On a class of cardinal densities

Ladislav Mišík[1], János T. Tóth[2]

National Supercomputing Center IT4Innovations[1]; University J. Selye[2]

ladislav.misik@osu.cz[1]; tothj@selueuni.sk[2]

Liptovský Ján
January 29, 2014

IT4Innovations
national01$#&0
supercomputing
center@#01%101

EUROPEAN UNION
EUROPEAN REGIONAL DEVELOPMENT FUND
INVESTING IN YOUR FUTURE

2007–13
OP Research and
Development for Innovation

Universal fuzzy measures

Definition and properties

- For $n \in \mathbb{N}$ denote $\mathbf{n} = \{1, 2, \ldots, n\}$. By **measure** on $\mathbf{n}$ we mean any monotone real function μ_n on $2^{\mathbf{n}}$ with $\mu_n(\emptyset) = 0,\ \mu_n(\mathbf{n}) = 1$.
- Mesiar, Valášková (2003): **Universal fuzzy measure** (UFM) is a sequence $\mu = (\mu_n)$ of measures on $\mathbf{n}$.
- UFM μ is called **regular** if $\mu_n(A) \leq \mu_m(A)$ for all $m < n$ and $A \subset \mathbf{m}$.
  UFM μ is called **proportional** if $\mu_n(A) = \frac{\mu_{n+k}(A)}{\mu_{n+k}(\mathbf{n})}$ for all $n, k \in \mathbb{N}$ and $A \subset \mathbf{n}$.
  Additive, sub-additive, super-additive UFM are defined in a usual way.
- **Example**: Let $d_n(A) = \frac{|A|}{n}$ for all $n \in \mathbb{N}$ and $A \subset \mathbf{n}$. Then $d = (d_n)$ possesses all the above listed properties.

Universal fuzzy measures

Definition and properties

- For $n \in \mathbb{N}$ denote $\mathbf{n} = \{1, 2, \ldots, n\}$. By **measure** on $\mathbf{n}$ we mean any monotone real function μ_n on $2^{\mathbf{n}}$ with $\mu_n(\emptyset) = 0,\ \mu_n(\mathbf{n}) = 1$.
- Mesiar, Valášková (2003): **Universal fuzzy measure** (UFM) is a sequence $\mu = (\mu_n)$ of measures on $\mathbf{n}$.
- UFM μ is called **regular** if $\mu_n(A) \leq \mu_m(A)$ for all $m < n$ and $A \subset \mathbf{m}$.
  UFM μ is called **proportional** if $\mu_n(A) = \frac{\mu_{n+k}(A)}{\mu_{n+k}(\mathbf{n})}$ for all $n, k \in \mathbb{N}$ and $A \subset \mathbf{n}$.
  Additive, sub-additive, super-additive UFM are defined in a usual way.
- **Example**: Let $d_n(A) = \frac{|A|}{n}$ for all $n \in \mathbb{N}$ and $A \subset \mathbf{n}$. Then $d = (d_n)$ possesses all the above listed properties.

Universal fuzzy measures

Definition and properties

- For $n \in \mathbb{N}$ denote $\mathbf{n} = \{1, 2, \ldots, n\}$. By **measure** on $\mathbf{n}$ we mean any monotone real function μ_n on $2^{\mathbf{n}}$ with $\mu_n(\emptyset) = 0,\ \mu_n(\mathbf{n}) = 1$.
- Mesiar, Valášková (2003): **Universal fuzzy measure** (UFM) is a sequence $\mu = (\mu_n)$ of measures on $\mathbf{n}$.
- UFM μ is called **regular** if $\mu_n(A) \leq \mu_m(A)$ for all $m < n$ and $A \subset \mathbf{m}$.
  UFM μ is called **proportional** if $\mu_n(A) = \frac{\mu_{n+k}(A)}{\mu_{n+k}(\mathbf{n})}$ for all $n, k \in \mathbb{N}$ and $A \subset \mathbf{n}$.
  Additive, sub-additive, super-additive UFM are defined in a usual way.
- **Example**: Let $d_n(A) = \frac{|A|}{n}$ for all $n \in \mathbb{N}$ and $A \subset \mathbf{n}$. Then $d = (d_n)$ possesses all the above listed properties.

