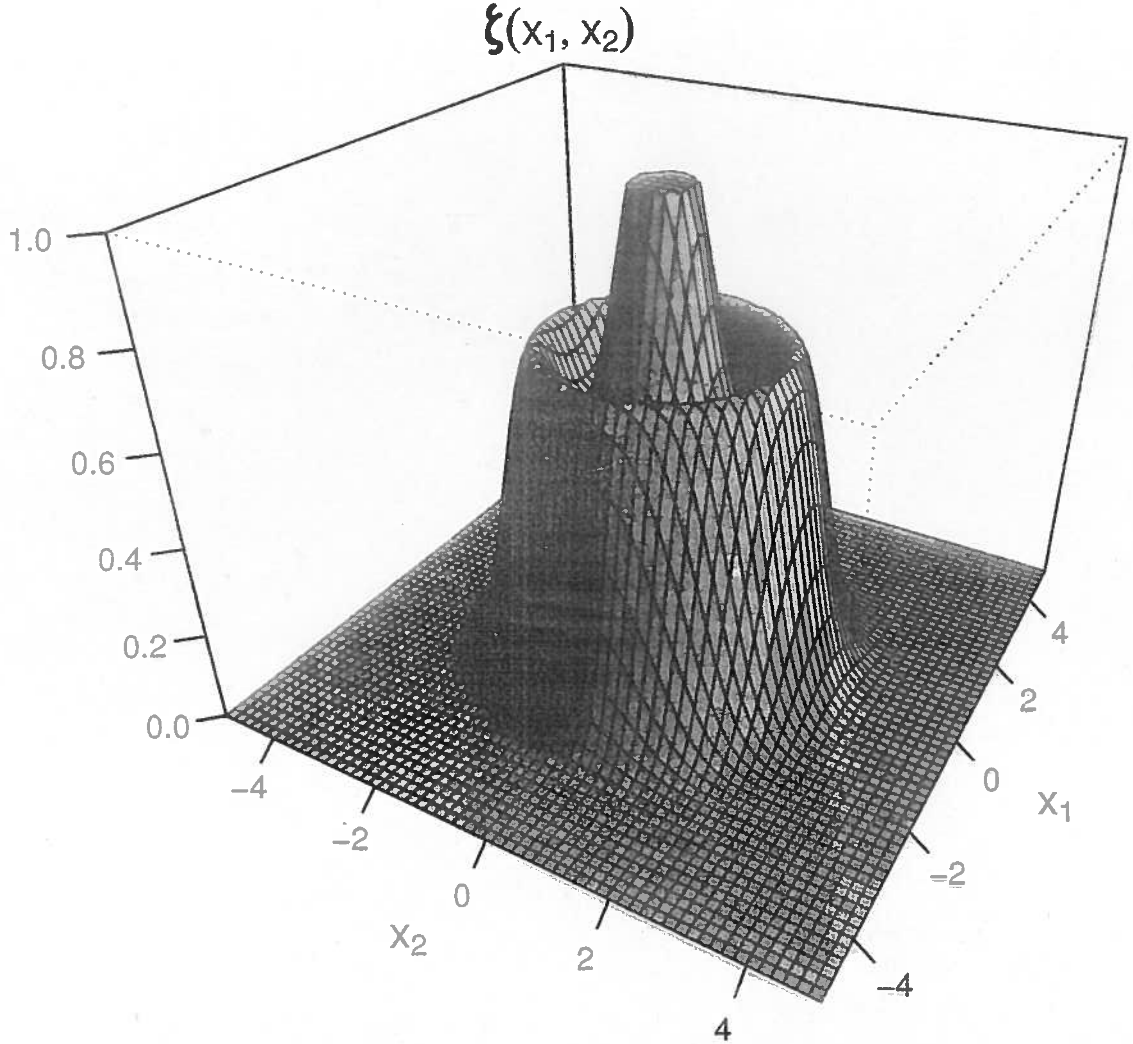
obeying

- $\forall \delta \in (0;1]$ the so-called δ-cut

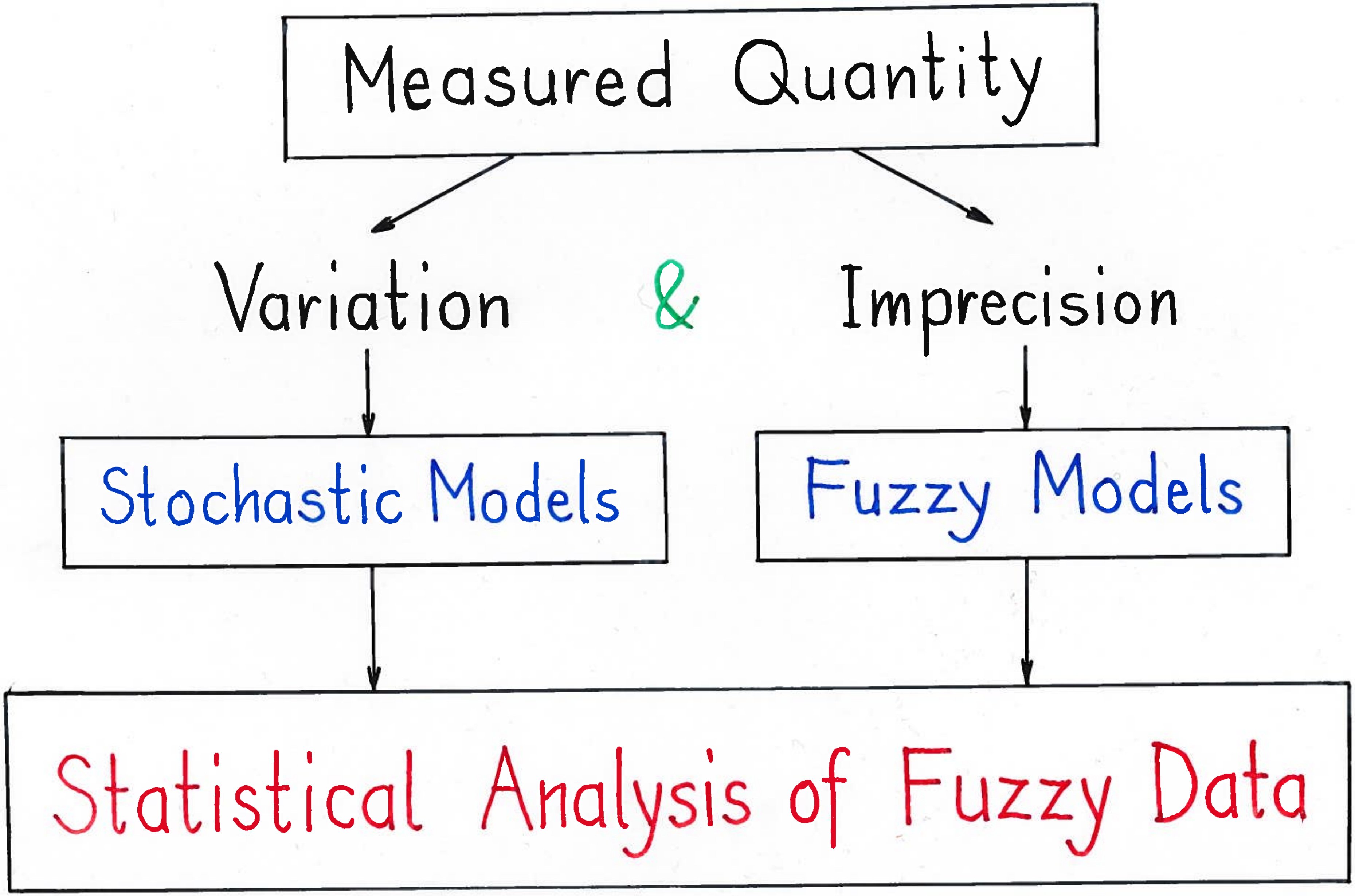
  $$C_\delta[\xi(\cdot)] := \{\underline{x} \in \mathbb{R}^k : \xi(\underline{x}) \geq \delta\} \neq \emptyset$$

  is a finite union of simply connected closed sets

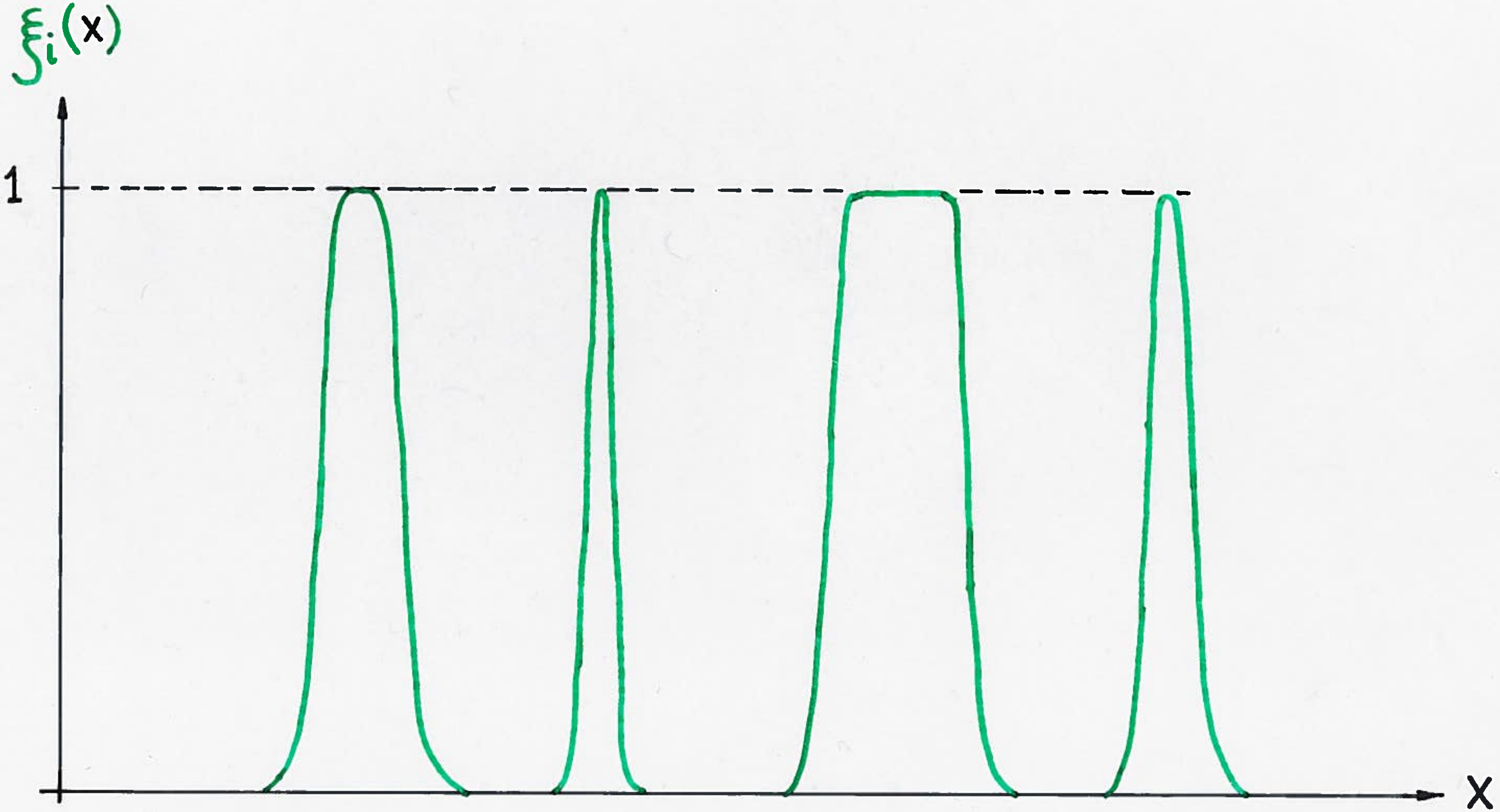
- $\mathrm{Supp}[\xi(\cdot)]$ is bounded



