ANNUITY VALUATION BY COPULAS

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THE TWELFTH INTERNATIONAL CONFERENCE
ON FUZZY SET THEORY AND APPLICATIONS

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Core Syllabus for Actuarial Training in Europe

- basic probability theory
- random variables and related concepts
- correlation and regression analysis
- simulation methods

Actuaries strive

- to understand stochastic outcomes of financial security systems
- to estimate of joint life mortality and multidecrement models

Copula - very important concept in life insurance

Copula

- a copula is a function that connects univariate marginal distribution functions to their full multivariate distribution function
- copulas are useful for examining the dependence structure of multivariate random variables

The range of copulas applications

- civil engineering- reliability of analysis of highway bridges
- climate and weather related research
- analysis of extremas in financial assets and returns
- failure of paired organs in health science
- **human mortality in insurance (actuarial science)**
  - **mortalities of spouses**
  - **mortalities of parents and children**
  - **mortality of twins (identical or non-identical)**

Distribution functions which are useful for modeling of age at death

Gompertz distribution function, $G(m;\sigma)$

$$F(x) = 1 - e^{\left[e^{\left(-\frac{m}{\sigma}\right)}\cdot\left(1-e^{\left(\frac{x}{\sigma}\right)}\right)\right]}$$

Weibull distribution function, $W(\gamma;c)$

$$F(x) = 1 - e^{-cx^{\gamma}}$$

Pareto distribution function, $Pa(\alpha;\lambda)$

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}$$

Modeling of dependence

Uniform Gompertz distibution functions

$$u = F_1(x) = 1 - e^{\left[e^{\left(-\frac{m_1}{\sigma_1}\right)} \cdot \left(1 - e^{\left(\frac{x}{\sigma_1}\right)}\right)\right]}$$

$$v = F_2(y) = 1 - e^{\left[e^{\left(-\frac{m_2}{\sigma_2}\right)} \cdot \left(1 - e^{\left(\frac{y}{\sigma_2}\right)}\right)\right]}$$

estimation of individual parameters

$$\theta = (m_1, \sigma_1, m_2, \sigma_2)$$

Bivariate distribution function - Copula

$$H(x,y) = C(F_1(x), F_2(y))$$

basic property

It is clear that if $F_1(x)$, $F_2(y)$ and C are known, then H can be determined.

Sklar (1959) proved a converse:

"If H is known and $F_1(x)$, $F_2(y)$ are known and continuous, then C is uniquely determined".

Copula C belongs to the Archimedean class of copulas if

$$C_\phi(u, v, \alpha) = \phi^{-1}\left(\phi(u) + \phi(v)\right)$$

a generator of the copula

$\phi :]0, 1] \to [0, \infty[$ is a convex, decreasing function satisfyig $\phi(1) = 0$

estimation of individual parameters

$$\theta = (m_1, \sigma_1, m_2, \sigma_2, \alpha)$$

Characteristic of Archimedean copulas

Family of copulas	Parameter α	Generator $\phi(t)$	Kendall's τ (Spearman's ρ)
Gumbel	$\alpha > 1$	$(-ln)^{\alpha}$	$\frac{\alpha-1}{\alpha}$ (no closed form)
Clayton	$\alpha > 0$	$\frac{t^{-\alpha}-1}{\alpha}$	$\frac{\alpha}{\alpha+2}$ (complicated form)
Frank	$\alpha \in R$	$-ln\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right)$	$1-\frac{4}{\alpha}(1-D_1(\alpha))$ $(1-\frac{12}{\alpha}(D_2(-\alpha)-D_1(-\alpha)))$

Kendall's τ and Spearman's ρ - measures of the association between two variables (X, Y)

Kendall's τ

For each pair of observations (x_1, y_1) and (x_2, y_2)
we consider it concordant if $\frac{x_1 - x_2}{y_1 - y_2} > 0$ and discordant if $\frac{x_1 - x_2}{y_1 - y_2} < 0$

$$\tau = \frac{C - D}{\frac{1}{n} n \cdot (n-1)}$$

Spearman's ρ

is the ordinary (Pearson) correlation coefficient of the transformed random variables $F_1(x)$ and $F_2(y)$.