Universal fuzzy measures

Definition and properties

- For $n \in \mathbb{N}$ denote $\mathbf{n} = \{1, 2, \ldots, n\}$. By **measure** on **n** we mean any monotone real function μ_n on $2^{\mathbf{n}}$ with $\mu_n(\emptyset) = 0,\ \mu_n(\mathbf{n}) = 1$.
- Mesiar, Valášková (2003): **Universal fuzzy measure** (UFM) is a sequence $\mu = (\mu_n)$ of measures on **n**.
- UFM μ is called **regular** if $\mu_n(A) \leq \mu_m(A)$ for all $m < n$ and $A \subset \mathbf{m}$.
  UFM μ is called **proportional** if $\mu_n(A) = \frac{\mu_{n+k}(A)}{\mu_{n+k}(\mathbf{n})}$ for all $n, k \in \mathbb{N}$ and $A \subset \mathbf{n}$.
  Additive, sub-additive, super-additive UFM are defined in a usual way.
- **Example**: Let $d_n(A) = \frac{|A|}{n}$ for all $n \in \mathbb{N}$ and $A \subset \mathbf{n}$. Then $d = (d_n)$ possesses all the above listed properties.

Extension of UFM

Asymptotic fuzzy measures

- Let $\mu = (\mu_n)$ be a UFM and $A \subset \mathbb{N}$. The sequence $(\mu_n(A \cap \mathbf{n}))$ contains some information on the size of A.
- Define

$$\underline{\mu}^*(A) = \liminf_{n\to\infty} \mu_n(A \cap \mathbf{n}), \quad \overline{\mu}^*(A) = \limsup_{n\to\infty} \mu_n(A \cap \mathbf{n})$$

and $\mu^*(A) = \lim_{n\to\infty} \mu_n(A \cap \mathbf{n})$ if the limit exists.

Extension of UFM

Asymptotic fuzzy measures

- Let $\mu = (\mu_n)$ be a UFM and $A \subset \mathbb{N}$. The sequence $(\mu_n(A \cap \mathbf{n}))$ contains some information on the size of A.
- Define

$$\underline{\mu}^*(A) = \liminf_{n\to\infty} \mu_n(A \cap \mathbf{n}), \quad \overline{\mu}^*(A) = \limsup_{n\to\infty} \mu_n(A \cap \mathbf{n})$$

and $\mu^*(A) = \lim_{n\to\infty} \mu_n(A \cap \mathbf{n})$ if the limit exists.

Extension of UFM

Asymptotic density

- **Example (continuation)**: Let $A \subset \mathbb{N}$. Define

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \mathbf{n}|}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \mathbf{n}|}{n}.$$

  We call $\underline{d}$ and $\overline{d}$ the *lower and upper asymptotic densities*, respectively. If $\underline{d}(A) = \overline{d}(A)$ the common value is referred as *asymptotic density* and denoted by $d(A)$.

- For a set $A \subset \mathbb{N}$ and positive real numbers $a < b$ we will denote by $A(m, n)$ the cardinality of the set $A \cap [a, b)$ in the sequel.

Extension of UFM

Asymptotic density

- **Example (continuation)**: Let $A \subset \mathbb{N}$. Define

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \mathbf{n}|}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \mathbf{n}|}{n}.$$

  We call $\underline{d}$ and $\overline{d}$ the *lower and upper asymptotic densities*, respectively. If $\underline{d}(A) = \overline{d}(A)$ the common value is referred as *asymptotic density* and denoted by $d(A)$.
- For a set $A \subset \mathbb{N}$ and positive real numbers $a < b$ we will denote by $A(m, n)$ the cardinality of the set $A \cap [a, b)$ in the sequel.