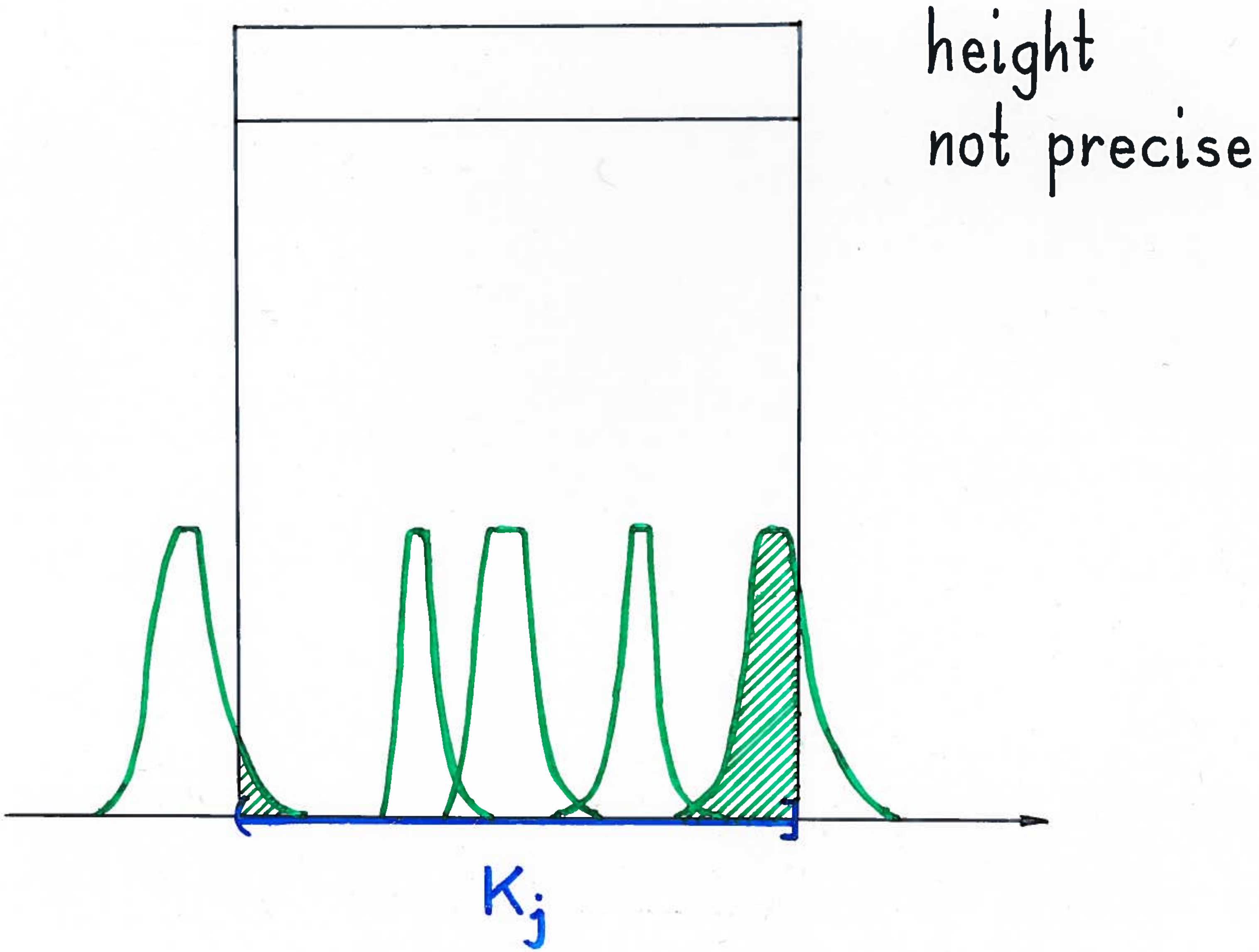
ξ(x1, x2)
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
x2
2
4
4
2
0
x1
−2
−4


Measured Quantity
Variation
&
Imprecision
Stochastic Models
Fuzzy Models
Statistical Analysis of Fuzzy Data

FUZZY SAMPLE



FUZZY HISTOGRAMS



CONSTRUCTION LEMMA

Let $(A_\delta ; \delta \in (0;1])$ be a nested family of subsets of a set M. Then the membership function of the corresponding fuzzy subset of M is given by

$$\xi(x) = \sup\{\delta \cdot \mathbb{1}_{A_\delta}(x) : \delta \in [0;1]\} \quad \forall x \in M$$

with $A_0 := M$

FUZZY FREQUENCY

n_j^* fuzzy absolute frequency of class K_j

Let $A_\delta(n_j^*) := [\underline{n}_\delta(K_j); \overline{n}_\delta(K_j)] \quad \forall \delta \in (0;1]$

where

$\overline{n}_\delta(K_j) = \#$ observ. with $C_\delta(\xi_i(\cdot)) \cap K_j \neq \emptyset$

$\underline{n}_\delta(K_j) = \#$ observ. with $C_\delta(\xi_i(\cdot)) \subseteq K_j$

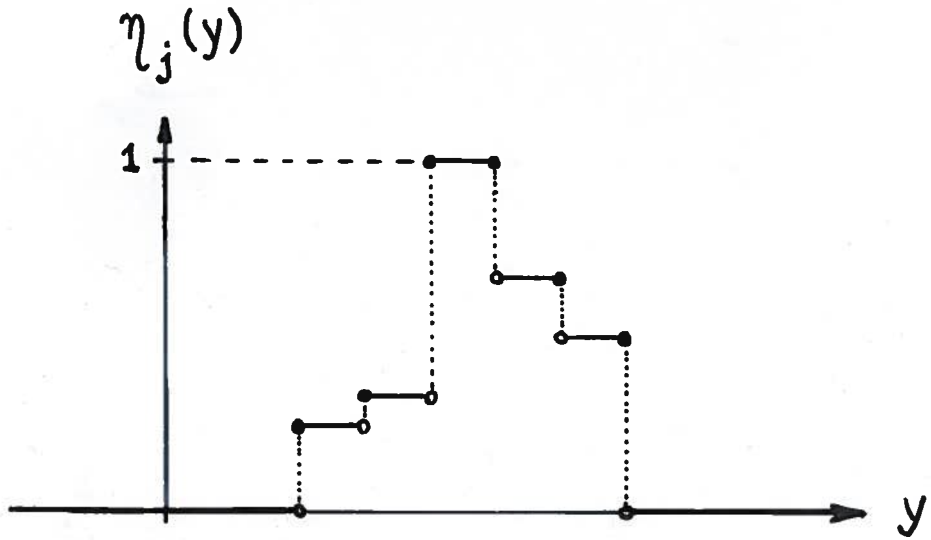
$\Rightarrow$ char. f. $\psi_j(\cdot)$ of n_j^* given by its values

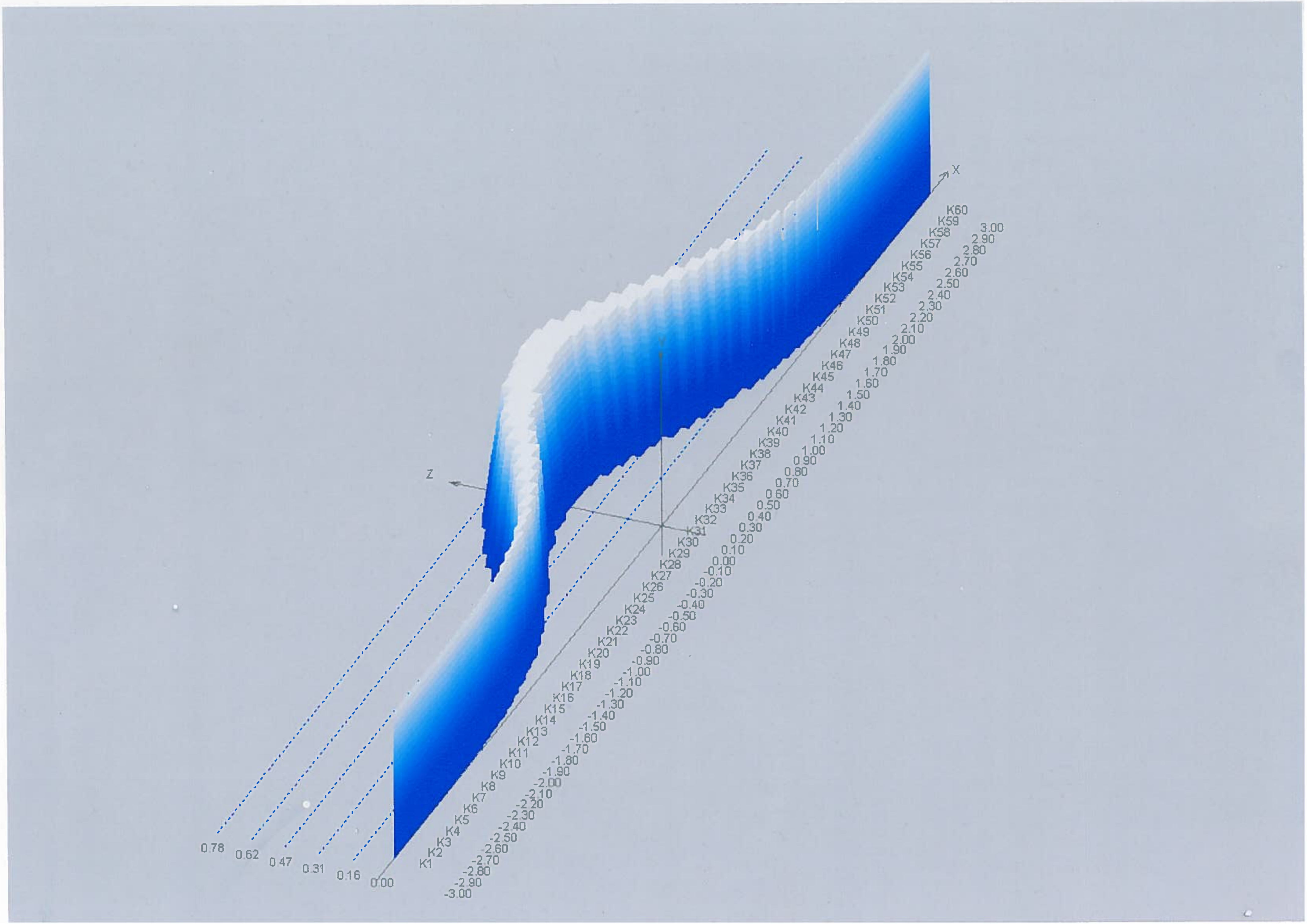
$$\psi_j(y) := \sup_{\delta \in [0,1]} \delta \cdot \mathbb{1}_{A_\delta(n_j^*)}(y) \quad \forall y \in \mathbb{R}$$

$h_j^* := \frac{n_j^*}{n}$ fuzzy relative frequency of class K_j

$\Rightarrow$ char. f. $\eta_j(\cdot)$ of h_j^* is given by

$$\eta_j(y) = \psi_j(ny) \quad \forall\, y \in \mathbb{R}$$





X
Y
Z
K1 K2 K3 K4 K5 K6 K7 K8 K9 K10 K11 K12 K13 K14 K15 K16 K17 K18 K19 K20 K21 K22 K23 K24 K25 K26 K27 K28 K29 K30 K31 K32 K33 K34 K35 K36 K37 K38 K39 K40 K41 K42 K43 K44 K45 K46 K47 K48 K49 K50 K51 K52 K53 K54 K55 K56 K57 K58 K59 K60
-3.00 -2.90 -2.80 -2.70 -2.60 -2.50 -2.40 -2.30 -2.20 -2.10 -2.00 -1.90 -1.80 -1.70 -1.60 -1.50 -1.40 -1.30 -1.20 -1.10 -1.00 -0.90 -0.80 -0.70 -0.60 -0.50 -0.40 -0.30 -0.20 -0.10 0.00 0.10 0.20 0.30 0.40 0.50 0.60 0.70 0.80 0.90 1.00 1.10 1.20 1.30 1.40 1.50 1.60 1.70 1.80 1.90 2.00 2.10 2.20 2.30 2.40 2.50 2.60 2.70 2.80 2.90 3.00
0.78 0.62 0.47 0.31 0.16 0.00

CALCULATIONS

Sums $\sum_{i=1}^{n} x_i^*$

Averages $\overline{x}_n^*$

Indicators and Indexes I^*

$$I^* = f(x_1^*, \ldots, x_n^*; w_1, \ldots, w_n)$$

Functions of Fuzzy Variables

Extension Principle

STANDARD STATISTICAL INFERENCE

$X \sim P_\theta ; \theta \in \Theta$, M_X Observation Space

$x_1, \cdots, x_n$ Sample , $x_i \in M_X \Rightarrow (x_1, \cdots, x_n) \in M_X^n$

M_X^n Sample Space

- Estimators $\vartheta(x_1, \cdots, x_n)$, $\vartheta : M_X^n \to \Theta$
- Confidence Regions $\kappa(x_1, \cdots, x_n)$
- Test Statistics $t(x_1, \cdots, x_n)$

Generalization for Fuzzy Data ?

