IT4I

Fuzzy Relational Compositions Based on Generalized Quantifiers

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Outline

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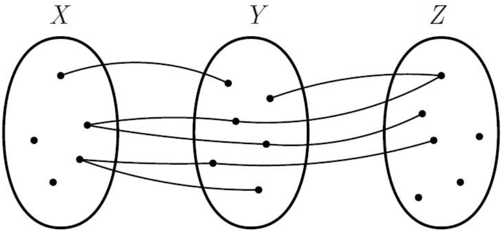
Compositions? Why? What for?

Motivation:



We know $R \subseteq X \times Y$, and $S \subseteq Y \times Z$.
But we do not know (and we would like to know) the relationship between elements from X and Z.

Compositions? Why? What for?

Formally:

$$\begin{array}{cccccccc} R & \subseteq & X & \times & Y & & \\ S & \subseteq & & & Y & \times & Z \\ \hline R@S & \subseteq & X & & & \times & Z. \end{array}$$

Composed relation $R@S$ is already a binary relation between elements from X and Z.

Compositions? Why? What for?

We follow the work of W. Bandler and L.J. Kohout from 70's including the medical diagnosis example but feel free to abstract from the example anytime during the talk.

- X – set of patients
- Y – set of symptoms
- Z – set of diseases

$(x, y) \in R$ – patient x has symptom y

$(y, z) \in S$ – symptom y belongs to disease z

$(x, z) \in R@S$ – patient x has some relationship (suspicion, diagnosis) to disease z (result of the composition)

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1. Basic composition $\circ$

Relation $R \circ S \subseteq X \times Z$ is given as follows

$$R \circ S = \{(x, z) \in X \times Z \mid \exists\, y \in Y : (x, y) \in R \;\&\; (y, z) \in S\}$$

which may be expressed with help of its characteristic function:

$$\chi_{R \circ S}(x, z) = \bigvee_{y \in Y} \left(\chi_R(x, y) \wedge \chi_S(x, y)\right)$$

The meaning of $(x, z) \in R \circ S$ (or $\chi_{R \circ S}(x, z) = 1$)

Patient x has at least one symptom of the disease z and therefore, there exists a suspicion of having this disease.

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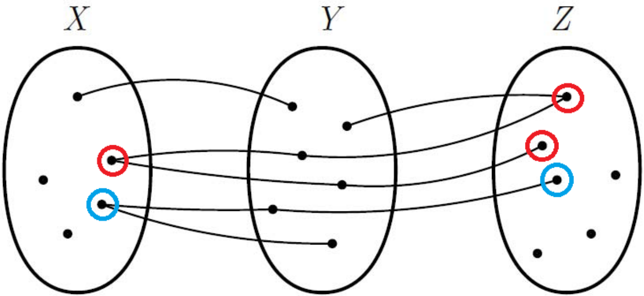
1. Basic composition ∘

Illustration of the meaning of $\chi_{R\circ S}(x,z)=1$



2. Bandler-Kohout subproduct $\lhd$

Relation $R \lhd S \subseteq X \times Z$ is given as follows

$$R \lhd S = \{(x, z) \in X \times Z \mid \forall\, y \in Y : (x, y) \in R \;\Rightarrow\; (y, z) \in S\}$$

which may be expressed with help of its characteristic function:

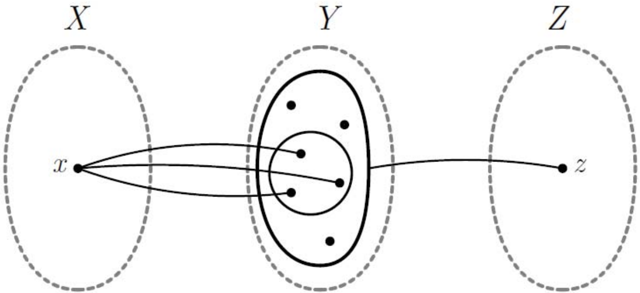
$$\chi_{R \lhd S}(x, z) = \bigwedge_{y \in Y} (\chi_R(x, y) \Rightarrow \chi_S(x, y))$$

The meaning of $\chi_{R \lhd S}(x, z) = 1$

All symptoms of patient x belong to the disease z which strengthens the suspicion.