Measures of the association between two variables expressed by copula function

Kendall's τ

$$\tau = 1 - 4 \int\int_{[0,1]^2} \frac{\partial C}{\partial u}(u,v) \frac{\partial C}{\partial v}(u,v) du dv$$

Spearman's ρ

$$\rho = 12 \int\int_{[0,1]^2} C(u,v) du dv - 3$$

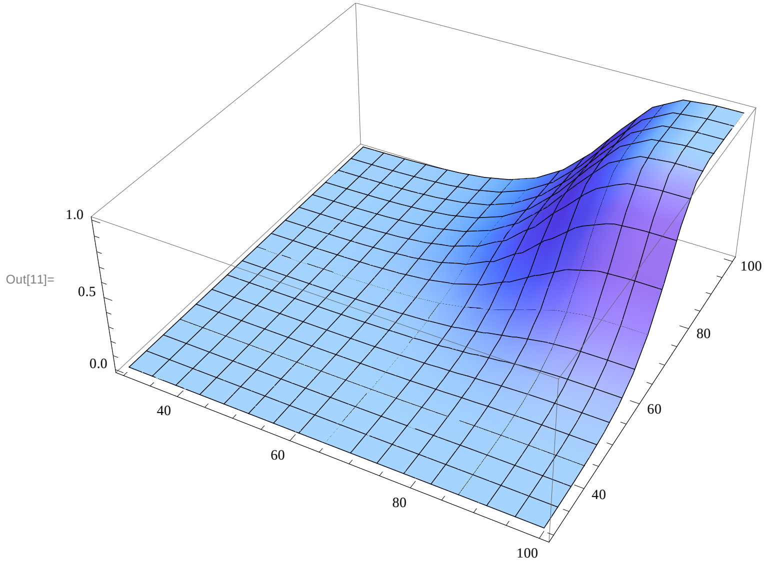
Frank's copula

$$C_F(u, v, \alpha) = \frac{1}{\alpha} ln \left[1 + \frac{(e^{\alpha u} - 1) \cdot (e^{\alpha v} - 1)}{e^{\alpha} - 1} \right]$$

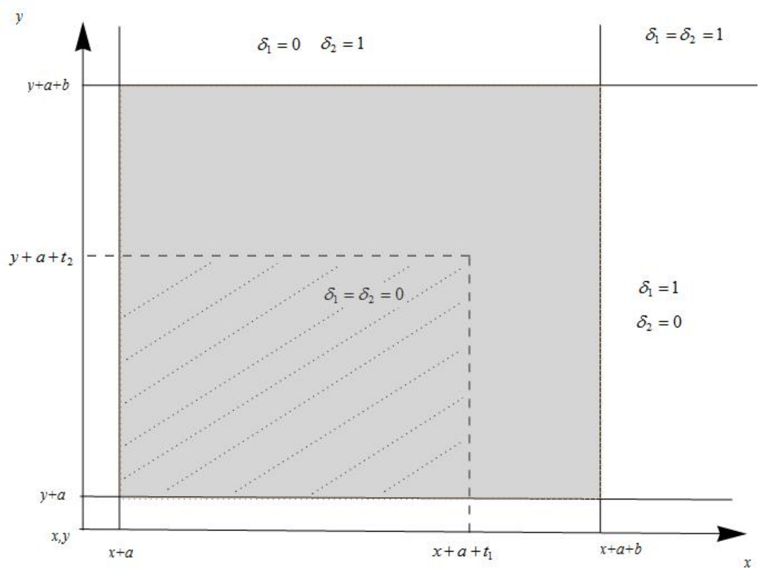
corresponding generator

$$\phi(t) = -ln \frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}$$

Figure : Frank's copula



Maximum Likelihood Estimation



Maximum Likelihood Estimation

Conditional distribution function of lifetime random variables

$$H_T(t_1, t_2) = P(T_1 \leq t_1, T_2 \leq t_2 | T_1 > 0, T_2 > 0)$$

$$\frac{H(x+a+t_1, y+a+t_2) - H(x+a+t_1, y+a) - H(x+a, y+a+t_2) + H(x+a, y+a)}{1 - H(x+a, \infty) - H(\infty, y+a) + H(x+a, y+a)}$$

