Structural analysis of semilattices and lattices by fuzzy sets

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The other basic feature of fuzzy sets is their **functional nature**, including properties of the ordered structure of membership values. There are many results concerning properties of cut sets and of fuzzy sets as functions, which are developed for different purposes. We use these techniques for investigations of different lattices, semilattices and functions on ordered sets, obtaining results in the classical set and order theory.

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In particular, we prove that $\approx$ is a diagonal relation if and only if $\mu(X)$ is meet-dense subset of L.
Next, if L is finite and distributive and $\mu(X)$ consists of (some) meet-irreducible elements of L, then $\approx$ is a congruence relation on lattice (semilattice) L.

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In particular, we prove that $\approx$ is a diagonal relation if and only if $\mu(X)$ is meet-dense subset of L.
Next, if L is finite and distributive and $\mu(X)$ consists of (some) meet-irreducible elements of L, then $\approx$ is a congruence relation on lattice (semilattice) L.
If L is not distributive, then the analogue property holds if $\mu(X)$ consists of particular special elements in L.

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Using this, **we classify all functions** – fuzzy sets in L^X, defining special equivalence relation on L^X. We describe equivalence classes in terms of collections of cuts of the corresponding functions and also using properties of the congruences on L defined above.

Ordered structures

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It is well known that for $p, q \in L$,

$$\text{from } p \leq q \text{ it follows that } \mu_q \subseteq \mu_p.$$

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Proposition

Let $\mathcal{F}$ be a closure system over a set X. Then there is a lattice L and an L-valued function $\mu : X \to L$, such that the collection μ_L of cuts of μ is $\mathcal{F}$.

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The required lattice L is the collection $\mathcal{F}$ ordered dually to inclusion, and $\mu : X \to L$ can be defined by:

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Moreover, for every $f \in \mathcal{F}$, the cut μ_f coincides with f : $\mu_f = f$.

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$(\mu_L, \leq)$ *is a complete lattice and*

$$\bigcap \{\mu_p \mid p \in L_1 \subseteq L\} = \mu_{\bigvee(p \mid p \in L_1)}.$$

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The order $\leq_{L/\approx}$ of classes in $L/\approx$ corresponds to the order of suprema of classes in L (we denote the order in L by $\leq_L$):

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$$[p]_\approx \leq_{L/\approx} [q]_\approx \text{ if and only if } \uparrow q \cap \mu(X) \subseteq \uparrow p \cap \mu(X).$$

The order $\leq_{L/\approx}$ of classes in $L/\approx$ corresponds to the order of suprema of classes in L (we denote the order in L by $\leq_L$):

Proposition

The poset $(L/\approx, \leq_{L/\approx})$ *is a complete lattice.*

Next we connect the lattice $(L/\approx, \leq_{L/\approx})$ and the lattice $(\mu_L, \leq)$ of cuts of μ; recall that the latter is ordered dually to inclusion.

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Proposition

Let $\mu : X \to L$ be an L-valued function on X. The lattice of cuts $(\mu_L, \leq)$ is isomorphic with the lattice $(L/\approx, \leq_{L/\approx})$ of $\approx$-classes in L under the mapping $\mu_p \mapsto [p]_\approx$.

Example

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$X = \{a, b, c\}$

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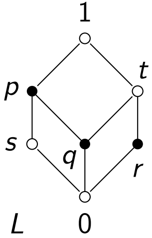

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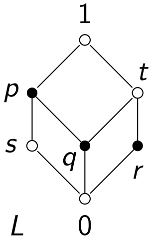

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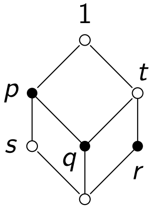

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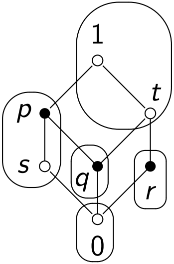
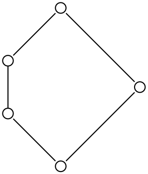


$L/\approx$



$(L\mu, \subseteq)$

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In this case L *is (up to an isomorphism) represented as a lattice of cuts of* μ.
Or, if $\approx$ is a congruence which is not a diagonal relation, we obtain that the lattice of cuts possesses the same properties as L (e.g., it is Boolean if L is, it is Heyting if L is and so on).

Problem

Let L be a complete lattice (semilattice, or Heyting semilattice), and M a nonempty subset of L. Let also $\approx_M$ be a relation on L defined by:

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Find conditions under which $\approx_M$ is a congruence relation on L.

In the following, L is a complete lattice, M is a nonempty subset of L, and $\approx_M$ is the above defined relation on L:

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for a given subset M of L.

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For any $p, q \in L$, ${\uparrow}p \cap {\uparrow}q = {\uparrow}(p \vee q)$.

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For every $p \in L$, if ${\uparrow}p \cap M \neq \emptyset$, then $[p]_{\approx_M}$ has the top element $\bigvee[p]_{\approx_M}$, and $\bigvee[p]_{\approx_M} \in [p]_{\approx_M}$.

Proposition

If $\approx_M$ is a congruence relation on L, then for every $x \in M$ and $p, q \in L$

$$x \leq p \wedge q \quad \textit{implies} \quad x \leq p.$$

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Let J be a minimal meet-dense subset of L and $\emptyset \neq M \subseteq J$. Then $\approx_M$ is a congruence relation on L.

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If M is a collection of (some) meet-irreducible elements in L, then $\approx_M$ is a congruence relation on L.