Other important densities

Pólya density

- Let $A \subset \mathbb{N}$ and $\delta > 0$. Define

$$\underline{p}_\delta(A) = \liminf_{n\to\infty} \frac{A(n\,,(1+\delta)n)}{\delta n},$$

$$\overline{p}_\delta(A) = \limsup_{n\to\infty} \frac{A(n\,,(1+\delta)n)}{\delta n}$$

and

- $$\underline{p}(A) = \lim_{\delta\to 0^+} \underline{p}_\delta(A) \quad \text{and} \quad \overline{p}(A) = \lim_{\delta\to 0^+} \overline{p}_\delta(A).$$

We call $\underline{p}$ and $\overline{p}$ the *lower and upper Pólya densities*, respectively. If $\underline{p}(A) = \overline{p}(A)$ the common value is called *Pólya density* and denoted by $p(A)$.

Other important densities

Pólya density

- Let $A \subset \mathbb{N}$ and $\delta > 0$. Define

$$\underline{p}_\delta(A) = \liminf_{n\to\infty} \frac{A(n\,,(1+\delta)n)}{\delta n},$$

$$\overline{p}_\delta(A) = \limsup_{n\to\infty} \frac{A(n\,,(1+\delta)n)}{\delta n}$$

and

- $$\underline{p}(A) = \lim_{\delta\to 0^+} \underline{p}_\delta(A) \quad \text{and} \quad \overline{p}(A) = \lim_{\delta\to 0^+} \overline{p}_\delta(A).$$

We call $\underline{p}$ and $\overline{p}$ the *lower and upper Pólya densities*, respectively. If $\underline{p}(A) = \overline{p}(A)$ the common value is called *Pólya density* and denoted by $p(A)$.

Other important densities

Banach (uniform) density

- Let $A \subset \mathbb{N}$ and $k \in \mathbb{N}$. Define

$$\underline{b}_k(A) = \liminf_{n\to\infty} \frac{A(n\,, n+k)}{k},$$

$$\overline{b}_k(A) = \limsup_{n\to\infty} \frac{A(n\,, n+k)}{k}$$

and

- $$\underline{b}(A) = \lim_{k\to\infty} \underline{b}_k \quad \text{and} \quad \overline{b}(A) = \lim_{k\to\infty} \overline{b}_k(A).$$

  We call $\underline{b}$ and $\overline{b}$ the *lower and upper Banach (uniform) densities*, respectively. If $\underline{b}(A) = \overline{b}(A)$ the common value is called *Banach density* and denoted by $b(A)$.

Other important densities

Banach (uniform) density

- Let $A \subset \mathbb{N}$ and $k \in \mathbb{N}$. Define

$$\underline{b}_k(A) = \liminf_{n\to\infty} \frac{A(n\,, n+k)}{k},$$

$$\overline{b}_k(A) = \limsup_{n\to\infty} \frac{A(n\,, n+k)}{k}$$

and

- $$\underline{b}(A) = \lim_{k\to\infty} \underline{b}_k \quad \text{and} \quad \overline{b}(A) = \lim_{k\to\infty} \overline{b}_k(A).$$

We call $\underline{b}$ and $\overline{b}$ the *lower and upper Banach (uniform) densities*, respectively. If $\underline{b}(A) = \overline{b}(A)$ the common value is called *Banach density* and denoted by $b(A)$.

Most used densities

Inequalities

- All the three mentioned densities are related by the well known chain of inequalities

$$0 \leq \underline{b}(A) \leq \underline{p}(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{p}(A) \leq \overline{b}(A) \leq 1$$

holding for every $A \subset \mathbb{N}$.

- Moreover, the equality $\underline{d}(A) = \overline{d}(A)$ implies also the equality $\underline{p}(A) = \overline{p}(A)$.

Most used densities

Inequalities

- All the three mentioned densities are related by the well known chain of inequalities

$$0 \le \underline{b}(A) \le \underline{p}(A) \le \underline{d}(A) \le \overline{d}(A) \le \overline{p}(A) \le \overline{b}(A) \le 1$$

holding for every $A \subset \mathbb{N}$.
- Moreover, the equality $\underline{d}(A) = \overline{d}(A)$ implies also the equality $\underline{p}(A) = \overline{p}(A)$.

