# On the category of lattice-valued bornological spaces

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# Outline

# Lattice-valued bornology

- There exist well-known concepts of functional analysis, namely, bornological space and bounded map, which provide a convenient tool to study "boundedness".
- The construct **Born** of bornological spaces and bounded maps has already found applications in Functional Analysis.
- In 2011, M. Abel and A. Šostak introduced the notions of $L$-bornological space and $L$-bounded map for a complete lattice $L$.
- M. Abel and A. Šostak showed that the construct $L$-**Born** of $L$-bornological spaces and $L$-bounded maps is topological, provided that the complete lattice $L$ is infinitely distributive.

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# Properties of lattice-valued bornology

This talk

- provides the necessary and sufficient condition on the complete lattice $L$ for the construct $L$-**Born** to be topological;
- shows that for "reasonable" lattices $L$, the construct $L$-**Born**$_s$ of strict $L$-bornological spaces (in the sense of M. Abel and A. Šostak) is a topological universe;
- introduces the category **L**-**Born** of variable-basis lattice-valued bornological spaces (in the sense of S. E. Rodabaugh) over a subcategory **L** of the category **Sup** of $\bigvee$-semilattices and $\bigvee$-preserving maps, and provides the necessary and sufficient conditions on **L** for the category **L**-**Born** to be topological.

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# Bornological spaces and bounded maps

Every map $X \xrightarrow{f} Y$ gives rise to the *forward powerset operator* $\mathcal{P}X \xrightarrow{f^{\rightarrow}} \mathcal{P}Y$, which is defined by $f^{\rightarrow}(S) = \{f(s) \mid s \in S\}$.

**Definition 1**

A *bornological space* is a pair $(X, \mathcal{B})$, where $X$ is a set, and $\mathcal{B}$ (a *bornology* on $X$) is a subfamily of $\mathcal{P}X$ (the elements of which are called *bounded sets*), which satisfy the following axioms:

1. $X = \bigcup \mathcal{B} (= \bigcup_{B \in \mathcal{B}} B)$;
2. if $B \in \mathcal{B}$ and $D \subseteq B$, then $D \in \mathcal{B}$;
3. if $\mathcal{S} \subseteq \mathcal{B}$ is finite, then $\bigcup \mathcal{S} \in \mathcal{B}$.

Given bornological spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is *bounded* provided that $f^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$. **Born** is the construct of bornological spaces and bounded maps.

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# $L$-bornological spaces and $L$-bounded maps

Given a complete lattice $L$, every map $X \xrightarrow{f} Y$ provides the *forward $L$-powerset operator* $L^X \xrightarrow{f_L^{\rightarrow}} L^Y$ with $(f_L^{\rightarrow}(B))(y) = \bigvee_{f(x)=y} B(x)$.

**Definition 2 (M. Abel and A. Šostak)**

An *$L$-bornological space* is a pair $(X, \mathcal{B})$, where $X$ is a set, and $\mathcal{B}$ (an *$L$-bornology* on $X$) is a subfamily of $L^X$ (the elements of which are called *bounded $L$-sets*), which satisfy the following axioms:

1. $\bigvee_{B \in \mathcal{B}} B(x) = \top_L$ for every $x \in X$;
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Given $L$-bornological spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is *$L$-bounded* provided that $f_L^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$. $L$-**Born** is the construct of $L$-bornological spaces and $L$-bounded maps.

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# Strict $L$-bornological spaces

Given $x \in X$ and $a \in L$, define a map $X \xrightarrow{\hat{x}_a} L$ by

$$\hat{x}_a(y) = \begin{cases} a, & y = x \\ \perp_L, & \text{otherwise.} \end{cases}$$

**Definition 3 (M. Abel and A. Šostak)**

$L$-$\mathbf{Born}_s$ is the full subconstruct of $L$-$\mathbf{Born}$, the objects of which (called *strict $L$-bornological spaces*) $(X, \mathcal{B})$ satisfy additionally the condition $\hat{x}_{\top_L} \in \mathcal{B}$ for every $x \in X$.

**Example 4**

Every crisp bornological space is strict.

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# Variable-basis lattice-valued forward powerset operator

- **Set** is the category of sets and maps.
- **Sup** is the category of $\bigvee$-semilattices and $\bigvee$-preserving maps.