2. Bandler-Kohout subproduct $\lhd$

Illustration of the meaning of $\chi_{R \lhd S}(x, z) = 1$



3. Bandler-Kohout superproduct $\rhd$

Relation $R \rhd S \subseteq X \times Z$ is given as follows

$$R \rhd S = \{(x, z) \in X \times Z \mid \forall\, y \in Y : (x, y) \in R \ \Leftarrow\ (y, z) \in S\}$$

which may be expressed with help of its characteristic function:

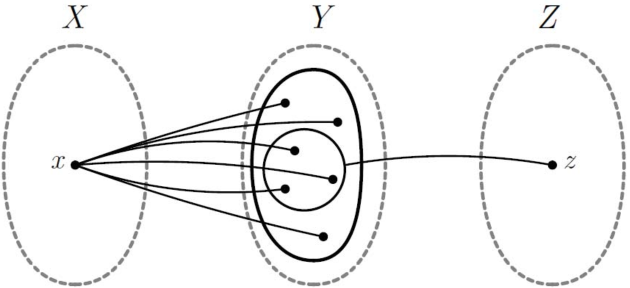
$$\chi_{R \rhd S}(x, z) = \bigwedge_{y \in Y} (\chi_R(x, y) \Leftarrow \chi_S(x, y))$$

The meaning of $\chi_{R \rhd S}(x, z) = 1$

Patient x has all symptoms belonging to the disease z which strengthens the suspicion.

3. Bandler-Kohout superproduct $\triangleright$

Illustration of the meaning of $\chi_{R \triangleright S}(x, z) = 1$



4. Bandler-Kohout square product $\square$

Relation $R \square S \subseteq X \times Z$ is given as follows

$$R \square S = \{(x,z) \in X \times Z \mid \forall\, y \in Y : (x,y) \in R \;\Leftrightarrow\; (y,z) \in S\}$$

which may be expressed with help of its characteristic function:

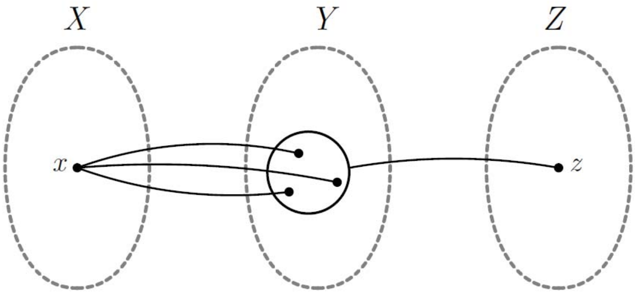
$$\chi_{R \square S}(x,z) = \bigwedge_{y \in Y} (\chi_R(x,y) \Leftrightarrow \chi_S(x,y))$$

The meaning of $\chi_{R \square S}(x,z) = 1$

Patient x has all symptoms belonging to the disease z and all patient's symptoms belong to disease z strengthens the suspicion – prototypical example from literature.

4. Bandler-Kohout square product $\square$

Illustration of the meaning of $\chi_{R\square S}(x,z) = 1$



Compositions of classical relations

$$R \subseteq X \times Y,\ S \subseteq Y \times Z,\ R@S \subseteq X \times Z$$

$$\chi_{(R \circ S)}(x,z) = \bigvee_{y \in Y} \left(\chi_R(x,y) \wedge \chi_S(y,z)\right),$$

$$\chi_{(R \lhd S)}(x,z) = \bigwedge_{y \in Y} \left(\chi_R(x,y) \Rightarrow \chi_S(y,z)\right),$$

$$\chi_{(R \rhd S)}(x,z) = \bigwedge_{y \in Y} \left(\chi_R(x,y) \Leftarrow \chi_S(y,z)\right),$$

$$\chi_{(R \square S)}(x,z) = \bigwedge_{y \in Y} \left(\chi_R(x,y) \Leftrightarrow \chi_S(y,z)\right).$$

Compositions of fuzzy relations

$$R \underset{\sim}{\subseteq} X \times Y,\ S \underset{\sim}{\subseteq} Y \times Z,\ R@S \underset{\sim}{\subseteq} X \times Z$$