Cardinal densities

Aim of the talk

- The aim of this contribution is to investigate a large class of fuzzy measures (or simply "densities") on $\mathbb{N}$ defined by use of cardinality of sets of positive integers. We will call them *cardinal densities*.

Cardinal densities

Φ-densities

- Denote by Φ the set of all non-decreasing functions defined on $\mathbb{N}$ with values in $\mathbb{R}^+$.
- For every $\phi \in \Phi$ and $A \subset \mathbb{N}$ define the lower and upper ϕ-density, respectively by

$$\underline{d}_\phi(A) = \liminf_{n\to\infty} \frac{A(n\,, n+\phi(n))}{\phi(n)}$$

$$\overline{d}_\phi(A) = \limsup_{n\to\infty} \frac{A(n\,, n+\phi(n))}{\phi(n)}.$$

Cardinal densities

Φ-densities

- Denote by Φ the set of all non-decreasing functions defined on $\mathbb{N}$ with values in $\mathbb{R}^+$.
- For every $\phi \in \Phi$ and $A \subset \mathbb{N}$ define the lower and upper ϕ-density, respectively by

$$\underline{d}_\phi(A) = \liminf_{n\to\infty} \frac{A(n\,, n+\phi(n))}{\phi(n)}$$

$$\overline{d}_\phi(A) = \limsup_{n\to\infty} \frac{A(n\,, n+\phi(n))}{\phi(n)}.$$

Cardinal densities

A special quantified partial order

- For $\phi, \psi \in \Phi$ define

$$q(\phi,\psi) = \inf\{c > 0 \mid \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N},\ n \geq n_0\ :$$

$$\phi(n + \lfloor \psi(n) \rfloor) \leq c\psi(n)\}.$$

- For $\phi, \psi \in \Phi$ denote $\phi \leq^* \psi$ if and only if $\phi(n) \leq \psi(n)$ for all sufficiently large $n \in \mathbb{N}$. Note that $q(\phi,\psi) \leq 1$ implies $\phi \leq^* \psi$.
- Also note that $q(\phi,\psi)q(\psi,\phi) \geq 1$.

Cardinal densities

A special quantified partial order

- For $\phi, \psi \in \Phi$ define

$$q(\phi,\psi) = \inf\{c > 0 \mid \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N},\ n \geq n_0\ :$$

$$\phi(n + \lfloor \psi(n) \rfloor) \leq c\psi(n)\}.$$

- For $\phi, \psi \in \Phi$ denote $\phi \leq^* \psi$ if and only if $\phi(n) \leq \psi(n)$ for all sufficiently large $n \in \mathbb{N}$. Note that $q(\phi,\psi) \leq 1$ implies $\phi \leq^* \psi$.
- Also note that $q(\phi,\psi)q(\psi,\phi) \geq 1$.

Cardinal densities

A special quantified partial order

- For $\phi, \psi \in \Phi$ define

$$q(\phi,\psi) = \inf\{c > 0 \mid \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N},\ n \geq n_0\ :$$

$$\phi(n + \lfloor \psi(n) \rfloor) \leq c\psi(n)\}.$$

- For $\phi, \psi \in \Phi$ denote $\phi \leq^* \psi$ if and only if $\phi(n) \leq \psi(n)$ for all sufficiently large $n \in \mathbb{N}$. Note that $q(\phi,\psi) \leq 1$ implies $\phi \leq^* \psi$.
- Also note that $q(\phi,\psi)q(\psi,\phi) \geq 1$.

Comparison of cardinal densities

Order results

- Very roughly spoken, $\underline{d}_\phi$ becomes greater with ϕ being greater, on the other hand $\overline{d}_\phi$ becomes smaller with greater ϕ. More precisely, we have the following lemma which is useful if $q(\phi, \psi)$ is close to 0.
- **Lemma.** Let $\phi, \psi \in \Phi$. Then $\underline{d}_\psi \geq \underline{d}_\phi - q(\phi, \psi)$ and $\overline{d}_\psi \leq \overline{d}_\phi + q(\phi, \psi)$.
- **Corollary.** If $q(\phi, \psi) = 0$ then $\underline{d}_\psi \geq \underline{d}_\phi$ and $\overline{d}_\psi \leq \overline{d}_\phi$.

