Conjugacy Relations via Group Action on the set of Implications

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Fuzzy Implication

Classical Implication: Truth Table

p	q	$p \rightarrow q$
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- A generalization of classical implication.

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Definition [Kitainik, 1993]

A function $I : [0,1]^2 \to [0,1]$ is called a **fuzzy implication(FI)** if

- I is decreasing in the first variable and increasing in the second variable.
- $I(0,0) = I(1,1) = I(0,1) = 1$ and $I(1,0) = 0$.

Fuzzy Implications: Examples

Basic Fuzzy Implications

Name	**Formula**
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1 - x + y)$
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$
Kleene-Dienes	$I_{\mathbf{KD}}(x,y) = \max(1 - x, y)$
Goguen	$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Yager	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$

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Largest FI	$\mathbf{I_1(x,y)} = \begin{cases} 1, & \text{if } x < 1 \text{ or } \; y > 0 \\ 0, & \text{if } x = 1 \text{ and } \; y = 0 \end{cases}$
Most strict	$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$

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- Φ - the set of all increasing bijections on [0,1].

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Fuzzy implications from fuzzy implications will lead to

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- Algebraic structures on $\mathbb{I}$ (often).

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- Richest algebraic structure available on $\mathbb{I}$ is a semigroup
- May obtain hitherto unknown Characterisations and Representations

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Theorem

$(\mathbb{I}, \Delta)$ is a semigroup.

We do not use any *external* operator.

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$(\mathbb{I}, \circledast)$ is NOT a group.

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- What are the invertible implications?

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$$I(x, J(x,y)) = y = J(x, I(x,y)) \ . \qquad (1)$$

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Theorem [Vemuri & Jayaram , FSS 2013]

$I \in \mathbb{I}$ is invertible w.r.to $\circledast$ if and only if

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \ , \\ \varphi(y), & \text{if } x > 0 \ , \end{cases}$$

where the function $\varphi \colon [0,1] \to [0,1]$ is an increasing bijection.

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$\mathbb{S}$ - the set of all invertible elements of $(\mathbb{I}, \circledast)$.

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$$K \circledast (I \Delta K^{-1}) = (K \circledast I) \Delta K^{-1}, \qquad I \in \mathbb{I}, K \in \mathbb{S}$$

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$$g_1 \sim g_2 \Longleftrightarrow g_1 = s \bullet g_2 \tag{2}$$

for some $s \in S$.

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So in our context...

- Can we get a partition on $\mathbb{I}$?
- Can we define an equivalence relation on $\mathbb{I}$?

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So in our context...

- Can we get a partition on $\mathbb{I}$?
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Yes we can !!!

Group action on the set $\mathbb{I}$-1

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Note

$[I]$ = transformations proposed by Jayaram & Mesiar. FSS(2009).

Group action on the set $\mathbb{I}$-2

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Conjugacy classes of FIs

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Group Action on f-implications

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1. An $I \in \mathbb{I}_{\mathbb{F},\infty}$ if and only if
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$\varphi \in \Phi$.
2. An $I \in \mathbb{I}_{\mathbb{F},1}$ if and only if $I(x,y) = \varphi\left(1 - x + x\varphi^{-1}(y)\right)$, for some $\varphi \in \Phi$.

g-implications

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Let $g\colon [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I\colon [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \qquad x, y \in [0,1]\ ,$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a g-implication, where the function $g^{(-1)}$ is the pseudo inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0,\ g(1)]\ , \\ 1, & \text{if } x \in [g(1),\ \infty]\ , \end{cases}$$

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$\varphi \in \Phi$.
2. An $I \in \mathbb{I}_{\mathbb{G},1}$ if and only if
$$I(x,y) = \begin{cases} 1, & \text{if } \varphi(x) \leq y, \\ \varphi\left(\frac{\varphi^{-1}(y)}{\varphi(x)}\right), & \text{if } \varphi(x) > y, \end{cases} \text{ for some } \varphi \in \Phi.$$

Summary

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- give algebraic connotation of conjugacy classes, transformation.

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- give algebraic connotation of conjugacy classes, transformation.
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Questions???