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Compositions of fuzzy relations

$$R \underset{\sim}{\subseteq} X \times Y,\ S \underset{\sim}{\subseteq} Y \times Z,\ R@S \underset{\sim}{\subseteq} X \times Z$$

$$(R \circ S)(x,z) = \bigvee_{y \in Y} \left(R(x,y) \wedge S(y,z)\right),$$

$$(R \lhd S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \Rightarrow S(y,z)\right),$$

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Compositions of fuzzy relations

$$R \underset{\sim}{\subset} X \times Y,\ S \underset{\sim}{\subset} Y \times Z,\ R@S \underset{\sim}{\subset} X \times Z$$

$$(R \circ_* S)(x,z) = \bigvee_{y \in Y} \left(R(x,y) * S(y,z)\right),$$

$$(R \lhd_* S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \rightarrow_* S(y,z)\right),$$

$$(R \rhd_* S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \leftarrow_* S(y,z)\right),$$

$$(R \square_* S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \leftrightarrow_* S(y,z)\right).$$

Compositions of fuzzy relations

If we fix the underlying residual structure

$$\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1 \rangle$$

we can omit "*" from the notation and simply write:

$$(R \circ S)(x,z) = \bigvee_{y \in Y} \left(R(x,y) * S(y,z) \right),$$

$$(R \lhd S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \rightarrow S(y,z) \right),$$

$$(R \rhd S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \leftarrow S(y,z) \right),$$

$$(R \square S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \leftrightarrow S(y,z) \right).$$

Further developments

- Deep analysis of properties (W. Bandler & L.J. Kohout, E. Kerre et al.)
- Images of fuzzy sets under fuzzy relations (derived from compositions $\circ, \lhd$) were used as fuzzy inference mechanisms (L.A. Zadeh, W. Pedrycz, B. Jayaram)
- Solvability of fuzzy relational equations (B. De Baets, A. Di Nola, S. Gottwald, B. Jayaram, F. Klawonn, L. Nosková, W. Pedrycz, K. Peeva, I. Perfilieva, E. Sanchez, S. Sessa)
- Original BK compositions were modified by assumption of existence of some connections (B. De Baets, E. Kerre)
- Complete analysis in higher-order fuzzy logic (Fuzzy Class Theory) (L. Běhounek, M. Daňková)

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Quantifiers?

$$(R \circ S)(x,z) = \bigvee_{y \in Y} (R(x,y) * S(y,z)),$$

$$(R \lhd S)(x,z) = \bigwedge_{y \in Y} (R(x,y) \to S(y,z)),$$

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- But there is a big gap between $\exists$ and $\forall$

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- But there is a big gap between $\exists$ and $\forall$

Gap between quantifiers? Example:

$\langle [0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ be the Łukasiewicz MV-algebra

Symptoms:

y_1 - tiredness; y_2 - cough; y_3 - fever; y_4 - blurred vision

Diseases:

z_1 - pulmonary hypertension; z_2 - sleeping sickness;
z_3 - malaria; z_4 - hangover; z_5 - influenza

R	y_1	y_2	y_3	y_4
x_1	0.9	1	0.8	0
x_2	0	0.9	0.8	0.1
x_3	0	0.8	0.9	0
x_4	0	0	1	0.9

S	z_1	z_2	z_3	z_4	z_5
y_1	1	1	0.1	0.9	0
y_2	0.9	0.2	0.9	0	1
y_3	0	1	0	1	1
y_4	1	0	0.7	0.1	0.9

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S	z_1	z_2	z_3	z_4	z_5
y_1	1	1	0.1	0.9	0
y_2	0.9	0.2	0.9	0	1
y_3	0	1	0	1	1
y_4	1	0	0.7	0.1	0.9

$$\begin{aligned}(R \circ S)(x_1, z_1) &= (0.9 \otimes 1) \vee (1 \otimes 0.9) \vee (0.8 \otimes 0) \vee (0 \otimes 1)\\ &= 0.9 \vee 0.9 \vee 0 \vee 0 = \mathbf{0.9}\\ (R \circ S)(x_1, z_4) &= (0.9 \otimes 0.9) \vee (1 \otimes 0) \vee (1 \otimes 0) \vee (0 \otimes 0.1)\\ &= ((0.9 + 0.9 - 1) \vee 0) \vee 0 \vee 0 \vee 0 = \mathbf{0.8}\end{aligned}$$