COMBINED FUZZY SAMPLE

Sample $x_1^*, \dots, x_n^*$

$\xi_1(\cdot), \dots, \xi_n(\cdot)$

x_i^* Fuzzy Element of Observation Space M

$M^n = \{ \underline{x} = (x_1, \dots, x_n) : x_i \in M \}$ Sample Space

$\underline{x}^*$ Fuzzy Element of M^n with VCF $\xi(\cdot)$

$$\xi(x_1, \dots, x_n) = T_n\big(\xi_1(x_1), \dots, \xi_n(x_n)\big) \quad \forall (x_1, \dots, x_n)$$

$\underline{x}^*$ Combined Fuzzy Sample

GENERALISED ESTIMATORS

$\hat{\theta} = \vartheta(x_1, \cdots, x_n)$ Classical Estimator

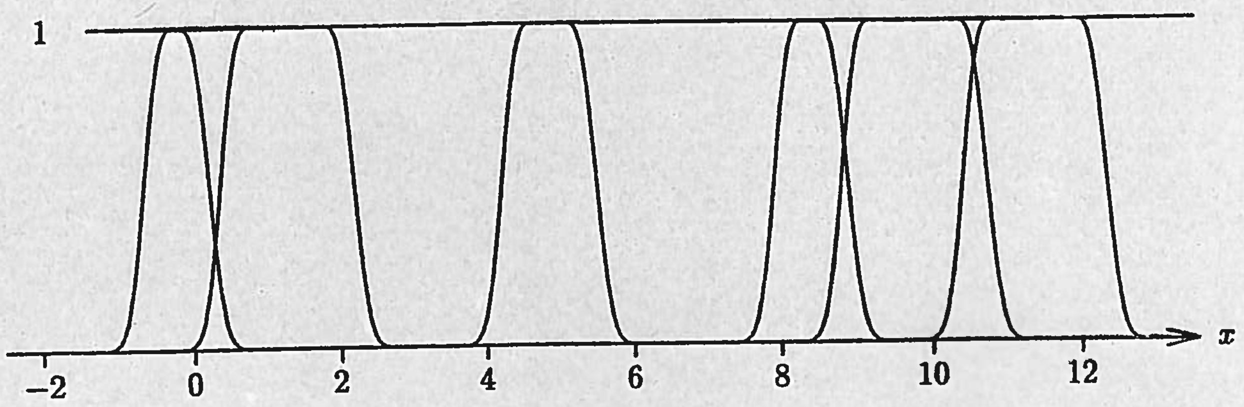
$\underline{x} = (x_1, \cdots, x_n) \in M_X^n\,, \quad \vartheta : M_X^n \to \Theta$

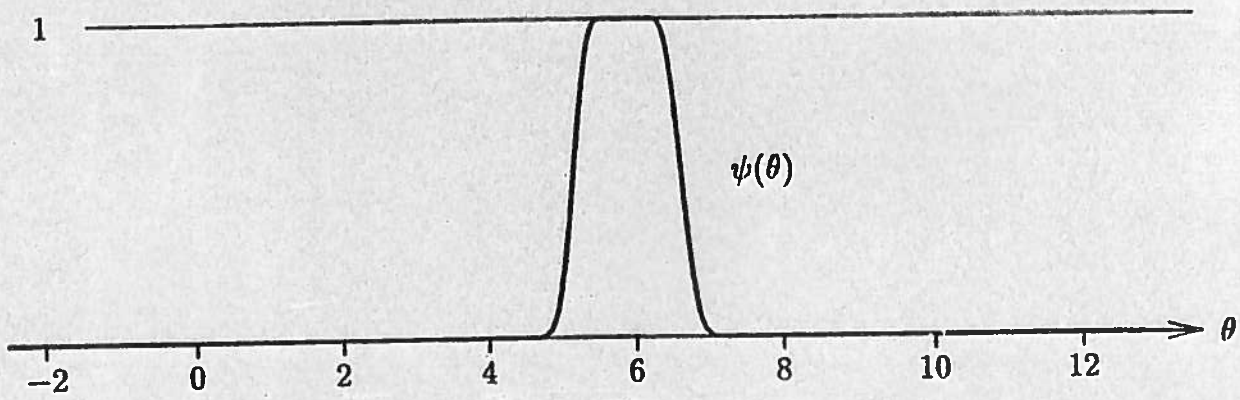
$\underline{x}^*$ Combined Fuzzy Sample $\triangleq \xi(\cdot, \cdots, \cdot)$

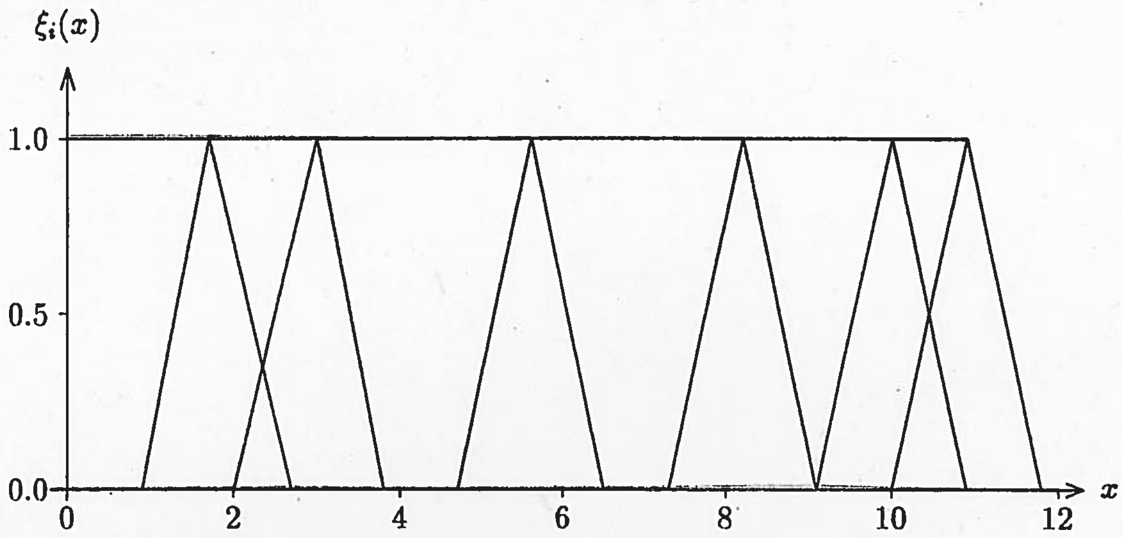
$\hat{\theta}^*$ Fuzzy Estimator $\triangleq \psi(\cdot)$

$$\psi(\theta) := \sup\left\{\xi(\underline{x}) : \underline{x} \in \vartheta^{-1}(\{\theta\})\right\} \quad \forall \theta \in \Theta$$

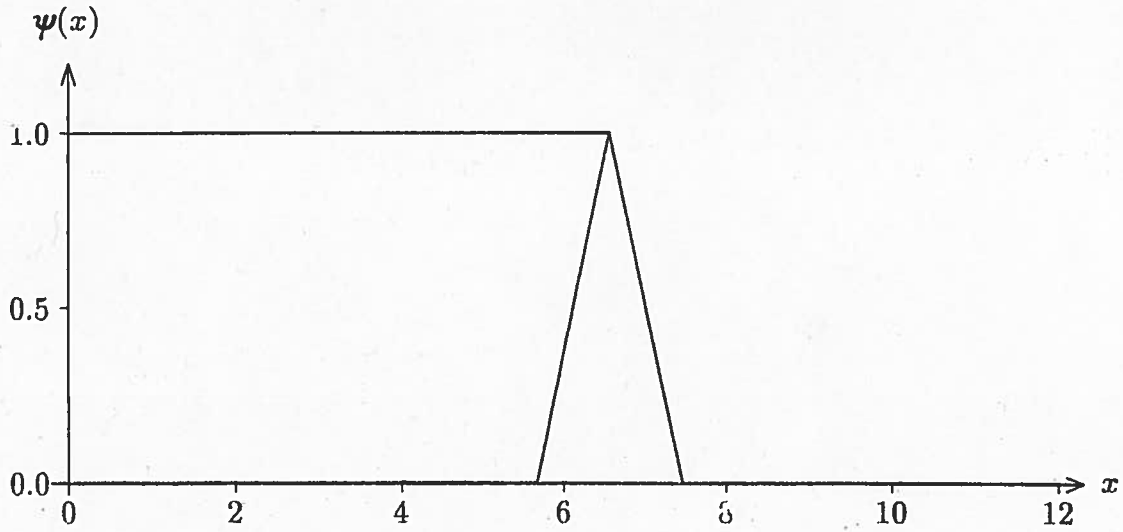
with $\vartheta^{-1}(\{\theta\}) = \left\{\underline{x} \in M_X^n : \vartheta(\underline{x}) = \theta\right\}$


1
x
−2
0
2
4
6
8
10
12


1
ψ(θ)
θ
−2
0
2
4
6
8
10
12



$\xi_i(x)$
1.0
0.5
0.0
0
2
4
6
8
10
12
x



$\psi(x)$
1.0
0.5
0.0
0
2
4
6
8
10
12
x

FUZZY CORRELATION

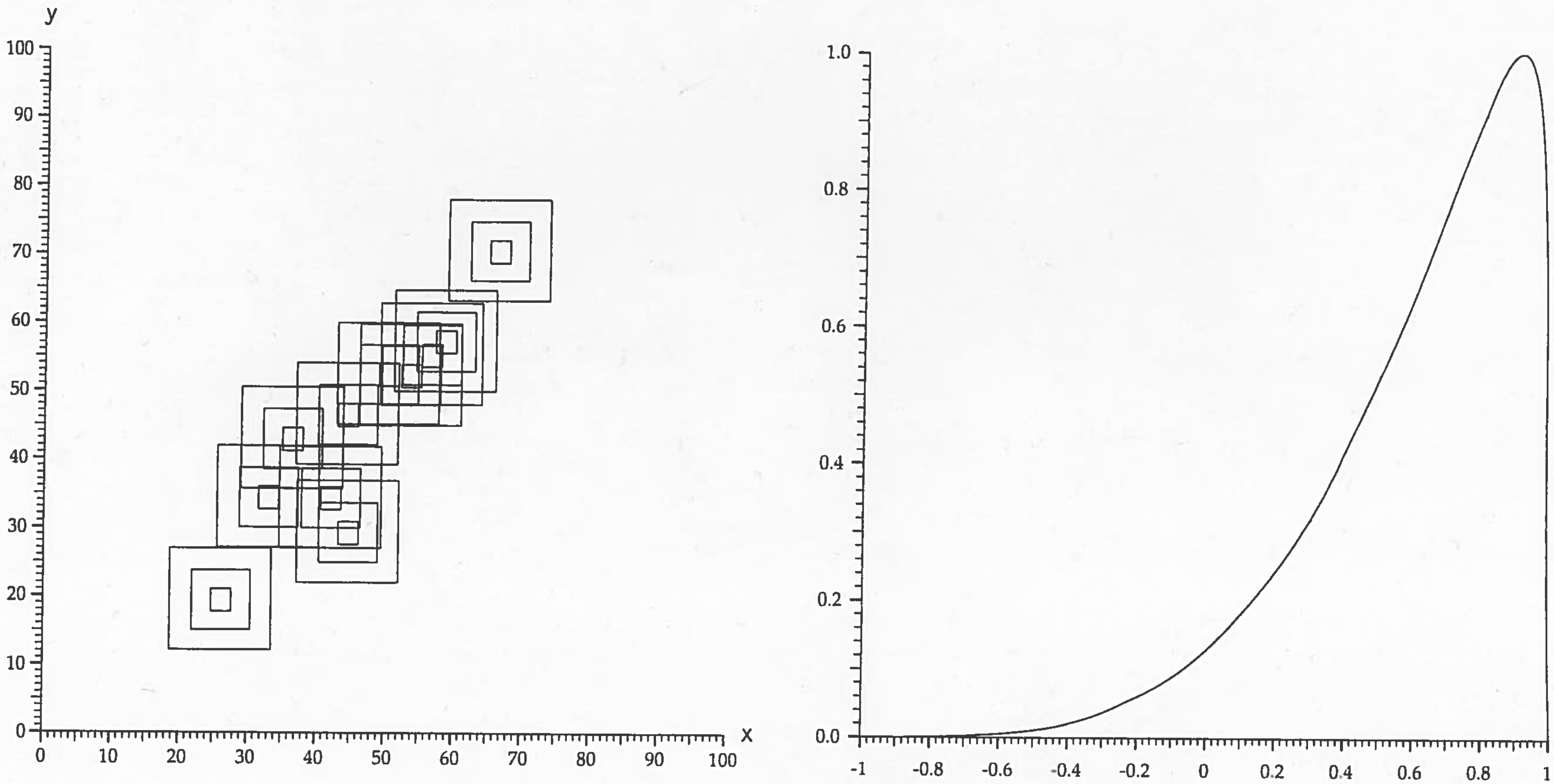


Figure *Fuzzy correlation coefficient.*

FUZZY CONFIDENCE REGIONS

$X \sim P_\theta \,,\quad \theta \in \Theta$

$\kappa(\cdot)$ Confidence function

$\underline{x}^*$ Combined fuzzy sample $\quad \xi(\cdot)$ VCF

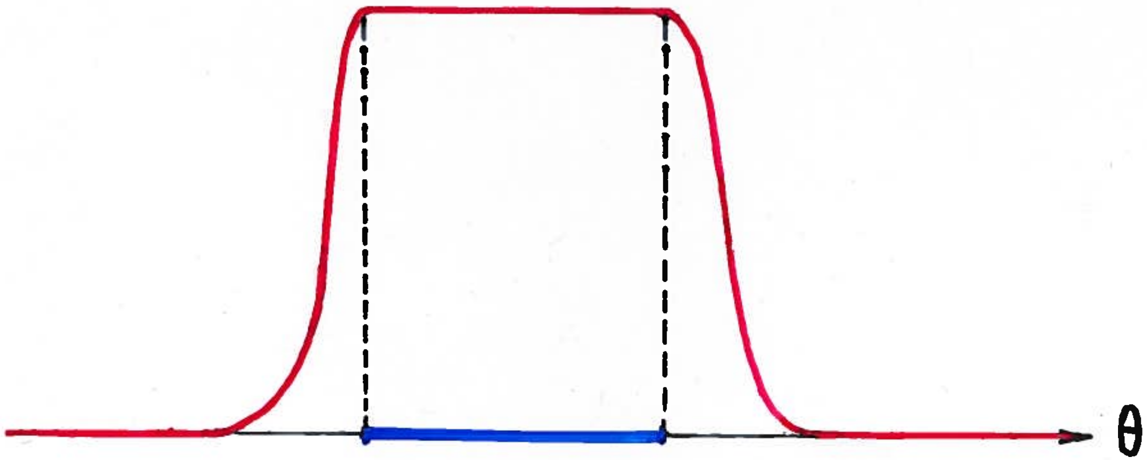
Generalized confidence set $\kappa(\underline{x}^*)$ is a fuzzy subset of Θ whose membership function $\varphi(\cdot)$ is defined by

$$\varphi(\theta) := \sup\left\{ \xi(\underline{x}) : \theta \in \kappa(\underline{x}) \right\} \quad \forall\, \theta \in \Theta$$

$$\underline{x} \in \text{support}\left[\xi(\cdot)\right]$$

The following holds:

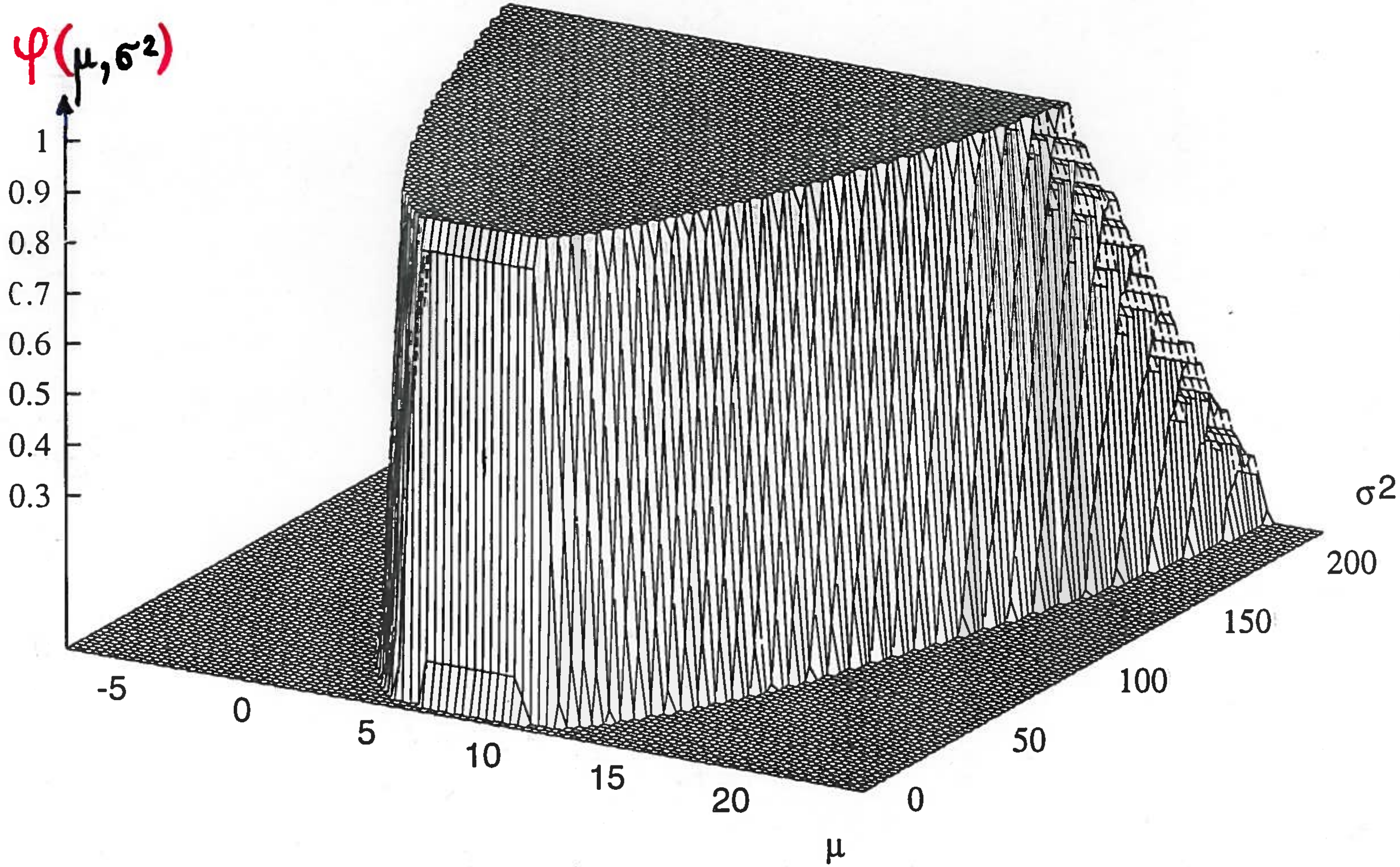
(1) $\varphi(\theta) = 1 \quad \forall\, \theta \in \bigcup_{\underline{x} : \xi(\underline{x}) = 1} K(\underline{x})$



$\bigcup_{\underline{x} : \xi(\underline{x}) = 1} K(\underline{x})$

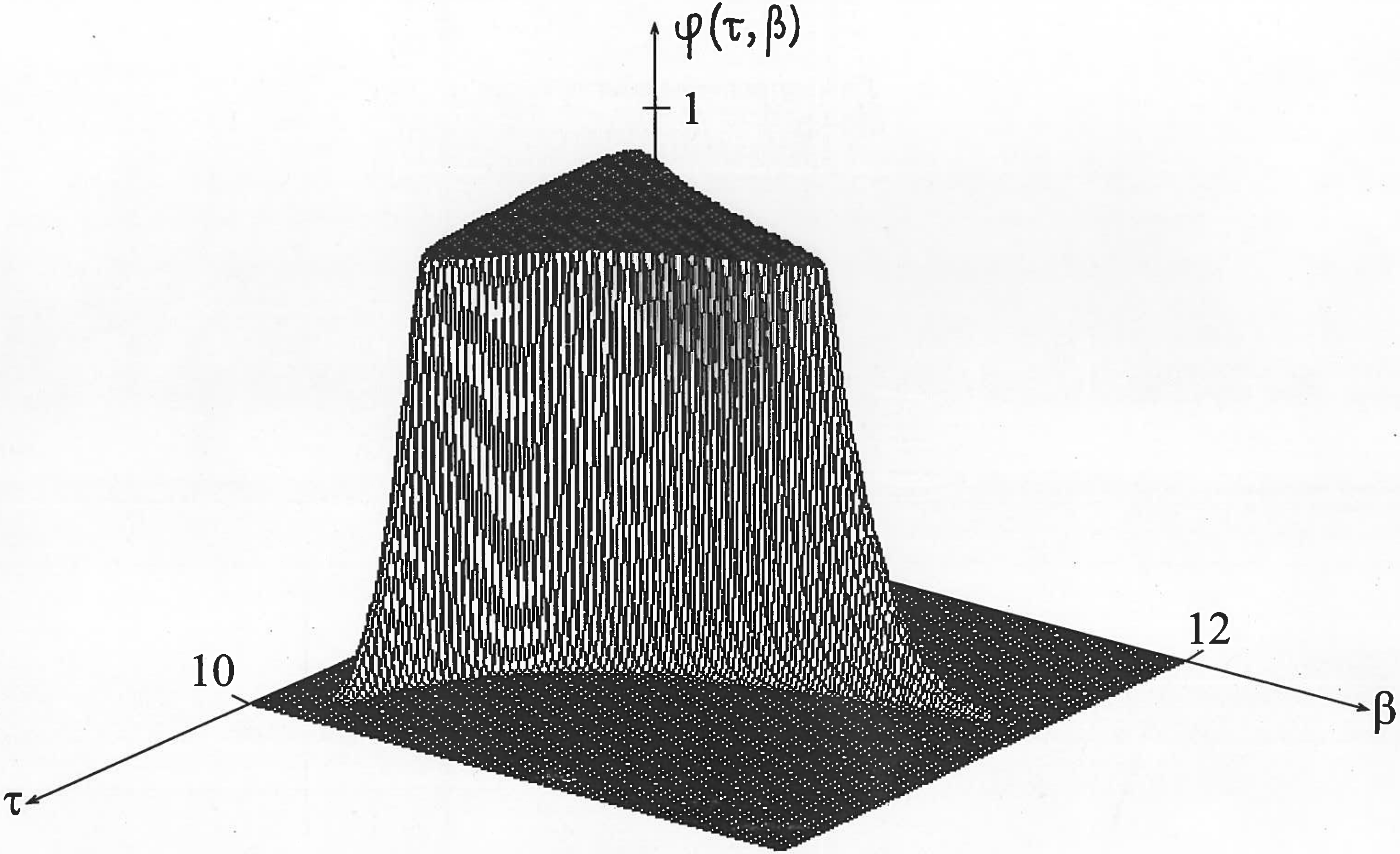
(2) For classical samples $\underline{x} = (x_1, \dots, x_n)$:

$\varphi(\cdot) = \mathbb{1}_{K(\underline{x})}(\cdot)$



φ(μ, σ²)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
-5
0
5
10
15
20
μ
0
50
100
150
200
σ2

Weibull (τ, β), $F(t) = 1 - e^{-\left(\frac{t}{\tau}\right)^{\beta}}$



STATISTICAL TESTS

$T = t(X_1, \cdots, X_n)$ Test Statistic

$x_1^*, \cdots, x_n^*$ Fuzzy Sample

$t^* = t(x_1^*, \cdots, x_n^*) \triangleq \eta(\cdot)$

Calculation of $\eta(\cdot)$ based on the combined fuzzy sample $\underline{x}^* \triangleq \xi(x_1, \cdots, x_n)$ and the extension principle:

$$t^* = t(\underline{x}^*)$$

FIRST SOLUTION

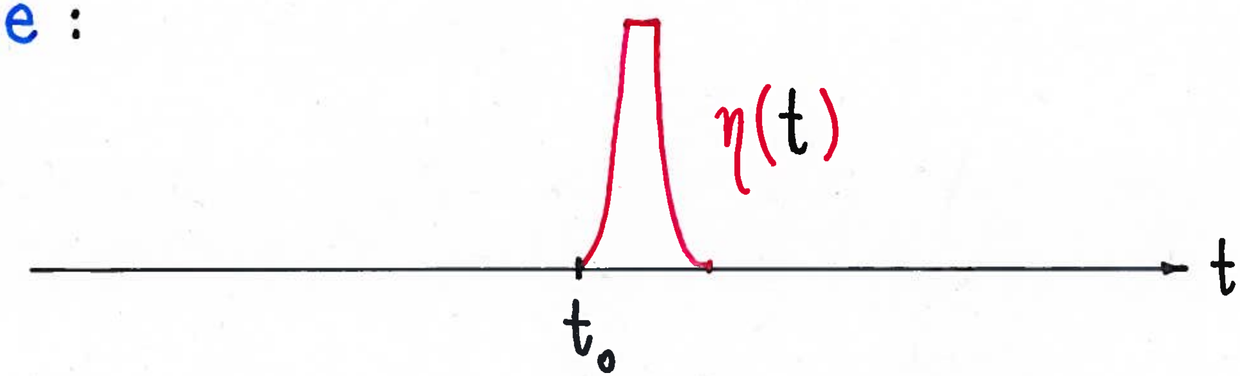
p-value approach

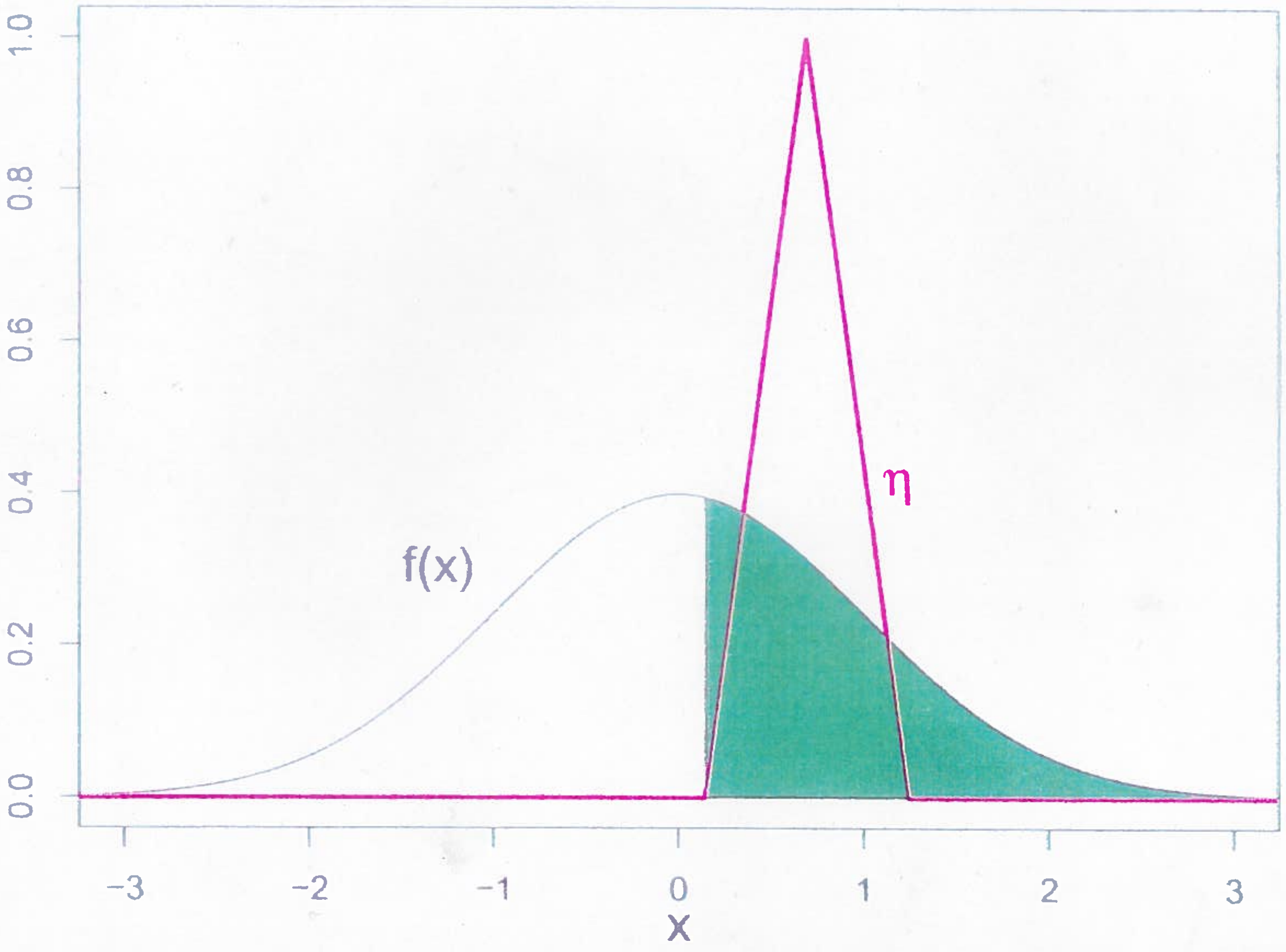
- For precise data $x_1, \dots, x_n \Rightarrow t = t(x_1, \dots, x_n)$

  p-value : largest error probability for which a hypothesis is rejected for t

- For fuzzy data $x_1^*, \dots, x_n^* \Rightarrow t^* \triangleq \eta(\cdot)$

  p-value :





