On the axiomatisation of logics for approximate reasoning

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Crisp reasoning

Model:
W, a set of worlds,
together with a Boolean algebra $\mathcal{B}$ of subsets of W.

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Meaning:
W is the set of distinguished situations;
each $A \in \mathcal{B}$ represents a property.

Classical propositional logic (CPL)

Language:

Propositional formulas are built up from $\varphi_0, \varphi_1, \ldots, \top, \bot$ by means of $\wedge, \vee, \neg$.

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Interpretation:

An evaluation v maps propositional formulas to $\mathcal{B}$, interpreting $\wedge, \vee, \neg$ by $\cap, \cup, \complement$.

The above statement is satisfied if

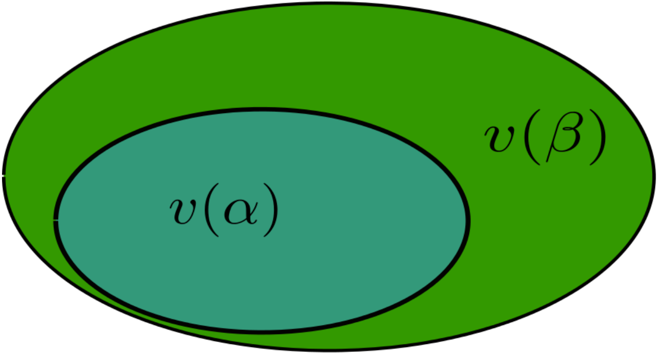
$$v(\alpha_1) \cap \ldots \cap v(\alpha_k) \subseteq v(\beta).$$

Illustration of CPL

$$\alpha \to \beta$$

is satisfied in CPL by an evaluation v if

$$v(\alpha) \subseteq v(\beta).$$



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W is endowed with a similarity relation $s\colon W \times W \to [0,1]$:

(S1) $s(u,u) = 1$ (*reflexivity*),

(S2) $s(u,v) = 1$ implies $u = v$ (*separability*),

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Meaning:
Two worlds $v, w \in W$ resemble each other to the degree $s(v,w)$.

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(Ll. Godo, F. Esteva, D. Dubois, H. Prade, R. Rodgríguez, ...)

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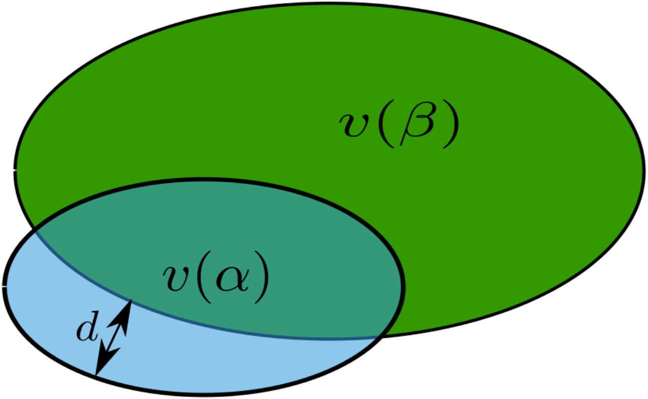

Rules for LAE

$$\frac{\Gamma, \alpha, \beta \xrightarrow{d} \gamma}{\Gamma, \alpha \wedge \beta \xrightarrow{d} \gamma} \qquad \frac{\Gamma \xrightarrow{d} \beta}{\Gamma, \alpha \xrightarrow{d} \beta} \qquad \frac{\Gamma, \alpha \xrightarrow{d} \gamma \quad \Gamma, \beta \xrightarrow{d} \gamma}{\Gamma, \alpha \vee \beta \xrightarrow{d} \gamma} \qquad \frac{\Gamma \xrightarrow{d} \alpha}{\Gamma \xrightarrow{d} \alpha \vee \beta}$$

$$\frac{\Gamma \xrightarrow{c} \alpha \quad \alpha \xrightarrow{d} \gamma}{\Gamma \xrightarrow{c \odot d} \gamma}$$

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Open problem

How can LAE be axiomatised?
The above rules are sound;
are these few rules actually already complete?