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$R \circ S$	z_1	z_2	z_3	z_4	z_5
x_1	**0.9**	0.9	0.9	**0.8**	1
x_2	0.8	0.8	0.8	0.8	0.9
x_3	0.7	0.9	0.7	0.9	0.9
x_4	0.9	1	0.6	1	1

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R	y_1	y_2	y_3	y_4
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x_2	0	0.9	0.8	0.1
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$$\begin{aligned}(R \lhd S)(x_1, z_1) &= (0.9 \to 1) \wedge (1 \to 0.9) \wedge (0.8 \to 0) \wedge (0 \to 1)\\ &= 1 \wedge 0.9 \wedge ((1 - 0.8 + 0) \wedge 1) \wedge 1 = \mathbf{0.2}\\ (R \lhd S)(x_1, z_4) &= (0.9 \to 0.9) \wedge (1 \to 0) \wedge (1 \to 0) \wedge (0 \to 0.1)\\ &= 1 \wedge 0 \wedge 0 \wedge 1 = \mathbf{0}\end{aligned}$$

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y_4	1	0	0.7	0.1	0.9

$$\begin{aligned}(R \triangleright S)(x_1, z_1) &= (0.9 \leftarrow 1) \wedge (1 \leftarrow 0.9) \wedge (0.8 \leftarrow 0) \wedge (0 \leftarrow 1)\\ &= 0.9 \wedge 1 \wedge 1 \wedge 0 = \mathbf{0}\\ (R \triangleright S)(x_1, z_4) &= (0.9 \leftarrow 0.9) \wedge (1 \leftarrow 0) \wedge (1 \leftarrow 0) \wedge (0 \leftarrow 0.1)\\ &= 1 \wedge 1 \wedge 1 \wedge ((1 - 0.1 + 0) \wedge 1) = \mathbf{0.9}\end{aligned}$$

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$$(R \square S) = (R \lhd S) \wedge (R \rhd S)$$

$R \square S$	z_1	z_2	z_3	z_4	z_5
x_1	**0**	0.2	0.2	**0**	0.1
x_2	0	0	0.2	0.1	0.2
x_3	0	0	0.1	0.1	0.1
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Gap between quantifiers? Example:

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x_2	0.8	0.8	0.8	0.8	0.9
x_3	0.7	0.9	0.7	0.9	0.9
x_4	0.9	1	0.6	1	1

Every patient is suspicious of having any disease. If we try to strengthen the suspicion, we get no suspicion anymore:

$R \square S$	z_1	z_2	z_3	z_4	z_5
x_1	0	0.2	0.2	0	0.1
x_2	0	0	0.2	0.1	0.2
x_3	0	0	0.1	0.1	0.1
x_4	0	0	0	0.1	0

Outline

Generalized quantifiers

Quantifiers such as `Most`, `Many` or `A Few`

They denote a quantity thus, their construction based on fuzzy measure is very natural

Definition

- $U = \{u_1, \ldots, u_n\}$ – non-empty finite universe
- $\mu : \mathcal{P}(U) \to [0,1]$ – normalized fuzzy measure, i.e., $\mu(\emptyset) = 0$ and $\mu(U) = 1$

μ is a fuzzy measure invariant w.r.t. cardinality if

$$\forall A, B \in \mathcal{P}(U) : |A| = |B| \Rightarrow \mu(A) = \mu(B)$$

We follow A. Dvořák, M. Holčapek, (FSS 2009):"L-fuzzy quantifiers of type $\langle 1 \rangle$ determined by fuzzy measures".

Generalized quantifiers

Example: Relative cardinality

$$\mu_{RC}(A) = \frac{|A|}{|U|}$$

is a fuzzy measure invariant w.r.t cardinality.

Example: Modified relative cardinality

Let $f : [0,1] \to [0,1]$ be a non-decreasing mapping with $f(0) = 0$ and $f(1) = 1$. Then $\mu(A) = f(\mu_{RC}(A))$ is also a fuzzy measure invariant w.r.t cardinality.

Remark: All fuzzy sets used to model evaluative linguistic expressions of the type `Big` fulfill the assumptions on f.