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Let L be an infinitely distributive lattice and $I \subseteq L$ the set of all meet-irreducible elements of L. Further, let $M \subseteq I$. Then $\approx_M$ is a congruence relation on L.

Let $(S, \wedge)$ be a meet-semilattice. We say that an element $a \in L$ is **distributive** in S if $a \geq b \wedge c$ implies that $a = b_1 \wedge c_1$, for some $b_1 \geq b$ and $c_1 \geq c$.

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A meet-semilattice is said to be **distributive** (Chajda), if

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Every relatively pseudocomplemented semilattice is distributive.

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Theorem

If S is a distributive or Heyting semilattice and $m \in S$ meet irreducible element, then for $M = \uparrow m$, $\approx_M$ is a congruence on S.

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$$\mu(x) = \bigvee \{p \in S \mid x \in \mu_p\}$$

for every $x \in X$, then the collection μ_S, ordered by inclusion, is a semi-closure system on X.

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If $\mathcal{F}$ is a semi-closure system over a nonempty set X, then there is a meet-semilattice S and a fuzzy set $\mu : X \to S$, such that the collection of cuts of μ coincides with $\mathcal{F}$.

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Let S be a complete join-semilattice and X be a nonempty set. If μ is a function in S^X defined by $\mu(x) = \bigvee\{p \in S \mid x \in \mu_p\}$ for every $x \in X$, then the collection μ_S, ordered by inclusion, is a dual semi-closure system on X.

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By the definition, L_μ consists of particular collections of images of μ in L and is a poset under inclusion.

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$F(\uparrow p \cap \mu(X)) := \uparrow \bigwedge \{\nu(x) \mid \mu(x) \geq p\} \cap \nu(X)$, $p \in L$.

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Theorem

Let $\mu, \nu : X \to L$. Then $\mu \sim \nu$ if and only if fuzzy sets μ and ν have equal collections of cuts.

Example

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$$\mu = \begin{pmatrix} x & y & z \\ p & q & r \end{pmatrix} \qquad \nu = \begin{pmatrix} x & y & z \\ p & q & t \end{pmatrix} \qquad \pi = \begin{pmatrix} x & y & z \\ p & r & t \end{pmatrix}$$

Example

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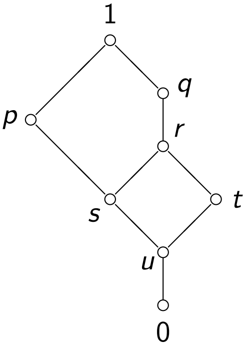

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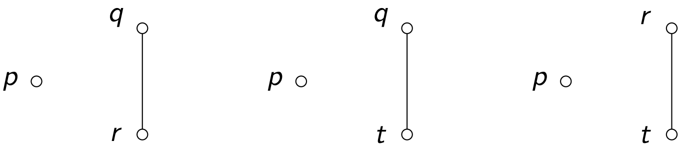
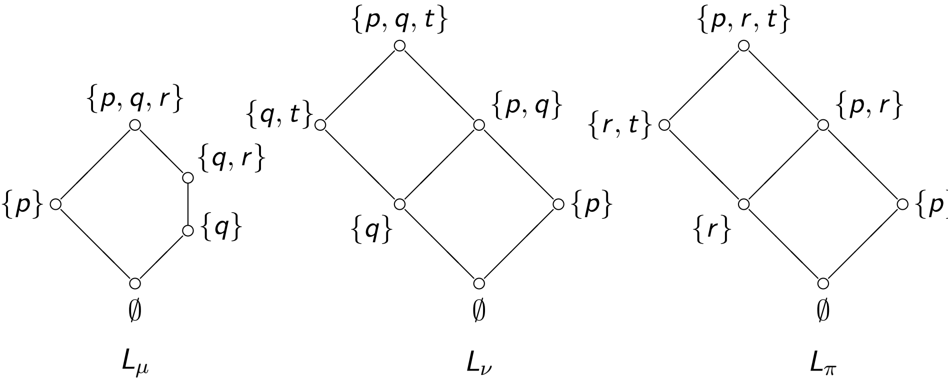


$\mu(X)$ $\nu(X)$ $\pi(X)$



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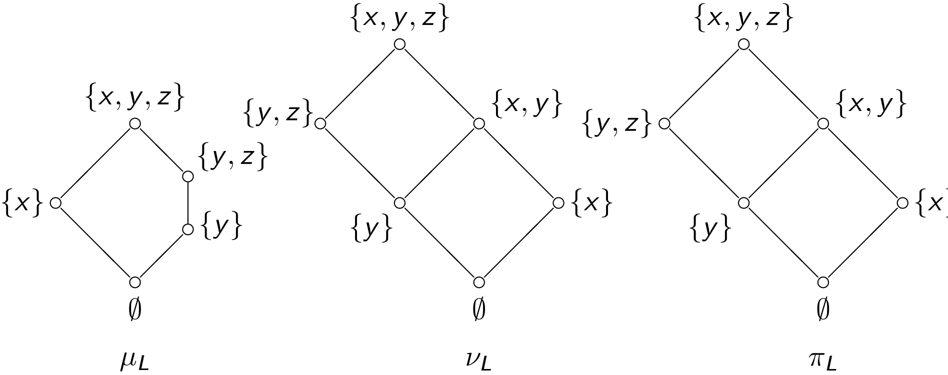

$$\mu_p = \mu^{-1}(\uparrow p); \quad \mu_L = \{\mu_p \mid p \in L\}$$

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Thank you, this was all!