States on IF-events

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Abstract

Recently, B. Riečan has shown that it is possible to develop a reasonable probability on *IF* structures introduced by K. T. Atanassov.

Starting with a structured system $\mathcal{F}$ of *IF* events, $\mathcal{F}$ can be "lifted" to an *MV*-algebra $\mathcal{M}$ and the corresponding probability theory on *MV*-algebras yields limit theorems for *IF* events. The construction is based on a representation theorem for states on *IF* events and their "lifting" to states on MV-algebras.

We give an alternative representation of $\mathcal{M}$ via a bold algebra $\mathcal{B}$, discuss a representation theorem for states on $\mathcal{B}$ and mention some directions of future research.

IF-sets

Definition

IF-set is a pair $A = (\mu_A; \nu_A) : \Omega \longrightarrow [0;1] \times [0;1]$ of fuzzy sets $\mu_A; \nu_A : \Omega \longrightarrow [0;1]$, where $\mu_A + \nu_A \leq 1$.
We call μ_A the membership function of A, ν_A the non membership function of A.

A fuzzy set $\varphi_A : \Omega \longrightarrow [0;1]$ can be considered as an IF-set, where $\mu_A = \varphi_A$; $\nu_A = 1 - \varphi_A$.
Here we have $\mu_A + \nu_A = 1$.

K. T. Atanassov, "Intuitionistic fuzzy sets: past, present and future," in EUSFLAT Conf., M. Wagenknecht and R. Hampel, Eds. University of Applied Sciences at Zittau/Görlitz, Germany, 2003, pp. 12–19.

Atanassov has posed a problem to develop a reasonable IF-probability.

Operations and ordering

B. Riečan: Analysis of fuzzy logic models. In: Intelligent Systems (ed. V. M. Koleshko), In Tech, Rijeka 2012, 219 - 244.

Riečan introduced operations on IF-sets as follows:
If $A = (\mu_A; \nu_A); B = (\mu_B; \nu_B)$ are two IF-sets, then we define
$A \oplus B = ((\mu_A + \mu_B) \wedge 1; (\nu_A + \nu_B - 1) \vee 0);$
$A \odot B = ((\mu_A + \mu_B - 1) \vee 0; (\nu_A + \nu_B) \wedge 1);$
$A \preceq B \Longleftrightarrow \mu_A \leq \mu_B; \nu_A \geq \nu_B:$

Observe that IF-sets are closed with respect to $\oplus$ and $\odot$, but the operation of a complement is problematic.
Riečan succeeded in proving limit theorems for IF-probability (avoiding complementation).

Extending to MV-algebra

Crucial is the embedding of IF-sets to a suitable MV-algebra.
MV-algebra $(M, \oplus, \odot, ^C, 0, 1)$

Theorem

Let $(\Omega; S)$ be a measurable space, $\mathcal{F}$ the family of all IF-sets $A = (\mu_A; \nu_A)$ be such that $\mu_A; \nu_A$ are S-measurable. Then there exists an MV-algebra $\mathcal{M}$ such that $\mathcal{F} \subset \mathcal{M}$, the operations $\oplus, \odot$ are extensions of operations on $\mathcal{F}$ and the ordering $\preceq$ is an extension of the ordering in $\mathcal{F}$.

$M = \{(\mu_A; \nu_A)\text{: measurable fuzzy sets } \mu_A; \nu_A : \Omega \longrightarrow [0; 1]\};$
$(\mu_A; \nu_A)^C = (1 - \mu_A; 1 - \nu_A).$

This theorem enables to use some results of the well developed probability theory on MV-algebras for IF-events.

To develop an IF-probability, it is necessary to define states:

Let (Ω, S) be a measurable space and $\mathcal{F}$ the family of all IF-sets $A = (\mu_A; \nu_A)$ such that $\mu_A; \nu_A : \Omega \longrightarrow [0; 1]$ are S-measurable.

Definition

A mapping $m : \mathcal{F} \longrightarrow [0; 1]$ is called a state if the following properties are satisfied:
(i) $m(1_\Omega; 0_\Omega) = 1; m(0_\Omega; 1_\Omega) = 0;$
(ii) $A \odot B = (0_\Omega; 1_\Omega) \Longrightarrow m((A \oplus B)) = m(A) + m(B);$
(iii) $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$

Representation theorem

For states Riečan proved the following representation theorem:

Theorem

For every state $m : \mathcal{F} \longrightarrow [0;1]$ *there exist probability measures* $P;Q : S \longrightarrow [0;1]$ *and* $\alpha \in [0;1]$ *such that*
$m((\mu_A;\nu_A)) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega(\mu_A + \nu_A)dQ).$

$P(A) - \alpha Q(A) \geq 0$ for $\forall A \in S$.

Theorem

Let $P;Q : S \longrightarrow [0;1]$ *be probability measures and* $\alpha \in [0;1]$ *such that* $P(A) - \alpha Q(A) \geq 0$ *for* $\forall A \in S$. *Then*
$m((\mu_A;\nu_A)) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega(\mu_A + \nu_A)dQ)$
is a state.

IF-events via bold algebras

- We show that IF-events can be embedded into a bold algebra.
- Bold algebras are probability domains of fuzzy probability. Fuzzy random events are measurable functions into [0,1].
- Operations on fuzzy random events and states on fuzzy random events are more transparent than operations on IF-events and states on IF-events and the corresponding MV-algebra. Relevant information about the notions of fuzzy probability can be found in

  Frič, R. and Papčo, M.: On probability domains II. Internat. J. Theoret. Phys. 50 (2011), 3778–3786.

Bold Algebra

Bold algebra is a special MV-algebra - fuzzy sets with Łukasiewicz operations.