Generalized quantifiers

Definition

- $U = \{u_1, \dots, u_n\}$ – non-empty finite universe
- μ – fuzzy measure invariant w.r.t. cardinality
- $*$ – left-continuous t-norm

Mapping $Q : \mathcal{F}(U) \to [0,1]$ defined by:

$$Q(C) = \bigvee_{D \in \mathcal{P}(U) \setminus \emptyset} \left(\left(\bigwedge_{u \in D} C(u) \right) * \mu(D) \right)$$

is a fuzzy quantifier determined by fuzzy measure μ

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Classical quantifiers as special cases

Example

Let us assume that the fuzzy measures μ defined as follows

$$\mu^{\forall}(D) = \begin{cases} 1 & D \equiv U \\ 0 & \text{otherwise,} \end{cases} \qquad \mu^{\exists}(D) = \begin{cases} 0 & D \equiv \emptyset \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Then the derived quantifiers $Q^{\forall}$ and $Q^{\exists}$ are exactly the classical universal and existential quantifiers, respectively.

Generalized quantifiers – computation

The definition is very inappropriate for computations (calculating over the whole potential set of U).

Theorem

$$Q(D) = \bigvee_{i=1}^{n} D(u_{\pi(i)}) * \mu(\{u_1, \ldots, u_i\})$$

where π is a permutation on U such that

$$D(u_{\pi(1)}) \geq D(u_{\pi(2)}) \geq \cdots \geq D(u_{\pi(n)}).$$

Example: Take Most, i.e. take $\mu(A) = \texttt{VeBi}(\mu_{RC}(A))$

$$Q(D) = \bigvee_{i=1}^{n} D(u_{\pi(i)}) * f(i/n) = \bigvee_{i=1}^{n} D(u_{\pi(i)}) * \mathrm{VeBi}(i/n)$$

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Outline

Compositions based on generalized quantifiers

The idea is to replace the standard quantifiers in the definitions of compositions, e.g. the universal quantifier in

$$R \lhd S = \{(x,z) \in X \times Z \mid \forall\, y \in Y : (x,y) \in R \;\Rightarrow\; (y,z) \in S\}$$

by a generalized quantifier Q defined on Y, in order to obtain the following composition:

$$R \lhd^{Q} S = \{(x,z) \in X \times Z \mid \mathcal{Q}\, y \in Y : (x,y) \in R \;\Rightarrow\; (y,z) \in S\}$$

Compositions based on generalized quantifiers

Definition

$$(R \circ^Q S)(x,z) = \bigvee_{D \in \mathcal{P}(U) \setminus \emptyset} \left(\left(\bigwedge_{y \in D} R(x,y) * S(y,z) \right) * \mu(D) \right),$$

$$(R \lhd^Q S)(x,z) = \bigvee_{D \in \mathcal{P}(U) \setminus \emptyset} \left(\left(\bigwedge_{y \in D} R(x,y) \rightarrow S(y,z) \right) * \mu(D) \right),$$

$$(R \rhd^Q S)(x,z) = \bigvee_{D \in \mathcal{P}(U) \setminus \emptyset} \left(\left(\bigwedge_{y \in D} R(x,y) \leftarrow S(y,z) \right) * \mu(D) \right),$$

$$(R \square^Q S)(x,z) = \bigvee_{D \in \mathcal{P}(U) \setminus \emptyset} \left(\left(\bigwedge_{y \in D} R(x,y) \leftrightarrow S(y,z) \right) * \mu(D) \right).$$

Computation with such compositions

Corollary

$$(R \circ^Q S)(x,z) = \bigvee_{i=1}^{n} \left((R(x, y_{\pi(i)}) * S(y_{\pi(i)}, z)) * f(i/n) \right),$$

$$(R \lhd^Q S)(x,z) = \bigvee_{i=1}^{n} \left((R(x, y_{\pi(i)}) \rightarrow S(y_{\pi(i)}, z)) * f(i/n) \right),$$

$$(R \rhd^Q S)(x,z) = \bigvee_{i=1}^{n} \left((R(x, y_{\pi(i)}) \leftarrow S(y_{\pi(i)}, z)) * f(i/n) \right),$$

$$(R \square^Q S)(x,z) = \bigvee_{i=1}^{n} \left((R(x, y_{\pi(i)}) \leftrightarrow S(y_{\pi(i)}, z)) * f(i/n) \right)$$

where π is a permutation such that (putting $\circledast \in \{*, \rightarrow, \leftarrow, \leftrightarrow\}$:

Equivalence to standard compositions

One may check that $R \circ S = R \circ^{\exists} S$ and that

$$R \lhd S = R \lhd^{\forall} S,\ R \rhd S = R \rhd^{\forall} S,\ R \square S = R \square^{\forall} S.$$

Indeed, $f^{\forall}(i/n) = 0$ for all $i < n$ and $f^{\forall}(1) = 1$ and thus

$$(R \lhd^{\forall} S)(x,z) = \big(R(x, y_{\pi(n)}) \to S(y_{\pi(n)}, z)\big) * f(n/n)$$

which due to the fact that

$$R(x, y_{\pi(n)}) \to S(y_{\pi(n)}, z) = \bigwedge_{i=1}^{n} (R(x, y_i) \to S(y_i, z))$$

confirms $R \lhd S = R \lhd^{\forall} S$

Does it help? Consider the previous example.

Use Roughly Big to construct quantifier Majority.

$(\mathrm{RoBi}(1/4) = 0,\ \mathrm{RoBi}(2/4) = 0,\ \mathrm{RoBi}(3/4) = 0.95,\ \mathrm{RoBi}(1) = 1)$

Symptoms:

y_1 - tiredness; y_2 - cough; y_3 - fever; y_4 - blurred vision

Diseases:

z_1 - pulmonary hypertension; z_2 - sleeping sickness;
z_3 - malaria; z_4 - hangover; z_5 - influenza

$R \square^Q S$	z_1	z_2	z_3	z_4	z_5
x_1	0.15	0.75	0.2	0.75	0.1
x_2	0.05	0.25	0.35	0.1	0.75
x_3	0	0.35	0.25	0.15	0.75
x_4	0	0.05	0.05	0.15	0.95

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Recall properties of standard compositions $\circ, \lhd, \rhd, \square$

1. $R \circ (S \circ T) = (R \circ S) \circ T$
2. $R \square S = (R \lhd S) \cap (R \rhd S)$
3. $R_1 \leq R_2 \;\Rightarrow\; (R_1 \circ S) \subseteq (R_2 \circ S)$ and
   $S_1 \leq S_2 \;\Rightarrow\; (R \circ S_1) \subseteq (R \circ S_2)$
4. $R_1 \leq R_2 \;\Rightarrow\; (R_1 \lhd S) \supseteq (R_2 \lhd S)$ and
   $(R_1 \rhd S) \subseteq (R_2 \rhd S)$
5. $(R_1 \cup R_2) \circ S = (R_1 \circ S) \cup (R_2 \circ S)$
6. $(R_1 \cap R_2) \lhd S = (R_1 \lhd S) \cup (R_2 \lhd S)$
7. $(R_1 \cup R_2) \rhd S = (R_1 \rhd S) \cup (R_2 \rhd S)$
8. $(R_1 \cap R_2) \circ S \subseteq (R_1 \circ S) \cap (R_2 \circ S)$
9. $(R_1 \cup R_2) \lhd S = (R_1 \lhd S) \cap (R_2 \lhd S)$
10. $(R_1 \cap R_2) \rhd S = (R_1 \rhd S) \cap (R_2 \rhd S)$

IRAFM

Are the properties preserved for $\circ^Q, \lhd^Q, \rhd^Q, \Box^Q$?

1. $R \circ (S \circ T) = (R \circ S) \circ T$
2. $R \Box S = (R \lhd S) \cap (R \rhd S)$
3. $R_1 \leq R_2 \;\Rightarrow\; (R_1 \circ S) \subseteq (R_2 \circ S)$ and
   $S_1 \leq S_2 \;\Rightarrow\; (R \circ S_1) \subseteq (R \circ S_2)$
4. $R_1 \leq R_2 \;\Rightarrow\; (R_1 \lhd S) \supseteq (R_2 \lhd S)$ and
   $(R_1 \rhd S) \subseteq (R_2 \rhd S)$
5. $(R_1 \cup R_2) \circ S = (R_1 \circ S) \cup (R_2 \circ S)$
6. $(R_1 \cap R_2) \lhd S = (R_1 \lhd S) \cup (R_2 \lhd S)$
7. $(R_1 \cup R_2) \rhd S = (R_1 \rhd S) \cup (R_2 \rhd S)$
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Are the properties preserved for $\circ^Q, \lhd^Q, \rhd^Q, \Box^Q$?