1.0
0.8
0.6
0.4
0.2
0.0
η
f(x)
−3
−2
−1
0
1
2
3
x

FUZZY P-VALUES

t^* fuzzy value of the test statistic T

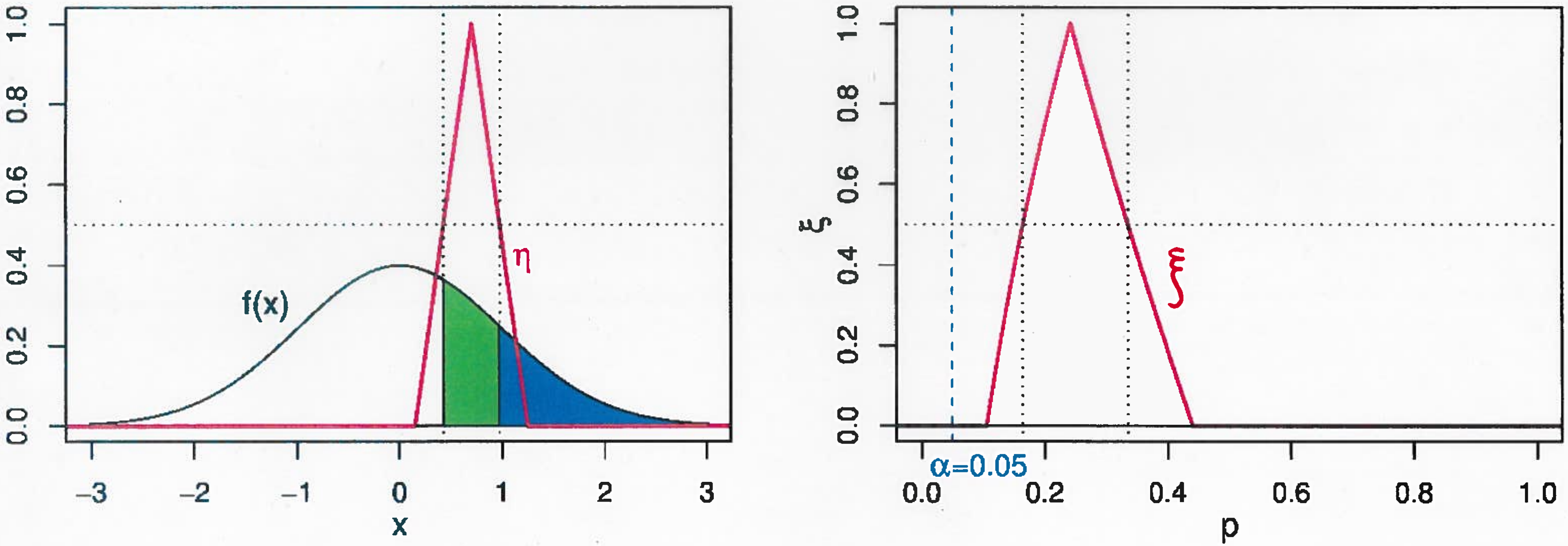
$$C_\delta(t^*) = [t_1(\delta); t_2(\delta)] \quad \forall \delta \in (0;1]$$

p^* fuzzy p-value

For one-sided tests

$$A_\delta(p^*) = [Pr(T \leq t_1(\delta)); Pr(T \leq t_2(\delta))]$$

$$\xi(p) = \sup\{\delta \cdot \mathbb{1}_{A_\delta}(p) : \delta \in [0;1]\} \quad \forall p \in \mathbb{R}$$



f(x)
η
x
ξ
α=0.05
p

BAYESIAN INFERENCE

$X \sim f(\cdot|\theta)$, $\theta \in \Theta$, $\tilde{\theta}$ Stochastic Qu.

$\pi(\cdot)$ a-priori distribution on Θ

$x_1, \cdots, x_n$ Sample information

Updating of the a-priori distribution

$$\pi(\theta|x_1,\cdots,x_n) = \frac{\pi(\theta)\cdot \ell(\theta; x_1,\cdots,x_n)}{\int_{\Theta} \pi(\theta)\cdot \ell(\theta; x_1,\cdots,x_n)\,d\theta} \quad \forall\, \theta \in \Theta$$

a-posteriori distribution

$$\ell(\theta; x_1,\cdots,x_n) = \prod_{i=1}^{n} f(x_i|\theta)$$

FOR FUZZY DATA ?

Fuzzy valued functions $f^*: M \to \mathcal{F}_I(\mathbb{R})$

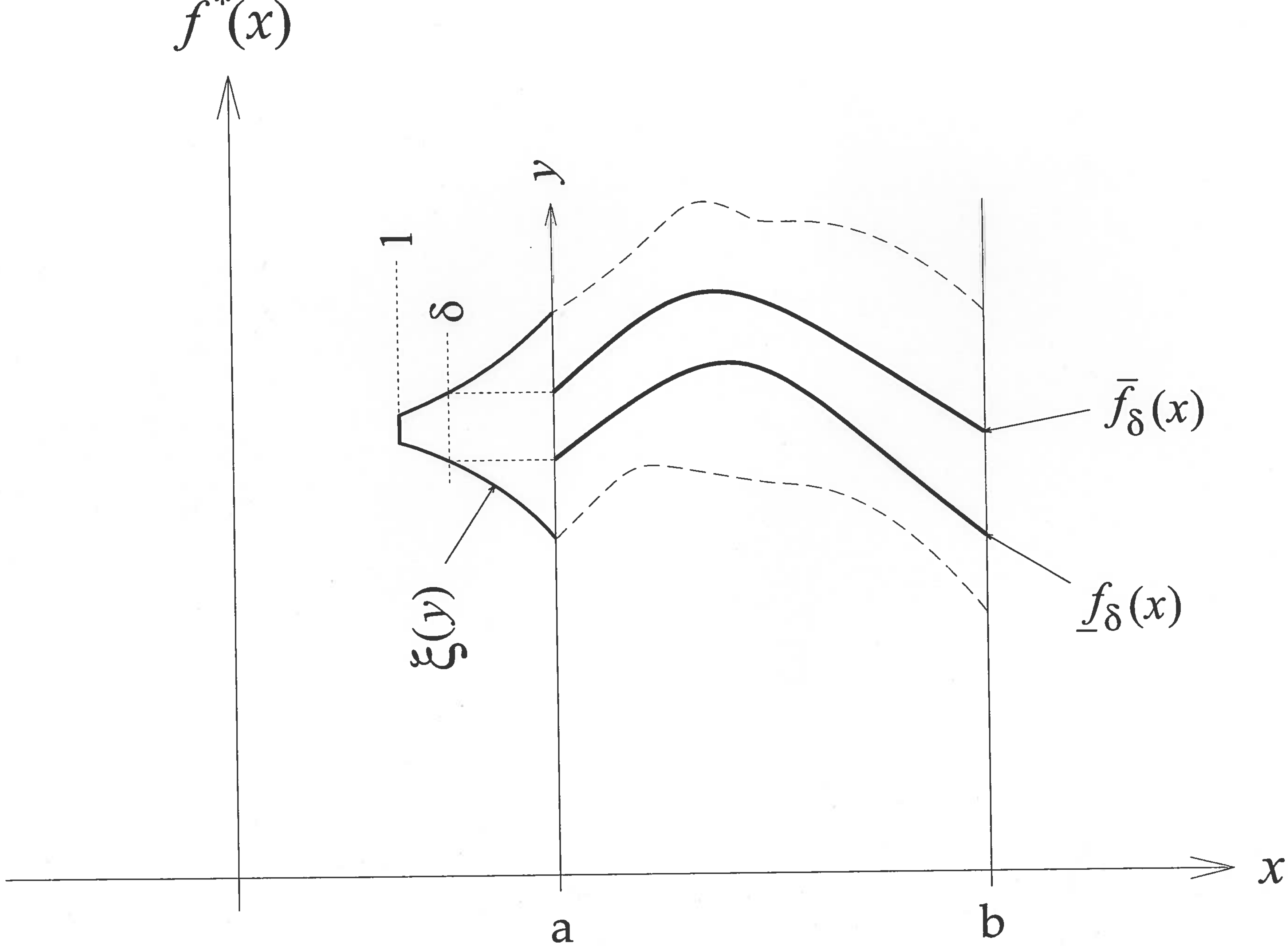
$$f^*(x) = y^* \;\hat{=}\; \xi_x(\cdot) \quad \forall x \in M$$

δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$

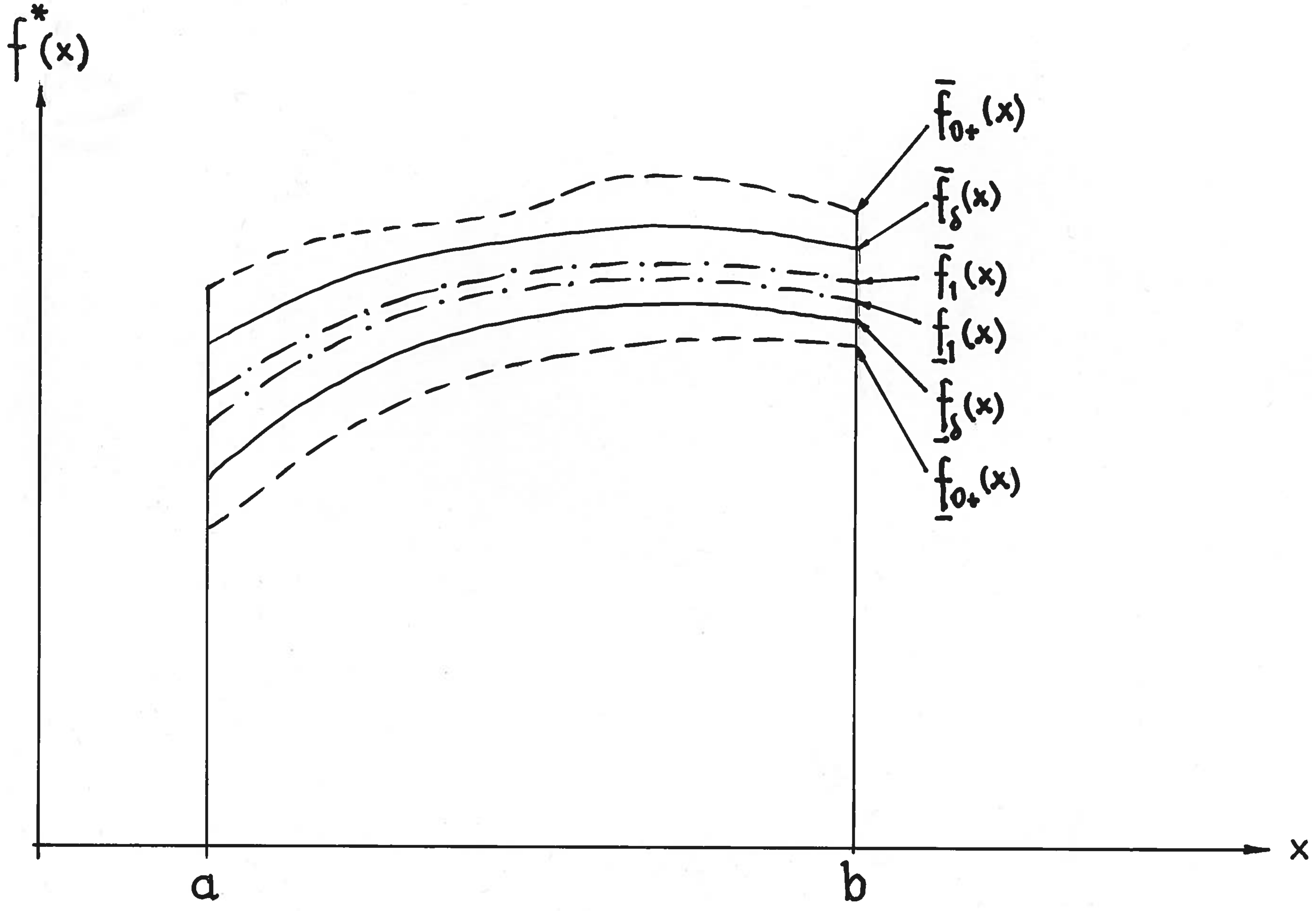
defined by $C_\delta[f^*(x)] = [\underline{f}_\delta(x) ; \overline{f}_\delta(x)] \quad \forall x \in M$

