Representation theorem of general states on IF-sets.

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Riesz spaces

Definition

Ordered quadrupplet $(V, +, \cdot, \leq)$ is called Riesz space iff

1. $(V, +, \cdot)$ is vector (linear) space over real numbers
2. $(V, \leq)$ is a lattice,
3. $(\forall a, b, c \in V)\,(a \leq b \Rightarrow a + c \leq b + c)$,
4. $(\forall a, b \in V)\,(\forall \mathbb{R} \ni \alpha \geq 0)\,(a \leq b \Rightarrow \alpha a \leq \alpha b)$

Monotonically convergency in Riesz spaces

Definition

Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of elements of Riesz space. We tell that v_n is monotonically converging to v iff is satisfied one of conditions

1. $v_1 \leq v_2 \leq v_3 \leq \cdots \leq v_n \leq v_{n+1} \leq \cdots$ & $v = \bigvee_{n=1}^{\infty} v_n$,
2. $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_n \geq v_{n+1} \geq \cdots$ & $v = \bigwedge_{n=1}^{\infty} v_n$

In case 1 we write $v_n \nearrow v$ and in case 2 we write $v_n \searrow v$.

IF-sets

Definition

By notion IF-subset of set X we understand an ordered pair

$$(f, g),$$

such that $f, g : X \to [0, 1]$ and $f + g \leq 1$

Monotonicaly convergency of IF-sets

Definition

Let $\{(f_n, g_n)\}_{n=1}^{\infty}$ be a sequence of IF-sets. We tell that sequence $\{(f_n, g_n)\}_{n=1}^{\infty}$ is monotonically converging to IF-set (f, g) iff is satisfied one of conditions:

1. $f_n \nearrow f$ & $g_n \searrow g$
2. $f_n \searrow f$ & $g_n \nearrow g$.

In case 1 we write $(f_n, g_n) \nearrow (f, g)$ and in case 2 we write $(f_n, g_n) \searrow (f, g)$.

Operations on IF-sets

Definition

We define these operations on IF-sets:

- $(f, g) \oplus (h, k) = (\min(f + h, 1), \max(g + k - 1, 0))$
- $(f, g) \odot (h, k) = (\max(f + h - 1, 0), \min(g + k, 1))$

Operation $\oplus$ is called Lukasiewicz sum, operation $\odot$ is called Lukasiewicz product.

Definition

By $\mathcal{F}$ we note a set of IF-subsets of set X closed under Lukasiewicz operations and monotonical limits of IF-sets.

States with values from Riesz interval

Definition

A mapping $m : \mathcal{F} \to I$ where $I = [0, u]$ is an interval in Riesz space is called state iff

1. $m((0,1)) = 0$,
2. $(\forall\,(f,g)\,,(h,k) \in \mathcal{F})\,((f,g) \odot (h,k) = (0,1) \Rightarrow$
   $\Rightarrow m((f,g) \oplus (h,k)) = m((f,g)) + m((h,k)))\,,$
3. $(\forall \{(f_n, g_n)\}_{n=1}^{\infty})\,(\forall\,(f,g) \in \mathcal{F})$
   $((f_n, g_n) \nearrow (f,g) \Rightarrow m((f_n, g_n)) \nearrow m((f,g)))$

Integral with values from Riesz space

Definition

Let V be a Riesz space , $\mathcal{S}$ be a $\sigma-$algebra of subsets of a set X. Then a set-mapping $\mu : \mathcal{S} \to \{v \in V; v \geq 0\}$ is called measure iff

$\mu(A) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(A_i)$, whenever

$A = \bigcup_{i=1}^{\infty} A_i \, (A_i \in \mathcal{S} \, (i = 1, 2, 3, \cdots) \, , A_i \cap A_j = \emptyset \, (i \neq j)) \, .$

Definition

Let $f : X \to \mathbb{R}_0^+$. Then f is called integrable iff $\left(\exists \mathbb{R}_0^+ \ni \alpha_i\right)\left(\exists A_i \in \mathcal{S}\right)$, such that

$$(\exists v \in V)\left(v = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i)\right)$$

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x)$$

Definition

We define the integral of integrable $f : X \to \mathbb{R}_0^+$ by equality

$$\int f \, \mathrm{d}\mu = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

Definition

We tell that function $h : X \to \mathbb{R}$ is integrable iff exist integrable $f, g : X \to \mathbb{R}_0^+$ such that $h = f - g$.

Definition

We define the integral of integrable $h : X \to \mathbb{R}$ by equality

$$\int h \mathrm{d}\mu = \int f \mathrm{d}\mu - \int g \mathrm{d}\mu.$$

Real case was proved by Prof. Riečan in folowing formulation

Theorem

For any real state on IF-set there are probability measures P, Q and real number α, such that

$$m((\mu_A, \nu_A)) = \int \mu_A \mathrm{d}P + \alpha \left(1 - \int (\mu_A + \nu_A) \, \mathrm{d}Q \right).$$

Representation of states by integral

For general case one must do some little change in formulation:

Theorem

Let V be a Riesz space, $0 < u \in V$ fixed positive element. Then for every state $m : \mathcal{F} \to [0, u]$, there exist measures $P, Q : \mathcal{S} \to [0, u]$ for which

$$m((\mu_A, \nu_A)) = \int \mu_A \mathrm{d}P + \int (1 - \mu_A - \nu_A)\, \mathrm{d}Q$$

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THANKS FOR ATTENTION