1. $R \circ^Q (S \circ^Q T) = (R \circ^Q S) \circ^Q T$
2. $R \,\Box^Q\, S \subseteq (R \lhd^Q S) \cap (R \rhd^Q S)$
3. $R_1 \leq R_2 \;\Rightarrow\; (R_1 \circ^Q S) \subseteq (R_2 \circ^Q S)$ and
   $S_1 \leq S_2 \;\Rightarrow\; (R \circ^Q S_1) \subseteq (R \circ^Q S_2)$
4. $R_1 \leq R_2 \;\Rightarrow\; (R_1 \lhd^Q S) \supseteq (R_2 \lhd^Q S)$ and
   $(R_1 \rhd^Q S) \subseteq (R_2 \rhd^Q S)$
5. $(R_1 \cup R_2) \circ^Q S = (R_1 \circ^Q S) \cup (R_2 \circ^Q S)$
6. $(R_1 \cap R_2) \lhd^Q S = (R_1 \lhd^Q S) \cup (R_2 \lhd^Q S)$
7. $(R_1 \cup R_2) \rhd^Q S = (R_1 \rhd^Q S) \cup (R_2 \rhd^Q S)$
8. $(R_1 \cap R_2) \circ^Q S \subseteq (R_1 \circ^Q S) \cap (R_2 \circ^Q S)$
9. $(R_1 \cup R_2) \lhd^Q S \subseteq (R_1 \lhd^Q S) \cap (R_2 \lhd^Q S)$
10. $(R_1 \cap R_2) \rhd^Q S \subseteq (R_1 \rhd^Q S) \cap (R_2 \rhd^Q S)$

Outline

Images of fuzzy sets under fuzzy relations

$$\begin{array}{cccccccc} A & \subseteq & \emptyset & \times & X & & \\ R & \subseteq & & & X & \times & Y \\ \hline A@R & \subseteq & \emptyset & & & \times & Y \end{array}$$

where $@ \in \{\circ, \lhd, \rhd, \Box\}$.

Example:

- X – set of symptoms,
- Y – set of patients,
- R – fuzzy relation on $X \times Y$,
- A – fuzzy sets specifying “searched” symptoms from X,
- $A@R$ – fuzzy sets of patients having searched symptoms.

Images of fuzzy sets under fuzzy relations

Example:

- $A \circ R$ – fuzzy set of patients having at least one from the searched symptoms.
- $A \lhd R$ – fuzzy set of patients having all searched symptoms.
- $A \rhd R$ – fuzzy set of patients for whose symptoms hold that all of them are among the searched ones (no symptoms out of the searched ones).
- $A \,\square\, R$ – fuzzy sets of patients having all searched symptoms and no other symptoms.

Relational databases

O. Pivert, P. Bosc, Fuzzy Preference Queries to Relational Databases, Imperial College Press 2012.

If r and s are two relations with respective schemas $R(A, X)$ and $S(B, Y)$ where A and B are compatible sets of attributes, the **division** is defined as follows

$$div(r, s, A, B) = \{x \mid \forall a, (a \in project(s, B)) \Rightarrow (\langle a, x\rangle \in r)\}$$

Nothing else but an image of a (fuzzy) set under a (fuzzy) relation, particularly:

$$S \subseteq K \times L, \quad S' \subseteq L \text{ is given as } S' = proj_L(S), \quad R \subseteq L \times M$$

Then the division is $S' \lhd R \subseteq M$.

Conclusions

- Compositions of classical and fuzzy binary relations were recalled.
- Motivation for introducing new compositions based on generalized quantifiers was provided.
- New compositions were defined and their use demonstrated.
- Validity of basic properties proved.
- Application potential lies e.g. in flexible query answering systems (images of fuzzy sets under fuzzy relations derived from the newly defined compositions) or may be also in fuzzy inference systems.

Thanksgiving

Thank You for Your Attention