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of 1-similar propositional formulas.

We then define a similarity between two propositions by

$$d(\alpha, \beta) \;=\; \sup\{t \in [0,1] : \mathcal{T} \vdash \alpha \xrightarrow{t} \beta\}.$$

The technical difficulties

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From $\varphi \xrightarrow{d} \psi$,
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From $\varphi \stackrel{t}{\to} \alpha \vee \beta$,
we cannot conclude
that there are φ_1, φ_2 such that
$\varphi_1 \stackrel{t}{\to} \alpha, \quad \varphi_2 \stackrel{t}{\to} \beta, \quad \varphi \stackrel{1}{\to} \varphi_1 \vee \varphi_2$.

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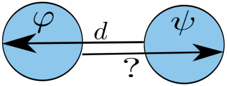
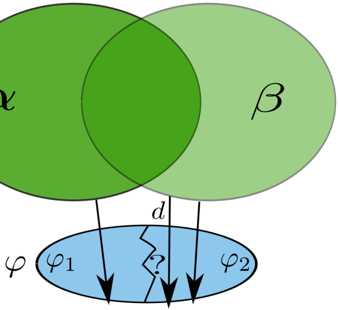
$\varphi \xrightarrow{d} \psi$



$\psi \xrightarrow{d?} \varphi$

$\varphi \xrightarrow{d} \alpha \vee \beta$



$\varphi_1 \xrightarrow{d?} \alpha$ $\varphi_2 \xrightarrow{d?} \beta$

First approach: finiteness
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We assume that there is a fixed finite number n of variables.

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We include to our axioms:

$$(\chi \stackrel{c}{\to} \chi') \;\to\; (\chi' \stackrel{c}{\to} \chi) \;\text{ if } \chi \text{ and } \chi' \text{ are m.e.c.'s}$$

$$(\chi \stackrel{c}{\to} \varphi \vee \psi) \;\leftrightarrow\; (\chi \stackrel{c}{\to} \varphi) \vee (\chi \stackrel{c}{\to} \psi) \;\text{ if } \chi \text{ is a m.e.c.}$$

A m.e.c. is of the form $(\neg)\varphi_1 \wedge \ldots \wedge (\neg)\varphi_n$.

Progress and drawbacks

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- LAEf solves both problems: "symmetry" and "conjunction";
- LAEf depends on a fixed finite number of variables.

Second approach: a further connective

We extend the language by a new connective:

$$\alpha \nearrow \beta$$

is interpreted by

$$\{w \in W : s(w, \alpha) \geq s(w, \beta)\},$$

i.e. those worlds that are more similar to α than to β.

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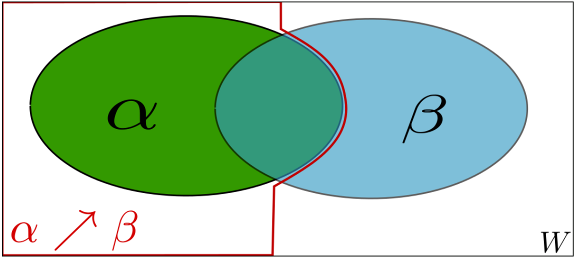

The logic LAEC

Model:

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Language: Propositional formulas from $\varphi_0, \varphi_1, \ldots$ by $\wedge, \vee, \neg$.
Conditional formulas of the form $\alpha_1, \ldots, \alpha_k \xrightarrow{d} \beta$.

Interpretation: in $\mathcal{B}$, interpreting $\wedge, \vee, \neg$ by $\cap, \cup, \complement$;
$v(\alpha \nearrow \beta) = \{w \in W \colon s(w, v(\alpha)) \geq s(w, v(\beta))\}$.