$\forall \delta \in (0;1]$

For $M = \mathbb{R}$ δ-level curves (real functions)



$f^*(x)$
y
1
δ
$\bar{f}_\delta(x)$
$\underline{f}_\delta(x)$
$\xi(y)$
a
b
x



$f^*(x)$
$\bar{f}_{0+}(x)$
$\bar{f}_\delta(x)$
$\bar{f}_1(x)$
$\underline{f}_1(x)$
$\underline{f}_\delta(x)$
$\underline{f}_{0+}(x)$
a
b
x

FUZZY PROBABILITY DENSITY

Generalized densities $f^*(\cdot)$ on $\mathbb{R}$:

$f^*(\cdot)$ fuzzy function with δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ integrable with

$$\int_{\mathbb{R}} \overline{f}_\delta(x)\,dx < \infty \quad \forall \delta \in (0;1]$$

and $\exists$ classical density $f(\cdot)$ on $\mathbb{R}$ with

$$\underline{f}_1(x) \leq f(x) \leq \overline{f}_1(x) \quad \forall x \in \mathbb{R}$$

The fuzzy probability $P^*(B)$ of $B \in \mathcal{B}$ is a fuzzy interval

FUZZY PROBABILITY

Fuzzy density $\pi^*(\cdot)$

δ - level curves $\underline{\pi}_\delta(\cdot)$ and $\overline{\pi}_\delta(\cdot)$

$$\mathcal{D}_\delta := \left\{ f : f \text{ density with } \underline{\pi}_\delta(\theta) \le f(\theta) \le \overline{\pi}_\delta(\theta) \quad \forall \theta \in \Theta \right\}$$

For $A \subseteq \Theta$ the fuzzy probability $P^*(A)$ defined by the generating sets

$$C_\delta(P^*(A)) = \left[\underline{P}_\delta(A), \overline{P}_\delta(A) \right] \quad \forall \delta \in (0;1]$$

where $\underline{P}_\delta(A)$ and $\overline{P}_\delta(A)$ are given by

$$\left.\begin{aligned} \overline{P}_{\delta}(A) &:= \sup_{f \in \mathcal{D}_{\delta}} \int_A f(\theta)\, d\theta \\ \underline{P}_{\delta}(A) &:= \inf_{f \in \mathcal{D}_{\delta}} \int_A f(\theta)\, d\theta \end{aligned}\right\} \quad \forall\, \delta \in (0;1]$$

Char. f. $\eta(\cdot)$ of $P^*(A)$:

$$\eta(x) = \sup_{\delta \in [0;1]} \delta \cdot I_{[\underline{P}_{\delta}(A)\,;\, \overline{P}_{\delta}(A)]}(x) \qquad \forall\, x \in \mathbb{R}$$

The following holds : $P^*(\emptyset) = [0;0] = 0$

$P^*(\Theta) = [1;1] = 1$

well motivated by fuzzy frequencies

For fuzzy probability distributions P^* :

If A and B are disjoint the following must hold

$$\left.\begin{aligned} \overline{P}_\delta(A \cup B) &\leq \overline{P}_\delta(A) + \overline{P}_\delta(B) \\ \underline{P}_\delta(A \cup B) &\geq \underline{P}_\delta(A) + \underline{P}_\delta(B) \end{aligned}\right\} \quad \forall \delta \in (0;1]$$

this is also justified by the analog for fuzzy frequencies (based on fuzzy data)

Remark: Classical probability distributions are special cases

LIKELIHOOD FOR FUZZY DATA

$\underline{x}^*$ combined fuzzy sample with v.c.f. $\xi(\cdot)$

$\ell^*(\theta; \underline{x}^*)$ fuzzy value of the likelihood $\ell(\theta; \underline{x})$ with c.f. $\eta_\theta(\cdot)$ defined by

$$\eta_\theta(y) = \begin{cases} \sup\{\xi(\underline{x}) : \ell(\theta; \underline{x}) = y\} & \text{if } \ell^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if } \ell^{-1}(\{y\}) = \emptyset \end{cases} \quad \forall y \in \mathbb{R}$$

Remark: For precise data $\underline{x}$ the indicator function of $\ell(\theta; \underline{x})$ is obtained

GENERALIZED BAYES' THEOREM

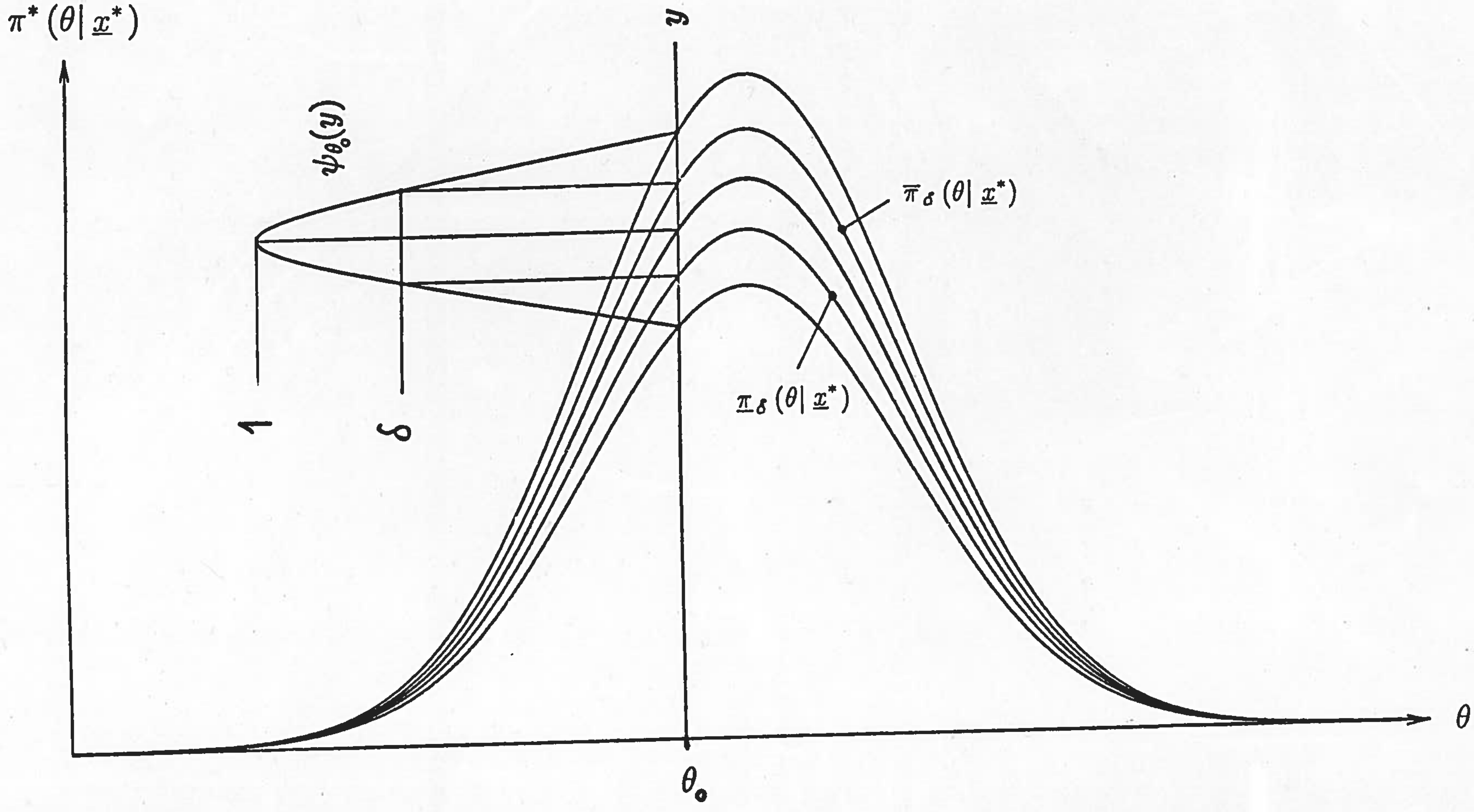
δ-level curves of the fuzzy a-posteriori density

$$\overline{\pi}_\delta(\theta \mid \underline{x}^*) = \frac{\overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta; \underline{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta; \underline{x}^*) + \overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta; \underline{x}^*)\right] d\theta}$$

$$\underline{\pi}_\delta(\theta \mid \underline{x}^*) = \frac{\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta; \underline{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta; \underline{x}^*) + \overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta; \underline{x}^*)\right] d\theta}$$

$$\forall\, \theta \in \Theta$$

Figure *Fuzzy a-posteriori density*



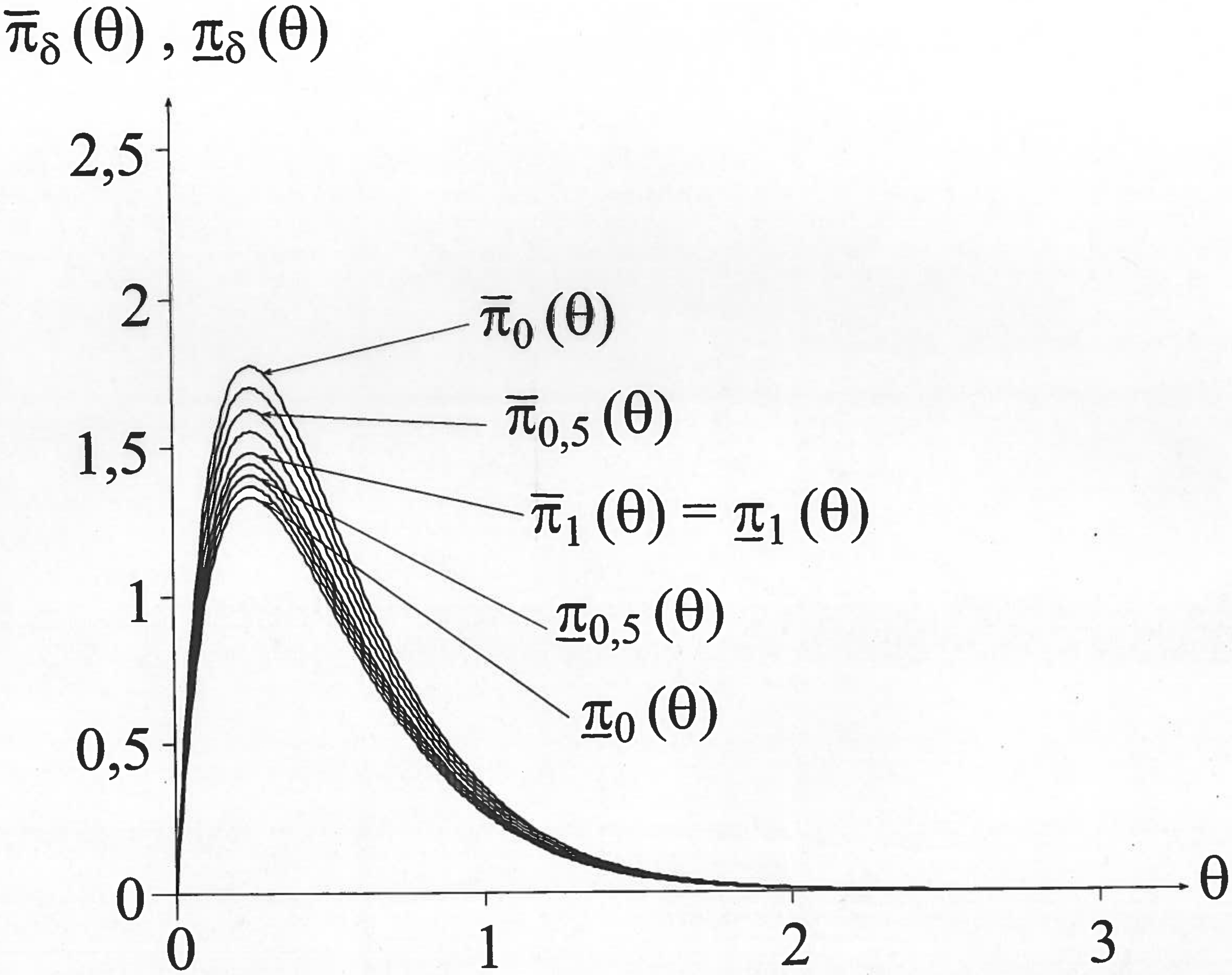
EXAMPLE $X \sim Ex_\theta$, $\theta \in \Theta = (0, \infty)$

$$f(x|\theta) = \frac{1}{\theta} \exp\left\{-\frac{x}{\theta}\right\} \cdot I_{(0,\infty)}(x)$$