Maximum Likelihood Estimation

$$\begin{aligned}
&ln\, L(x, y, t_1, t_2, \delta_1, \delta_2, a, b) = \\
&= (1-\delta_1)\cdot(1-\delta_2)\cdot ln\, h(x+a+t_1, y+a+t_2)+ \\
&+(1-\delta_1)\cdot\delta_2\cdot ln\,(H_1(x+a+t_1,\infty) - H_1(x+a+t_1, y+a+b)) + \\
&+\delta_1\cdot(1-\delta_2)\cdot ln\,(H_2(\infty, y+a+t_2) - H_2(x+a+b, y+a+t_2)) + \\
&+\delta_1\cdot\delta_2\cdot \\
&ln\,(1 - H(x+a+b,\infty) - H(\infty, y+a+b) + H(x+a+b, y+a+b)) - \\
&-ln\,(1 - H(x+a,\infty) - H(\infty, y+a) + H(x+a, y+a))
\end{aligned}$$

log-likelihood function for the data set

$$ln\mathcal{L} = \sum_{i=1}^{n} ln\, L(x_i, y_i, t_{1i}, t_{2i}, \delta_{1i}, \delta_{2i}, a_i, b_i)$$

Net single premium for independent lives aged x and y

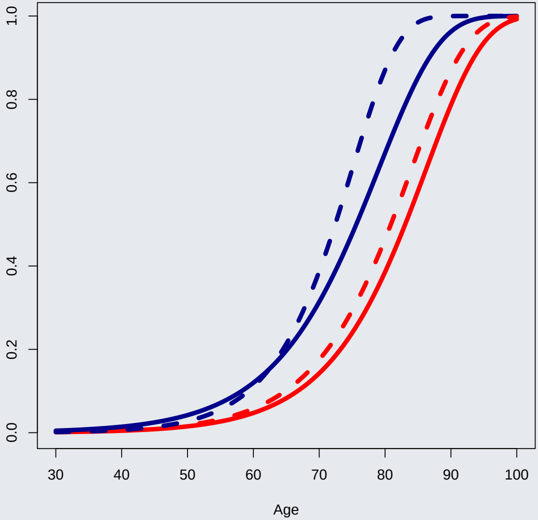
$$\ddot{a}_x = \sum_{t=0}^{\infty} \nu^t \cdot {}_t p_x \qquad {}_t p_x = 1 - H_T(t, \infty)$$

$$\ddot{a}_y = \sum_{t=0}^{\infty} \nu^t \cdot {}_t p_y \qquad {}_t p_y = 1 - H_T(\infty, t)$$

Net single premium for a joint and last-survivor annuity issued to lives aged x and y

$$\ddot{a}_{\overline{xy}} = \sum_{t=0}^{\infty} \nu^t \cdot {}_t p_{\overline{xy}} \qquad {}_t p_{\overline{xy}} = 1 - H_T(t, t), \;\; a = 0$$

Figure : Distribution functions



Estimated parameters of the copula function

$$\theta = (m_1 = 75, \sigma_1 = 7,0; m_2 = 84, \sigma_1 = 8,5; \alpha = -0.13)$$

Yearly pension annuity from accummulated sum S

$$P(\alpha) = \frac{S}{\ddot{a}_{\bar{x}\bar{y}}(\alpha)} \qquad P = \frac{S}{\ddot{a}_{\bar{x}\bar{y}}}$$

Ratios of Dependent to Independent Joint and last survivor pension annuities

Source: E. W. Frees, J. Carriere, E. Valdez: *Annuity valuation with dependent mortality, 1995* (5 % p.a.)

	Female age		
Male age	60	65	70
60	0.95	0.94	0.94
65	0.96	0.94	0.93
70	0.97	0.95	0.94

Source: own construction (2 % p.a.)

	Female age		
Male age	60	65	70
60	0.76	0.79	0.84
65	0.76	0.88	0.83
70	0.87	0.85	0.85