Satisfaction if $v(\alpha_1) \cap \ldots \cap v(\alpha_k) \subseteq U_d(v(\beta))$.

Proof system for LAEC

$$\frac{\Gamma, \alpha, \beta \xrightarrow{d} \gamma}{\Gamma, \alpha \wedge \beta \xrightarrow{d} \gamma} \qquad \frac{\Gamma \xrightarrow{d} \beta}{\Gamma, \alpha \xrightarrow{d} \beta} \qquad \frac{\Gamma, \alpha \xrightarrow{d} \gamma \quad \Gamma, \beta \xrightarrow{d} \gamma}{\Gamma, \alpha \vee \beta \xrightarrow{d} \gamma} \qquad \frac{\Gamma \xrightarrow{d} \alpha}{\Gamma \xrightarrow{d} \alpha \vee \beta}$$

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$$\alpha \xrightarrow{1} \alpha \nearrow \beta \qquad \alpha \nearrow \beta,\ \beta \nearrow \gamma \xrightarrow{1} \alpha \nearrow \gamma$$

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Progress and drawbacks

Call a theory $\mathcal{T}$ *consistent* if $\mathcal{T} \nvdash \top \xrightarrow{1} \bot$.

Theorem (Th.V.)

Let $\mathcal{T}$ be a consistent finite theory of LAEC;
let $\alpha \xrightarrow{d} \beta$ be a conditional formula.

Then $\mathcal{T}$ proves $\alpha \xrightarrow{d} \beta$

if and only if

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- LAEC depends on an additional connective.

Third approach: an embedding theorem

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Question

Can we embed in some sense a quasi-similarity space
into a similarity space?

Hausdorff quasi-similarity in similarity spaces

Definition

Let (Y, d) be a similarity space.
For $A, B \subseteq Y$, we call

$$q_d(A, B) \;=\; \sup_{a \in A} \inf_{b \in B} d(a, b)$$

the Hausdorff quasi-similarity of B from A.

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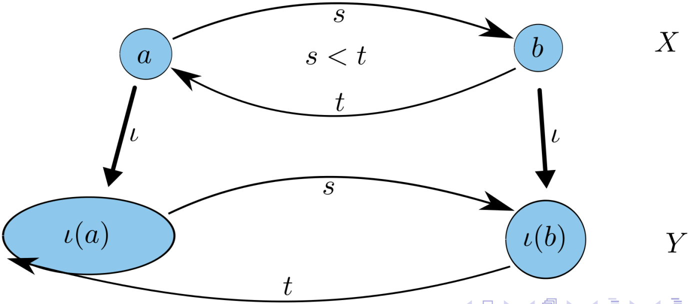

Embedding theorem

Theorem (Th.V.)

Let (X, q) be a quasi-similarity space.
Then there is a similarity space (Y, d) and a mapping

$$\iota\colon X \to \mathcal{P}(Y),$$

such that distinct points map to disjoint subsets and

$$q(a, b) \;=\; q_d(\iota(a), \iota(b))$$

for any $a, b \in X$.

Idea of the proof

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Let $Y = X \times {}^{U}2$, where

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We let d be the largest similarity on Y such that:

$$\begin{aligned} d(\ (a,(i_1,\ldots)),(b,(i_1,\ldots))\ ) &= q(a,b) \vee q(b,a); \\ d(\ (a,(\ldots,0,\ldots)),(b,(\ldots,1,\ldots))\ ) &= q(a,b) \text{ if } (a,b) \in U \end{aligned}$$

$\uparrow$ position (a,b) $\uparrow$

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  Two problems need to be overcome.

- A modification: we restrict the number of variables.
- An extension: we add a comparative connective.
- An embedding: we enlarge quasi-similarity spaces to similarity spaces.

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- Concerning LAEC:
  - The completeness proof should be simplified.
  - The embedding theorem should help to get rid of the "asymmetry".
- Ultimate aim:
  - an axiomatisation of the unmodified LAE.