Comparison of cardinal densities

Order results

- Very roughly spoken, $\underline{d}_\phi$ becomes greater with ϕ being greater, on the other hand $\overline{d}_\phi$ becomes smaller with greater ϕ. More precisely, we have the following lemma which is useful if $q(\phi, \psi)$ is close to 0.
- **Lemma.** Let $\phi, \psi \in \Phi$. Then $\underline{d}_\psi \geq \underline{d}_\phi - q(\phi, \psi)$ and $\overline{d}_\psi \leq \overline{d}_\phi + q(\phi, \psi)$.
- **Corollary.** If $q(\phi, \psi) = 0$ then $\underline{d}_\psi \geq \underline{d}_\phi$ and $\overline{d}_\psi \leq \overline{d}_\phi$.

Comparison of cardinal densities

Order results

- Very roughly spoken, $\underline{d}_\phi$ becomes greater with ϕ being greater, on the other hand $\overline{d}_\phi$ becomes smaller with greater ϕ. More precisely, we have the following lemma which is useful if $q(\phi,\psi)$ is close to 0.
- **Lemma.** Let $\phi,\psi \in \mathit{\Phi}$. Then $\underline{d}_\psi \geq \underline{d}_\phi - q(\phi,\psi)$ and $\overline{d}_\psi \leq \overline{d}_\phi + q(\phi,\psi)$.
- **Corollary.** If $q(\phi,\psi) = 0$ then $\underline{d}_\psi \geq \underline{d}_\phi$ and $\overline{d}_\psi \leq \overline{d}_\phi$.

Construction of general cardinal densities

$\wedge$ and $\vee$-densities

- Let $\mathcal{F} \subset \Phi$. For every $A \subset \mathbb{N}$ define

$$\wedge\underline{d}_{\mathcal{F}}(A) = \inf\{\underline{d}_\phi(A) \mid \phi \in \mathcal{F}\},$$

$$\vee\underline{d}_{\mathcal{F}}(A) = \sup\{\underline{d}_\phi(A) \mid \phi \in \mathcal{F}\}$$

and

-
$$\wedge\overline{d}_{\mathcal{F}}(A) = \inf\{\overline{d}_\phi(A) \mid \phi \in \mathcal{F}\},$$

$$\vee\overline{d}_{\mathcal{F}}(A) = \sup\{\overline{d}_\phi(A) \mid \phi \in \mathcal{F}\}.$$

Construction of general cardinal densities

$\wedge$ and $\vee$-densities

- Let $\mathcal{F} \subset \Phi$. For every $A \subset \mathbb{N}$ define

$$\wedge\underline{d}_{\mathcal{F}}(A) = \inf\{\underline{d}_{\phi}(A) \mid \phi \in \mathcal{F}\},$$

$$\vee\underline{d}_{\mathcal{F}}(A) = \sup\{\underline{d}_{\phi}(A) \mid \phi \in \mathcal{F}\}$$

  and

- $$\wedge\overline{d}_{\mathcal{F}}(A) = \inf\{\overline{d}_{\phi}(A) \mid \phi \in \mathcal{F}\},$$
  $$\vee\overline{d}_{\mathcal{F}}(A) = \sup\{\overline{d}_{\phi}(A) \mid \phi \in \mathcal{F}\}.$$

Classification of cardinal densities

Four natural classes

- We have seen that the values of both the lower and upper ϕ-densities are somehow dependent on the order of growth of the function ϕ. Thus we will classify the elements of the set $\mathcal{F}$ as follows. Let

- $$\Phi_1 = \{\phi \in \Phi \mid \phi \text{ is bounded}\},$$

- $$\Phi_2 = \{\phi \in \Phi \mid \phi \text{ is unbounded and } \phi(n) = o(n)\},$$

- $$\Phi_3 = \{\phi \in \Phi \mid \phi \approx n,$$
  $$\text{i.e. } \exists\, 0 < c < C \mid \forall n \in \mathbb{N}\ cn \leq \phi(n) \leq Cn\},$$