Definition

Let X be a non-empty set. Bold Algebra is an algebra $\mathcal{B} = (B, \boxplus, \boxdot, ^*, 0_x, 1_x)$ such that:

- $B \subseteq [0;1]^X$
- $f \boxplus g = \min\{f + g, 1\}$
- $f \boxdot g = \max\{f + g - 1, 0\}$
- $f^* = 1 - f$.

Example

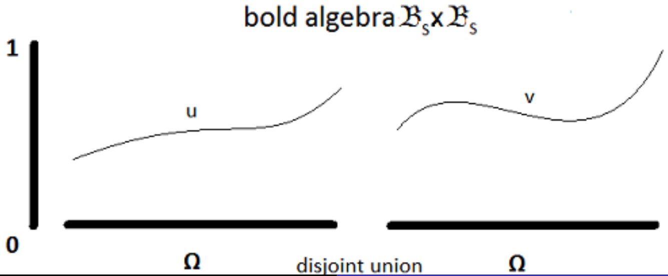
(Ω, S) - measurable space
$\mathcal{B}_S$ - bold algebra of all S-measurable functions $f : \Omega \longrightarrow [0;1]$.

Product of bold algebras

IF-events are not closed with respect to a complement and complementation is achieved by their embedding into an abstract MV-algebra. We show that such MV-algebras can be represented as products of bold algebras.

Definition

Let $\mathcal{B}_1, \mathcal{B}_2$ be bold algebras. Then all couples (u, v), where $u \in \mathcal{B}_1$, $v \in \mathcal{B}_2$, and operations are coordinatewise, "form" a bold algebra $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, which is called a product of $\mathcal{B}_1$ and $\mathcal{B}_2$.



MV-isomorphism

There is a natural way how to represent MV-algebra $\mathcal{M}$ via bold algebra $\mathcal{B}_S \times \mathcal{B}_S$:

Theorem

A mapping $\Phi : \mathcal{M} \longrightarrow \mathcal{B}_S \times \mathcal{B}_S$ *defined* $\Phi(\mu_A; \nu_A) = \langle \mu_A; 1 - \nu_A \rangle$ *is a MV-isomorphism between* $\mathcal{M}$ *and* $\mathcal{B}_S \times \mathcal{B}_S$*.*

- Φ is a bijection;
- $\Phi(0_\Omega; 1_\Omega) = \langle 0_\Omega; 0_\Omega \rangle$
- $\Phi(1_\Omega; 0_\Omega) = \langle 1_\Omega; 1_\Omega \rangle$;
- $\Phi((\mu_A; \nu_A) \oplus (\mu_B; \nu_B)) = \Phi(\mu_A; \nu_A) \boxplus \Phi(\mu_B; \nu_B)$
- $\Phi((\mu_A; \nu_A) \odot (\mu_B; \nu_B)) = \Phi(\mu_A; \nu_A) \boxdot \Phi(\mu_B; \nu_B)$

Remark

$\Phi(\mathcal{F}) = \{\langle \mu_A; \nu_A \rangle \in \mathcal{B}_S \times \mathcal{B}_S : \mu_A \leq \nu_A\}$

From theory of bold algebras:

- States on bold algebras are sequentially continuous D-homomorphisms into $[0, 1]$.
- Such homomorphisms are exactly integrals with respect to probability measures.

Theorem

A mapping $s : \mathcal{B}^2_S \longrightarrow [0; 1]$ is a state on a bold algebra $\mathcal{B}^2_S$ if and only if $s \circ \Phi$ is a state on $\mathcal{M}$.

Theorem

Let p, q be probability measures on (Ω, S) and $a \in [0,1]$. Then a mapping $s\langle \mu_A; \nu_A \rangle = a \int_\Omega \mu_A dp + (1-a) \int_\Omega \nu_A dq$ is a sequentially continuous D-homomorphism (e.g. state) on $\mathcal{B}_S \times \mathcal{B}_S$.

Such mappings will be called convex combinations of states.

Theorem

For every sequentially continuous D-homomorphism s on $\mathcal{B}_S \times \mathcal{B}_S$ there exist probability measures p, q and a number $a \in [0,1]$ such that $s\langle \mu_A; \nu_A \rangle = a \int_\Omega \mu_A dp + (1-a) \int_\Omega \nu_A dq$.

If $a \in (0,1)$, then p and q are unique.

Comparison

Note that from Riečan's representation it also follows:
$s\langle\mu_A;\nu_A\rangle = m\circ\Phi^{-1}\langle\mu_A;\nu_A\rangle = m(\mu_A;1-\nu_A) =$
$\int_\Omega \mu_A dP + \alpha(\int_\Omega(\nu_A-\mu_A)dQ)$

Corollary

We have the following representations of states:

I. $s\langle\mu_A;\nu_A\rangle = a\int_\Omega \mu_A dp + (1-a)\int_\Omega \nu_A dq$
II. $s\langle\mu_A;\nu_A\rangle = \int_\Omega \mu_A dP + \alpha(\int_\Omega(\nu_A-\mu_A)dQ)$

where

- $Q(A) = q(A) = \frac{s\langle 0_\Omega;\chi_A\rangle}{\alpha}$
- $P(A) = s\langle\chi_A;\chi_A\rangle$
- $p(A) = \frac{s\langle\chi_A;0_\Omega\rangle}{a}$
- $\alpha = 1-a = s\langle 0_\Omega;1_\Omega\rangle$

for all $A\in S$ *provided* $a,\alpha\neq\{0;1\}$.

Future research

- Probability domains cogenerated by powers of bold algebras (such as $\mathcal{B}_S \times \mathcal{B}_S$).
- Applications of such probability domains.