## Proposition 5

*Given a subcategory* $\mathbf{L}$ *of* $\mathbf{Sup}$, *there is a functor* $\mathbf{Set} \times \mathbf{L} \xrightarrow{(-)^{\rightarrow}} \mathbf{Sup}$, *which is defined by* $((X_1, L_1) \xrightarrow{(f,\psi)} (X_2, L_2))^{\rightarrow} = L_1^{X_1} \xrightarrow{(f,\psi)^{\rightarrow}} L_2^{X_2}$ *with* $((f,\psi)^{\rightarrow}(B))(x_2) = \bigvee_{f(x_1)=x_2} \psi \circ B(x_1)$.

# Variable-basis lattice-valued bornology

## Definition 6

Given a subcategory **L** of **Sup**, **L-Born** is the category, which is concrete over the product category **Set** $\times$ **L**, whose

objects are triples $(X, L, \mathcal{B})$, where $L$ is an **L**-object, and $(X, \mathcal{B})$ is an $L$-bornological space; and whose

morphisms $(X_1, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2, L_2, \mathcal{B}_2)$ (called **L**-*bounded maps*) consist of a map $X_1 \xrightarrow{f} X_2$ and an **L**-morphism $L_1 \xrightarrow{\psi} L_2$ such that $(f, \psi)^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$.

# Ideal complete distributivity at the top element

## Definition 7

Given a complete lattice $L$, a subset $S \subseteq L$ is called a *lattice ideal* of $L$ provided that

1. if $a \in L$ and $a \leqslant b$ for some $b \in S$, then $a \in S$;
2. if $T \subseteq S$ is finite, then $\bigvee T \in S$.

## Definition 8

A complete lattice $L$ is called *ideally completely distributive at $\top_L$* provided that for every non-empty family $\{S_i \mid i \in I\}$ of lattice ideals of $L$, $\bigwedge_{i \in I}(\bigvee S_i) = \top_L$ implies $\bigvee_{h \in H}(\bigwedge_{i \in I} h(i)) = \top_L$, where $H$ is the set of choice functions on $\bigcup_{i \in I} S_i$, which are maps $I \xrightarrow{h} \bigcup_{i \in I} S_i$ such that $h(i) \in S_i$ for every $i \in I$.

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# Examples of ideal complete distributivity

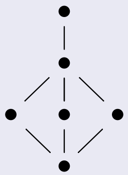
## Example 9

1. Every completely distributive lattice is ideally completely distributive at the top element.
2. Every complete lattice $L$ such that $\bigvee(L\backslash\{\top_L\}) < \top_L$ is ideally completely distributive at $\top_L$.
3. Every continuous lattice is ideally completely distributive at the top element.

# Ideal complete distributivity versus distributivity

## Remark 10

The lattice $L$, which is given by the following Hasse diagram



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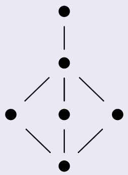

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# $L$-**Born** is a topological construct

**Theorem 12**

*The construct $(L\text{-}\mathbf{Born}, |-|)$ is topological iff $L$ is ideally completely distributive at $\top_L$.*

**Proof.**

"$\Longleftarrow$": Given an $|-|$-structured source $\mathcal{S} = (X \xrightarrow{f_i} |(X_i, \mathcal{B}_i)|)_{i \in I}$, the required $|-|$-initial structure on $X$ w.r.t. $\mathcal{S}$ is provided by $\mathcal{B} = \{B \in L^X \mid f_{i_L}^{\rightarrow}(B) \in \mathcal{B}_i \text{ for every } i \in I\}$.

- By the above theorem, the constructs 2-**Born** (crisp approach) and $[0,1]$-**Born** (fuzzy approach) are topological.
- Moreover, an infinitely distributive complete lattice $L$ does not necessarily provide a topological construct $L$-**Born**.

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- Moreover, an infinitely distributive complete lattice $L$ does not necessarily provide a topological construct $L$-**Born**.

# $L$-$\mathbf{Born}_s$ is a topological construct

From now on, assume that the complete lattice $L$ is ideally completely distributive at $\top_L$, and also that $L$ is a *frame*, i.e., satisfies the condition $(\bigvee S) \wedge a = \bigvee_{s \in S}(s \wedge a)$ for every $S \subseteq L$, $a \in L$.

**Theorem 13**

*$L$-$\mathbf{Born}_s$ is a topological construct, and therefore, is (co)complete.*

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# Binary products of strict $L$-bornological spaces

Given maps $X_1 \xrightarrow{B_1} L$ and $X_2 \xrightarrow{B_2} L$, define a map $X_1 \times X_2 \xrightarrow{B_1 \otimes B_2} L$ by $(B_1 \otimes B_2)(x_1, x_2) = B_1(x_1) \wedge B_2(x_2)$.