- $$\Phi_4 = \{\phi \in \Phi \mid n = o(\phi(n))\}.$$

Classification of cardinal densities

Four natural classes

- We have seen that the values of both the lower and upper ϕ-densities are somehow dependent on the order of growth of the function ϕ. Thus we will classify the elements of the set $\mathcal{F}$ as follows. Let
- $$\Phi_1 = \{\phi \in \Phi \mid \phi \text{ is bounded}\},$$
- $$\Phi_2 = \{\phi \in \Phi \mid \phi \text{ is unbounded and } \phi(n) = o(n)\},$$
- $$\Phi_3 = \{\phi \in \Phi \mid \phi \approx n,$$ $$\text{i.e. } \exists\, 0 < c < C \mid \forall n \in \mathbb{N}\ cn \le \phi(n) \le Cn\},$$
- $$\Phi_4 = \{\phi \in \Phi \mid n = o(\phi(n))\}.$$

Classification of cardinal densities

Four natural classes

- We have seen that the values of both the lower and upper ϕ-densities are somehow dependent on the order of growth of the function ϕ. Thus we will classify the elements of the set $\mathcal{F}$ as follows. Let

- $$\Phi_1 = \{\phi \in \Phi \mid \phi \text{ is bounded}\},$$

- $$\Phi_2 = \{\phi \in \Phi \mid \phi \text{ is unbounded and } \phi(n) = o(n)\},$$

- $$\Phi_3 = \{\phi \in \Phi \mid \phi \approx n,$$
  $$\text{i.e. } \exists\ 0 < c < C \mid \forall n \in \mathbb{N}\ cn \le \phi(n) \le Cn\},$$

- $$\Phi_4 = \{\phi \in \Phi \mid n = o(\phi(n))\}.$$

Classification of cardinal densities

Four natural classes

- We have seen that the values of both the lower and upper ϕ-densities are somehow dependent on the order of growth of the function ϕ. Thus we will classify the elements of the set $\mathcal{F}$ as follows. Let

- $$\Phi_1 = \{\phi \in \Phi \mid \phi \text{ is bounded}\},$$

- $$\Phi_2 = \{\phi \in \Phi \mid \phi \text{ is unbounded and } \phi(n) = o(n)\},$$

- $$\Phi_3 = \{\phi \in \Phi \mid \phi \approx n,$$ $$\text{i.e. } \exists\ 0 < c < C \mid \forall n \in \mathbb{N}\ cn \leq \phi(n) \leq Cn\},$$

- $$\Phi_4 = \{\phi \in \Phi \mid n = o(\phi(n))\}.$$

Classification of cardinal densities

Four natural classes

- We have seen that the values of both the lower and upper ϕ-densities are somehow dependent on the order of growth of the function ϕ. Thus we will classify the elements of the set $\mathcal{F}$ as follows. Let
- $$\Phi_1 = \{\phi \in \Phi \mid \phi \text{ is bounded}\},$$
- $$\Phi_2 = \{\phi \in \Phi \mid \phi \text{ is unbounded and } \phi(n) = o(n)\},$$
- $$\Phi_3 = \{\phi \in \Phi \mid \phi \approx n,$$
  $$\text{i.e. } \exists\ 0 < c < C \mid \forall n \in \mathbb{N}\ cn \leq \phi(n) \leq Cn\},$$
- $$\Phi_4 = \{\phi \in \Phi \mid n = o(\phi(n))\}.$$

Classification of cardinal densities

Classification and order of densities

- It can be easily seen that for every $i \in \{1, 2, 3\}$ and every choice of $\phi \in \Phi_i,\ \psi \in \Phi_{i+1}$ we have $\phi \leq^* \psi$. Moreover, also $q(\phi, \psi) = 0$ holds in this case.
- **Corollary** Let $1 \leq i < j \leq 4$ and $\phi \in \Phi_i$ and $\psi \in \Phi(j)$. Then $\underline{d}_\psi \geq \underline{d}_\phi$ and $\overline{d}_\psi \leq \overline{d}_\phi$.