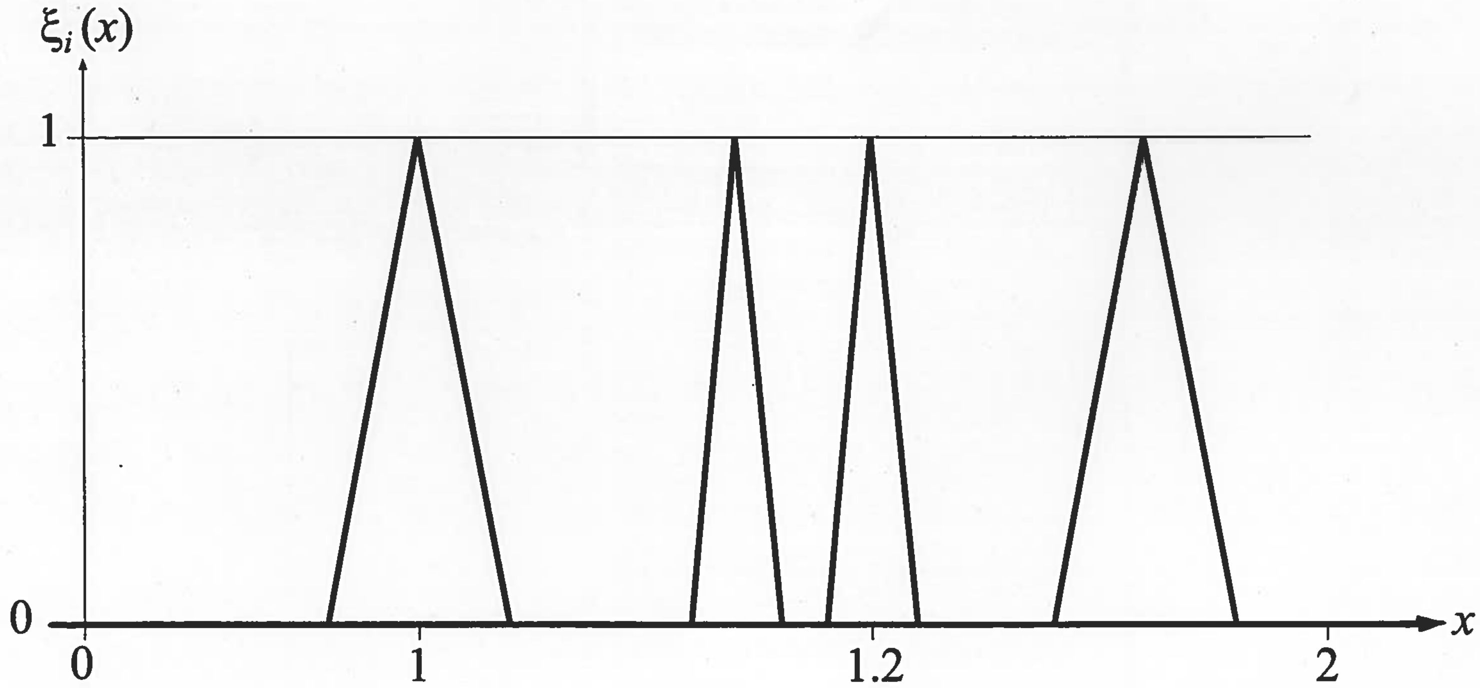
Fuzzy a-priori distribution

$\pi^*(\cdot)$ fuzzy gamma density

$\overline{\pi}_\delta(\cdot)$ upper
$\underline{\pi}_\delta(\cdot)$ lower
} δ-level curves



$\overline{\pi}_\delta(\theta)$, $\underline{\pi}_\delta(\theta)$
2,5
2
1,5
1
0,5
0
$\overline{\pi}_0(\theta)$
$\overline{\pi}_{0,5}(\theta)$
$\overline{\pi}_1(\theta) = \underline{\pi}_1(\theta)$
$\underline{\pi}_{0,5}(\theta)$
$\underline{\pi}_0(\theta)$
0
1
2
3
θ



$\xi_i(x)$
1
0
0
1
1.2
2
x

COMBINED FUZZY SAMPLE

$$\underline{x}^* = (x_1, x_2, x_3, x_4)^*$$

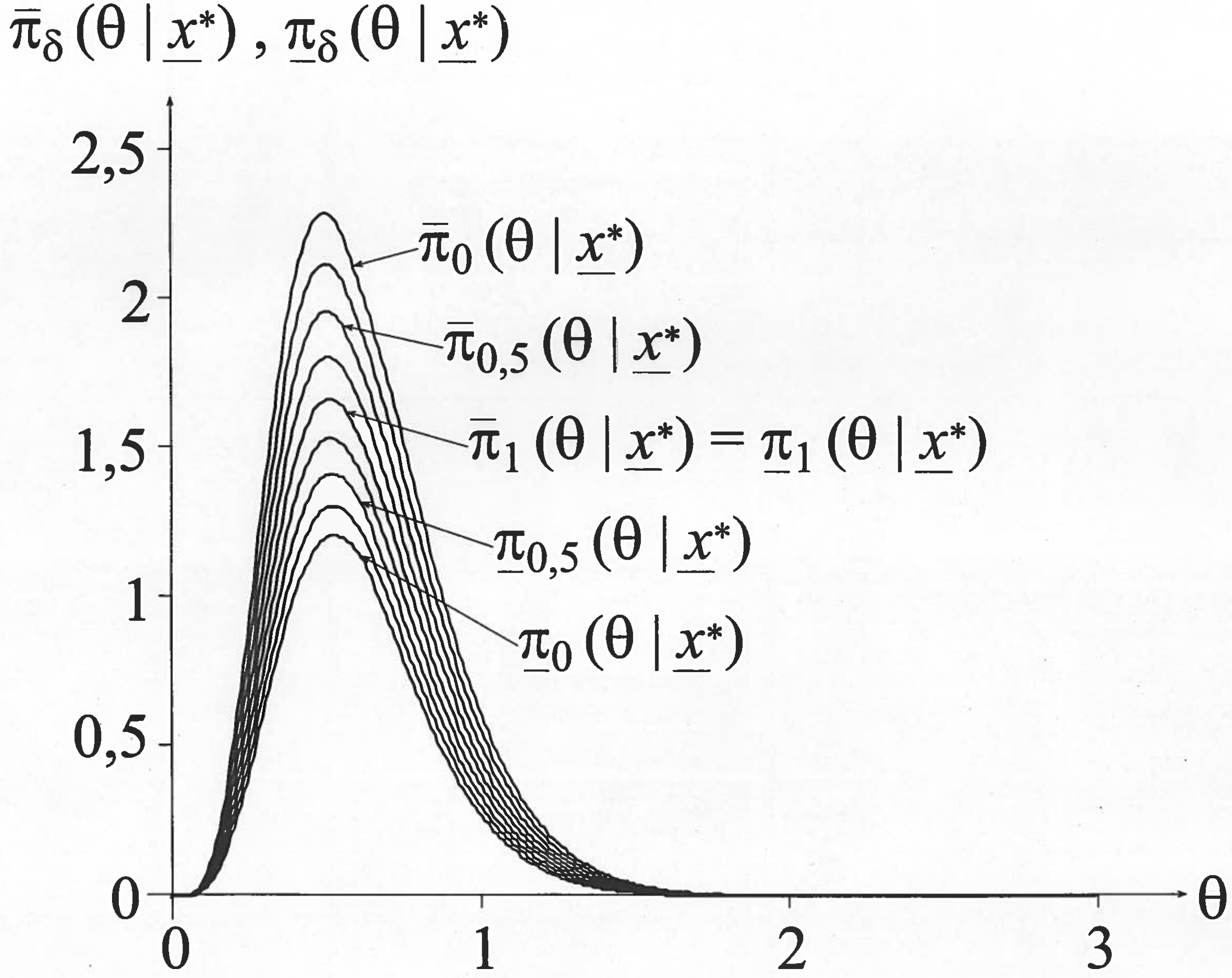
vector char. function $\xi(\cdot, \cdot, \cdot, \cdot)$

$$\xi(x_1, x_2, x_3, x_4) = \min\{\xi_1(x_1), \xi_2(x_2), \xi_3(x_3), \xi_4(x_4)\}$$

$\overline{\pi}_\delta(\cdot \mid \underline{x}^*)$

$\underline{\pi}_\delta(\cdot \mid \underline{x}^*)$

by gen. Bayes' theorem



$\bar{\pi}_\delta(\theta \mid \underline{x}^*)$, $\underline{\pi}_\delta(\theta \mid \underline{x}^*)$
2,5
2
1,5
1
0,5
0
$\bar{\pi}_0(\theta \mid \underline{x}^*)$
$\bar{\pi}_{0,5}(\theta \mid \underline{x}^*)$
$\bar{\pi}_1(\theta \mid \underline{x}^*) = \underline{\pi}_1(\theta \mid \underline{x}^*)$
$\underline{\pi}_{0,5}(\theta \mid \underline{x}^*)$
$\underline{\pi}_0(\theta \mid \underline{x}^*)$
0
1
2
3
θ

HPD - Regions

$\pi(\cdot|D)$ a-posteriori Density

$1-\alpha$ Confidence level

$\Theta_{1-\alpha} \subseteq \Theta$ obeying:

1) $\int_{\Theta_{1-\alpha}} \pi(\theta|D)\,d\theta = 1-\alpha$

2) $\pi(\theta|D)$ max. on $\Theta_{1-\alpha}$

GENERALIZED HPD-Regions

$\pi^*(\cdot|D^*)$ Fuzzy a-posteriori Density

$$\mathcal{D}_\delta := \{ g : g \text{ density with } \underline{\pi}_\delta(\theta) \le g(\theta) \le \overline{\pi}_\delta(\theta) \ \forall \theta \in \Theta \}$$

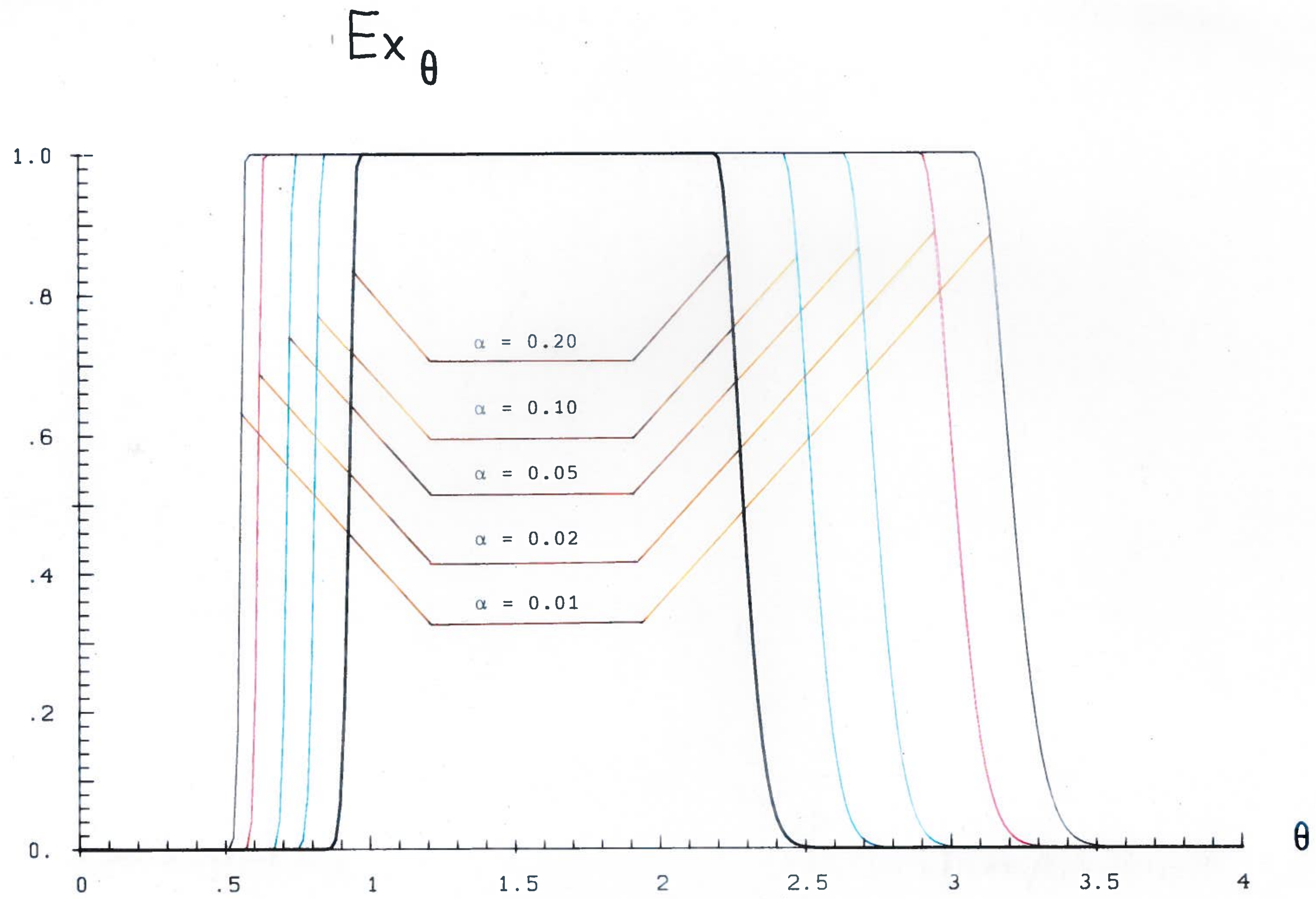
${}^\delta HPD_{1-\alpha}(g)$ HPD-Region based on g

$$A_\delta := \bigcup_{g \in \mathcal{D}_\delta} {}^\delta HPD_{1-\alpha}(g) \qquad \forall \delta \in (0;1]$$