## Proposition 14

*The product of two strict $L$-bornological spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ is given by the source $((X_1, \mathcal{B}_1) \xleftarrow{\pi_1} (X_1 \times X_2, \mathcal{B}^{\otimes}) \xrightarrow{\pi_2} (X_2, \mathcal{B}_2))$, in which $X_1 \times X_2 \xrightarrow{\pi_i} X_i$ is the $i$-th projection map, and $\mathcal{B}^{\otimes} = \{B \in L^{X_1 \times X_2} \mid$ there exists a finite set $J$ such that $B \leqslant \bigvee_{j \in J}(B_{1_j} \otimes B_{2_j})$, and $B_{1_j} \in \mathcal{B}_1, B_{2_j} \in \mathcal{B}_2$ for every $j \in J\}$.*

The proposition is valid for the category $L$-**Born** as well.

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# Binary products of bornological spaces

## Example 15

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# $L$-$\mathbf{Born}_s$ is cartesian closed

**Theorem 16**

*The construct $L$-$\mathbf{Born}_s$ is cartesian closed.*

**Proof.**

- Given a strict $L$-bornological space $(X_1, \mathcal{B}_1)$, one has to show that the functor $L\text{-}\mathbf{Born} \xrightarrow{(X_1,\mathcal{B}_1)\times -} L\text{-}\mathbf{Born}$ has a right adjoint.
- Given a strict $L$-bornological space $(X_2, \mathcal{B}_2)$, one defines $H = L\text{-}\mathbf{Born}_s((X_1, \mathcal{B}_1), (X_2, \mathcal{B}_2))$ and also $\mathcal{B} = \{S \in L^H \mid \bigvee_{h \in H}(S(h) \wedge h_L^{\rightarrow}(B_1)) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1\}$, which provides then a strict $L$-bornological space $(H, \mathcal{B})$.
- Define an $L$-bounded map $X_1 \times H \xrightarrow{ev} X_2$ by $ev(x_1, h) = h(x_1)$.

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- Given a strict $L$-bornological space $(X_1, \mathcal{B}_1)$, one has to show that the functor $L$-$\mathbf{Born} \xrightarrow{(X_1,\mathcal{B}_1)\times -} L$-$\mathbf{Born}$ has a right adjoint.
- Given a strict $L$-bornological space $(X_2, \mathcal{B}_2)$, one defines $H = L$-$\mathbf{Born}_s((X_1, \mathcal{B}_1), (X_2, \mathcal{B}_2))$ and also $\mathcal{B} = \{S \in L^H \mid \bigvee_{h\in H}(S(h) \wedge h_L^{\rightarrow}(B_1)) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1\}$, which provides then a strict $L$-bornological space $(H, \mathcal{B})$.
- Define an $L$-bounded map $X_1 \times H \xrightarrow{ev} X_2$ by $ev(x_1, h) = h(x_1)$.

# $L$-$\mathbf{Born}_s$ is cartesian closed cont.

## Proof cont.

- The map $(X_1,\mathcal{B}_1)\times(H,\mathcal{B})\xrightarrow{ev}(X_2,\mathcal{B}_2)$ provides the required co-universal arrow for $(X_2,\mathcal{B}_2)$, i.e., given an $L$-bounded map $(X_1,\mathcal{B}_1)\times(X_3,\mathcal{B}_3)\xrightarrow{f}(X_2,\mathcal{B}_2)$, there exists a unique $L$-bounded map $(X_3,\mathcal{B}_3)\xrightarrow{\bar{f}}(H,\mathcal{B})$, making the next triangle commute

$$\begin{tikzcd}
(X_1,\mathcal{B}_1)\times(X_3,\mathcal{B}_3) \arrow[d, dotted, "1_{X_1}\times\bar{f}"'] \arrow[dr, "f"] & \\
(X_1,\mathcal{B}_1)\times(H,\mathcal{B}) \arrow[r, "ev"'] & (X_2,\mathcal{B}_2).
\end{tikzcd}$$

- The required map $X_3\xrightarrow{\bar{f}}H$ can be then defined by $X_1\xrightarrow{\bar{f}(x_3)}X_2$ with $(\bar{f}(x_3))(x_1)=f(x_1,x_3)$.

# Natural bornological space

**Corollary 17**

*The construct* **Born** *is cartesian closed.*

$L$-$\mathbf{Born}_s$ (and thus, also **Born**) is not concretely cartesian closed, since (in general) the set $H$ is different from $\mathbf{Set}(X_1, X_2)$.