$\Rightarrow (A_\delta ; \delta \in (0;1])$ nested family of subsets of Θ

Construction Lemma for Membership Functions:

$$\varphi(\theta) := \sup\{ \delta \cdot \mathbb{1}_{A_\delta}(\theta) : \delta \in [0;1] \} \quad \forall \theta \in \Theta$$



Ex θ
1.0
.8
.6
.4
.2
0.
α = 0.20
α = 0.10
α = 0.05
α = 0.02
α = 0.01
0
.5
1
1.5
2
2.5
3
3.5
4
θ

PREDICTIVE DENSITIES

$X \sim f(\cdot|\theta)$, $\theta \in \Theta$ Stochastic Model

$\pi(\cdot)$ a-priori density

$(x_1, \cdots, x_n) = D$ data

$\Rightarrow$ $\pi(\cdot|D)$ a-posteriori density

$p(\cdot|D)$ predictive density

$$p(x|D) = \int_{\Theta} f(x|\theta) \cdot \pi(\theta|D)\, d\theta \quad \forall x \in M_X$$

FUZZY PREDICTIVE DENSITY

$$p^*(\cdot \mid D^*)$$

$$p^*(x \mid D^*) = \int_{\Theta} f(x \mid \theta) \odot \pi^*(\theta \mid D^*)\, d\theta \qquad \forall x \in M_X$$

$$\mathcal{D}_\delta := \{ g(\cdot) \text{ density on } \Theta : \underline{\pi}_\delta(\theta) \leq g(\theta) \leq \overline{\pi}_\delta(\theta) \ \forall \theta \in \Theta \}$$

$$a_\delta := \inf \left\{ \int_{\Theta} f(x \mid \theta)\, g(\theta)\, d\theta : g(\cdot) \in \mathcal{D}_\delta \right\}$$

$$\forall \delta \in (0; 1]$$

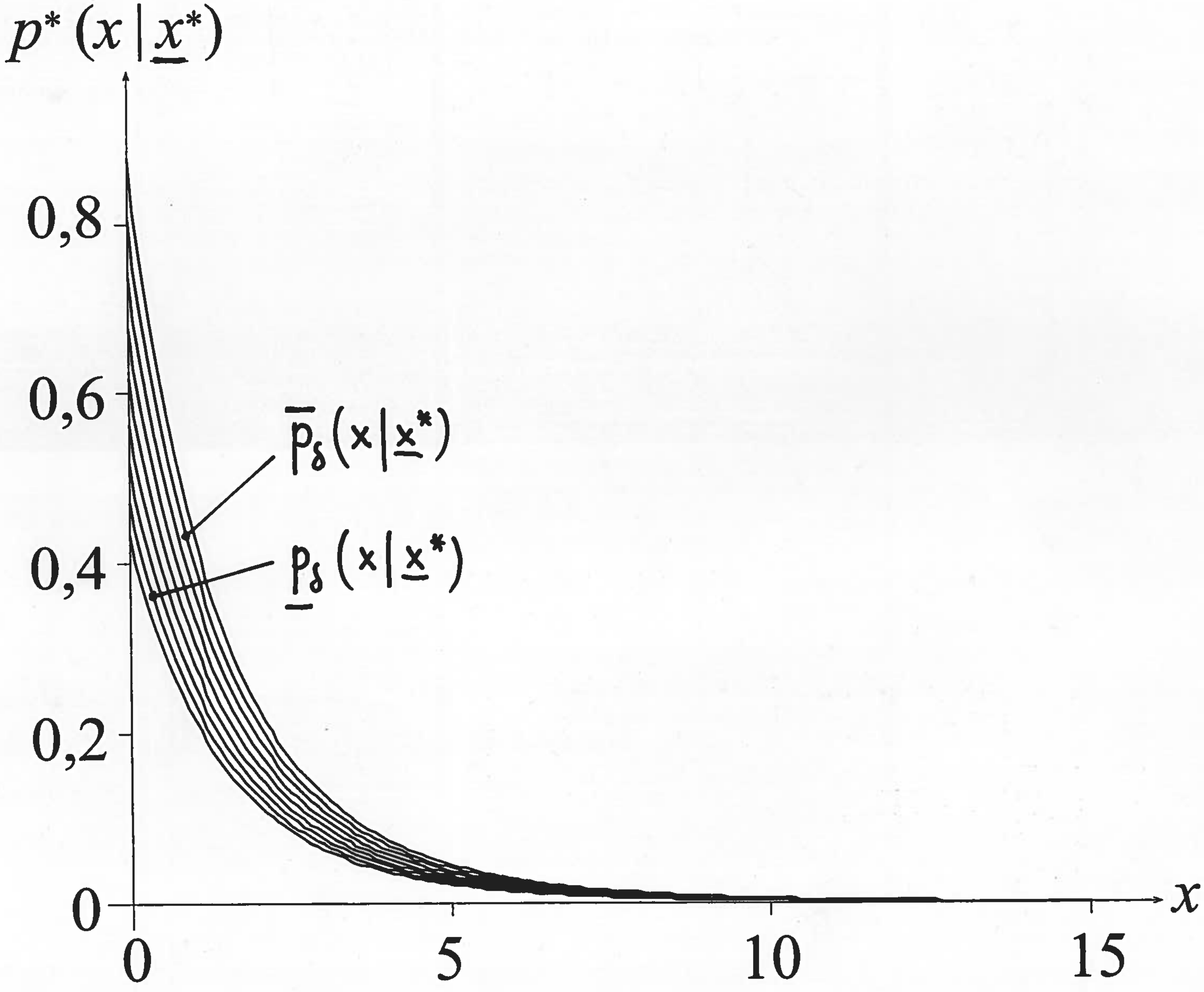
$$b_\delta := \sup \left\{ \int_{\Theta} f(x \mid \theta)\, g(\theta)\, d\theta : g(\cdot) \in \mathcal{D}_\delta \right\}$$

The nested family $\left([a_\delta ; b_\delta] ; \delta \in (0;1]\right)$ determins a characterizing function $\psi_x(\cdot)$ by

$$\psi_x(y) := \sup\left\{ \delta \cdot \mathbb{1}_{[a_\delta ; b_\delta]}(y) : \delta \in [0;1] \right\} \quad \forall y \in \mathbb{R}$$

$$p^*(x|D^*) \triangleq \psi_x(\cdot) \quad \forall x \in M_X$$

For variable x this is a fuzzy probability density called fuzzy predictive density with δ-level functions $\underline{p}_\delta(\cdot|D^*)$ and $\overline{p}_\delta(\cdot|D^*)$.


$p^*(x\,|\,\underline{x}^*)$
0,8
0,6
0,4
0,2
0
$\overline{p}_\delta(x\,|\,\underline{x}^*)$
$\underline{p}_\delta(x\,|\,\underline{x}^*)$
0
5
10
15
x

GENERAL FUZZY PROBABILITIES

$(A_i \,;\, i \in I)$ Event System

$P^*: (A_i \,;\, i \in I) \longrightarrow \mathcal{F}_I(\mathbb{R})$ obeying (1) to (4):

(1) $P^*(A_i)$ is a fuzzy interval with char. F. $\xi_i(\cdot)$ such as $\{x \in \mathbb{R} : \xi(x) > 0\} \subseteq [0; 1]$

(2) For all finite families of pairwise exclusive events $A_1, \dots, A_n$ the following holds $\forall \delta \in (0;1]$:

Let $C_\delta[P^*(A_i)] = [a_{i,\delta} \,;\, b_{i,\delta}] \quad \forall i = 1(1)n$
and $C_\delta[P^*(\bigvee_{i=1}^{n} A_i)] = [c_\delta \,;\, d_\delta]$, then

$$c_\delta \geq \sum_{i=1}^{n} a_{i,\delta} \quad \text{and} \quad d_\delta \leq \sum_{i=1}^{n} b_{i,\delta}$$

(3) For any complete system of events $A_{j_1}, \dots, A_{j_k}$ it follows

$$P^*\left(\bigvee_{i=1}^{k} A_{j_i}\right) \text{ has c.F. } \mathbb{1}_{\{1\}}(\cdot)$$

(4) The probability of the impossible event has c.F. $\mathbb{1}_{\{0\}}(\cdot)$

Remark: Condition (2) is the appropriate generalisation of finite additivity, called <u>lower superadditivity</u> and <u>upper subadditivity</u>

CONCLUSIONS

- Fuzziness can be described quantitatively
- Statistics based on fuzzy information is possible: Two different uncertainties
- Kolmogorov's probability concept has to be generalized
- Hybrid approach: Fuzzy and Stochastics

SOFTWARE

- Some Programs

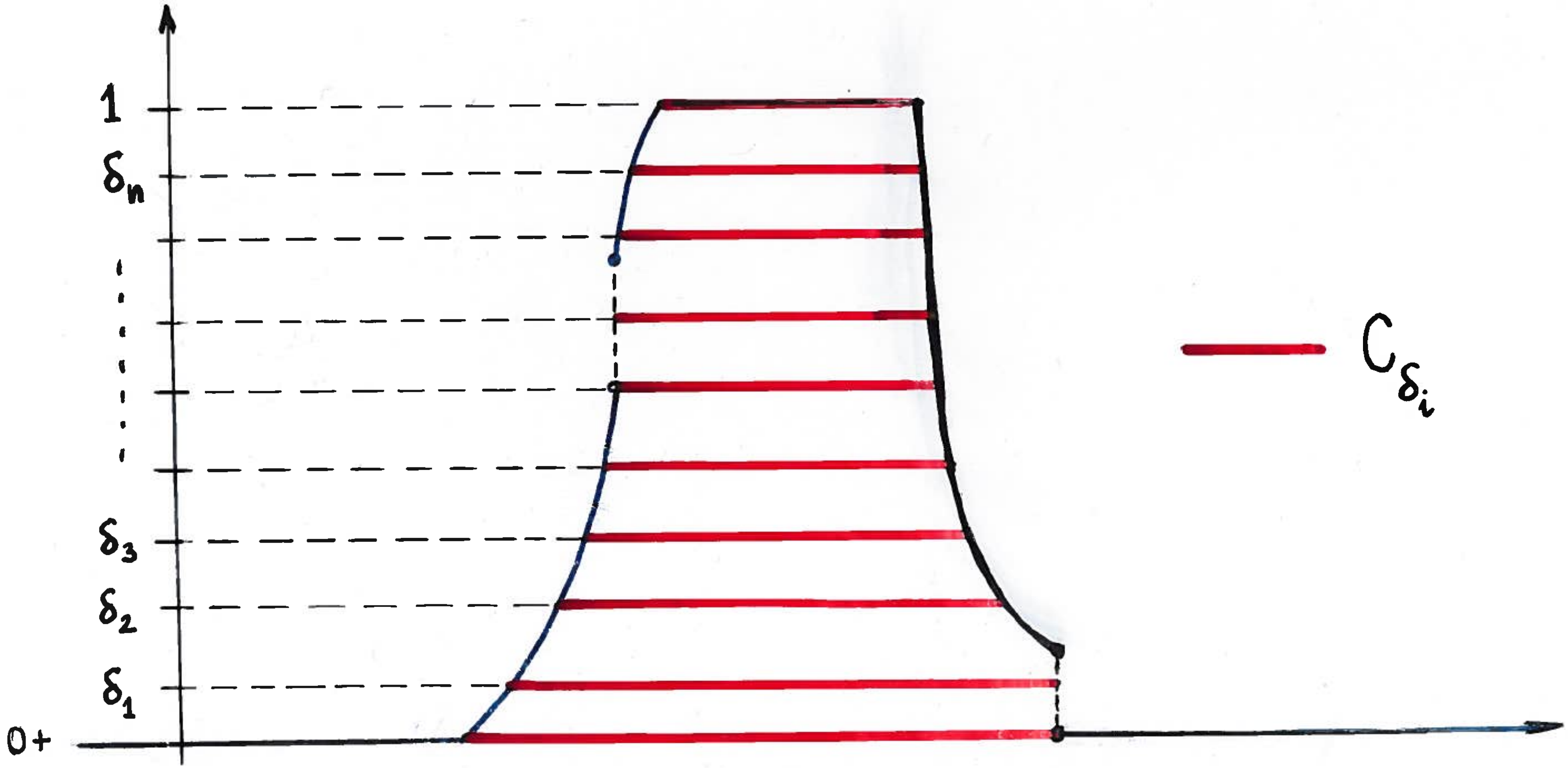
  C++, R

- Under Development:

  SAFD, ECSC at Mieres

Fuzzy Data in Databases

Storing δ- Cuts



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