**Example 18**

Given bornological spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$, it follows that $H = \mathbf{Born}((X_1, \mathcal{B}_1), (X_2, \mathcal{B}_2))$ and $\mathcal{B} = \{S \in \mathcal{P}H \mid \bigcup_{h \in S} h^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1\}$, i.e., $(H, \mathcal{B})$ is the *natural bornological space*.

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# Extremal monomorphisms in $L$-$\mathbf{Born}_s$

## Remark 19

Given an $L$-bounded map $(X_1, \mathcal{B}_1) \xrightarrow{m} (X_2, \mathcal{B}_2)$ in $L$-$\mathbf{Born}_s$, the following are equivalent:

1. $m$ is an extremal monomorphism;
2. $m$ is an embedding (an initial monomorphism);
3. $m$ is injective and $\mathcal{B}_1 = \{B \in L^{X_1} \mid m_L^{\rightarrow}(B) \in \mathcal{B}_2\}$.

# $L$-$\mathbf{Born}_s$ has representable extremal partial morphisms

Given maps $X_1 \xrightarrow{B_1} L$, $X_2 \xrightarrow{B_2} L$, define a map $X_1 \uplus X_2 \xrightarrow{B_1 \oplus B_2} L$ by

$$(B_1 \oplus B_2)(x) = \begin{cases} B_1(x), & x \in X_1 \\ B_2(x), & x \in X_2. \end{cases}$$

**Theorem 20**

*$L$-$\mathbf{Born}_s$ has representable extremal partial morphisms.*

**Proof.**

- Given a strict $L$-bornological space $(X, \mathcal{B})$, let $X^* = X \uplus \{*\}$ and $\mathcal{B}^* = \{C \in L^{X^*} \mid C \leqslant B \oplus \hat{*}_a \text{ for some } B \in \mathcal{B},\ a \in L\}$.
- The map $X \xrightarrow{m_{(X,\mathcal{B})}} X^*$, given by $m_{(X,\mathcal{B})}(x) = x$, provides an $L$-bounded map $(X, \mathcal{B}) \xrightarrow{m_{(X,\mathcal{B})}} (X^*, \mathcal{B}^*)$, which is an embedding.
- $m_{(X,\mathcal{B})}$ represents extremal partial morphisms into $(X, \mathcal{B})$.

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- $m_{(X,\mathcal{B})}$ represents extremal partial morphisms into $(X, \mathcal{B})$.

# $L$-$\mathbf{Born}_s$ is a quasitopos

**Corollary 21**

*The construct* **Born** *has representable extremal partial morphisms.*

**Example 22**

Given a bornological space $(X, \mathcal{B})$, it follows that $\mathcal{B}^* = \{C \in \mathcal{P}(X^*) \mid C \subseteq B \uplus \{*\}$ for some $B \in \mathcal{B}\}$.

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# Topological universes

**Definition 24**

A construct is called *well-fibred* provided that it is fibre-small, and for every set with at most one element, the corresponding fibre has exactly one element.

**Definition 25**

A well-fibred topological construct $(\mathbf{C}, |-|)$, for which $\mathbf{C}$ is a quasitopos, is called a *topological universe*.

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# $L$-$\mathbf{Born}_s$ is a topological universe

**Proposition 26**

*The construct $L$-$\mathbf{Born}_s$ is a topological universe.*

Proof.

- One shows that there exists precisely one strict $L$-bornology $\mathcal{B}$ on both the empty set $\varnothing$ and the singleton set $1 = \{\infty\}$.
- To show the latter case, notice that since the $L$-bornological space $(1, \mathcal{B})$ should be strict, $\widehat{\infty}_{\top_L} \in \mathcal{B}$, and therefore, $\mathcal{B} = L^1$.

The construct $[0,1]$-**Born** is not well-fibred, since $\mathcal{B}_1 = [0,1]^1$ and $\mathcal{B}_2 = \{\hat{\infty}_a \mid a \in [0,1)\}$ provide two different $L$-bornologies on 1.

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# (finite $\bigvee$)-closed complete distributivity at $\top_L$

## Definition 27

A complete lattice $L$ is called *(finite $\bigvee$)-closed completely distributive at $\top_L$* provided that for every non-empty family $\{S_i \mid i \in I\}$ of (finite $\bigvee$)-closed subsets of $L$, $\bigwedge_{i\in I}(\bigvee S_i) = \top_L$ implies $\bigvee_{h\in H}(\bigwedge_{i\in I} h(i)) = \top_L$, with $H$ the set of choice functions on $\bigcup_{i\in I} S_i$.

## Proposition 28

*Given a complete lattice $L$, the following are equivalent:*

1. *$L$ is (finite $\bigvee$)-closed completely distributive at $\top_L$;*
2. *$L$ is ideally completely distributive at $\top_L$.*

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# A particular category of $\bigvee$-semilattices

Given a $\bigvee$-semilattice homomorphism $L_1 \xrightarrow{\psi} L_2$, there exists a $\bigwedge$-preserving map $L_2 \xrightarrow{\psi^{\vdash}} L_1$ with $\psi^{\vdash}(b) = \bigvee\{a \in L_1 \mid \psi(a) \leqslant b\}$.

**Definition 29**

$\mathbf{L}^{\vdash}$ is the subcategory of **Sup**, whose objects $L$ are (finite $\bigvee$)-closed completely distributive at $\top_L$, and whose morphisms $L_1 \xrightarrow{\psi} L_2$ are such that the map $L_2 \xrightarrow{\psi^{\vdash}} L_1$ has the following property:

$$\psi^{\vdash}(\bigvee S) = \bigvee_{s \in S} \psi^{\vdash}(s) \text{ for every } S \subseteq L_2 \text{ such that } \bigvee S = \top_{L_2}.$$

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# **L**-**Born** is a topological category

**Theorem 30**

*The concrete category* $(\mathbf{L}\text{-}\mathbf{Born}, |-|)$ *is topological over the product category* $\mathbf{Set} \times \mathbf{L}$ *iff* $\mathbf{L}$ *is a subcategory of* $\mathbf{L}^{\vdash}$.

*Proof.*

"$\Longleftarrow$": Given an $|-|$-structured source $\mathcal{S} = ((X, L) \xrightarrow{(f_i,\psi_i)} |(X_i, L_i, \mathcal{B}_i)|)_{i \in I}$, the $|-|$-initial structure on $(X, L)$ w.r.t. $\mathcal{S}$ is provided by $\mathcal{B} = \{B \in L^X \mid (f_i, \psi_i)^{\rightarrow}(B) \in \mathcal{B}_i$ for every $i \in I\}$.

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# Example of topological **L-Born**

**Example 31**

An example of **L**, which gives a topological category **L-Born**, is the subcategory of **Sup**, whose objects satisfy the condition of the category $\mathbf{L}^{\vdash}$, and whose morphisms are $\bigvee$-semilattice isomorphisms.

# Example of non-topological **L-Born**

## Example 32

- Let $L_1 = \{\bot, \top\}$ with $\bot \neq \top$, let $L_2 = [0, 1]$ (the unit interval), and let $L_1 \xrightarrow{\psi} L_2$ be a $\bigvee$-semilattice homomorphism, which is defined by $\psi(\bot) = 0$ and $\psi(\top) = 1$.
- Every category $\mathbf{L}$, which contains $L_1 \xrightarrow{\psi} L_2$, provides a non-topological category **L-Born**.
- Both $L_1$ and $L_2$ satisfy the object condition of the category $\mathbf{L}^{\vdash}$, but $\psi$ does not satisfy the morphism condition of $\mathbf{L}^{\vdash}$, since $\psi^{\vdash}(\bigvee[0, 1)) = \psi^{\vdash}(1) = \top \neq \bot = \bigvee_{b\in[0,1)} \psi^{\vdash}(b)$.

# Conclusion

- This talk considered the categories $L$-**Born** and **L**-**Born** of fixed-basis and variable-basis lattice-valued bornological spaces.
- We showed the necessary and sufficient conditions on both $L$ and $\mathbf{L}$ for the categories $L$-**Born** and **L**-**Born** to be topological.
- We also showed that the category $L$-$\mathbf{Born}_s$ of strict $L$-bornological spaces is a topological universe for "fruitful" lattices $L$.

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# Open problem

## Definition 33

$\mathbf{L}$-$\mathbf{Born}_s$ is the full subcategory of $\mathbf{L}$-$\mathbf{Born}$, whose objects (*strict* **L**-*bornological spaces*) $(X, L, \mathcal{B})$ are strict $L$-bornological spaces.

## Problem 34

Is there a fruitful variable-basis subcategory of the category $\mathbf{L}$-$\mathbf{Born}_s$, which is a quasitopos?

Since $\mathbf{L}$-$\mathbf{Born}_s$ (in general) is not a construct, the notion of topological universe is not applicable in the variable-basis setting.

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# Thank you for your attention!