Classification of cardinal densities

Classification and order of densities

- It can be easily seen that for every $i \in \{1,2,3\}$ and every choice of $\phi \in \Phi_i,\ \psi \in \Phi_{i+1}$ we have $\phi \leq^* \psi$. Moreover, also $q(\phi,\psi) = 0$ holds in this case.
- **Corollary** Let $1 \leq i < j \leq 4$ and $\phi \in \Phi_i$ and $\psi \in \Phi(j)$. Then $\underline{d}_\psi \geq \underline{d}_\phi$ and $\overline{d}_\psi \leq \overline{d}_\phi$.

- The following theorem shows that the well-known and frequently used densities form natural bounds of d_ϕ densities for ϕ belonging to above defined classes.
- **Theorem.**

  (i) $\underline{b} = \vee \underline{d}_{\Phi_1}, \quad \overline{b} = \wedge \overline{d}_{\Phi_1}.$

  (ii) $\underline{b} = \wedge \underline{d}_{\Phi_2}, \quad \overline{b} = \vee \overline{d}_{\Phi_2}, \quad \underline{p} = \vee \underline{d}_{\Phi_2}, \quad \overline{p} = \wedge \overline{d}_{\Phi_2}$

  (ii) $\underline{p} = \wedge \underline{d}_{\Phi_3}, \quad \overline{p} = \vee \overline{d}_{\Phi_3} \quad \underline{d} = \vee \underline{d}_{\Phi_3}, \quad \overline{d} = \wedge \overline{d}_{\Phi_3}$

  (iv) $\underline{d} = \wedge \underline{d}_{\Phi_4}, \quad \overline{d} = \vee \overline{d}_{\Phi_4},$

  and for all $A \subset \mathbb{N}$

$$\{\underline{d}_\phi(A) \mid \phi \in \Phi_4\} = \{\overline{d}_\phi(A) \mid \phi \in \Phi_4\} = [\underline{d}(A), \overline{d}(A)].$$

- The following theorem shows that the well-known and frequently used densities form natural bounds of d_ϕ densities for ϕ belonging to above defined classes.
- **Theorem.**
  (i) $\underline{b} = \vee \underline{d}_{\Phi_1}, \quad \overline{b} = \wedge \overline{d}_{\Phi_1}.$
  (ii) $\underline{b} = \wedge \underline{d}_{\Phi_2}, \quad \overline{b} = \vee \overline{d}_{\Phi_2}, \quad \underline{p} = \vee \underline{d}_{\Phi_2}, \quad \overline{p} = \wedge \overline{d}_{\Phi_2}$
  (ii) $\underline{p} = \wedge \underline{d}_{\Phi_3}, \quad \overline{p} = \vee \overline{d}_{\Phi_3} \quad \underline{d} = \vee \underline{d}_{\Phi_3}, \quad \overline{d} = \wedge \overline{d}_{\Phi_3}$
  (iv) $\underline{d} = \wedge \underline{d}_{\Phi_4}, \quad \overline{d} = \vee \overline{d}_{\Phi_4},$

  and for all $A \subset \mathbb{N}$

$$\{\underline{d}_\phi(A) \mid \phi \in \Phi_4\} = \{\overline{d}_\phi(A) \mid \phi \in \Phi_4\} = [\underline{d}(A), \overline{d}(A)].$$

- The following theorem shows that the well-known and frequently used densities form natural bounds of d_ϕ densities for ϕ belonging to above defined classes.
- **Theorem.**
  (i) $\underline{b} = \vee \underline{d}_{\Phi_1}, \quad \overline{b} = \wedge \overline{d}_{\Phi_1}.$
  (ii) $\underline{b} = \wedge \underline{d}_{\Phi_2}, \quad \overline{b} = \vee \overline{d}_{\Phi_2}, \quad \underline{p} = \vee \underline{d}_{\Phi_2}, \quad \overline{p} = \wedge \overline{d}_{\Phi_2}$
  (ii) $\underline{p} = \wedge \underline{d}_{\Phi_3}, \quad \overline{p} = \vee \overline{d}_{\Phi_3} \quad \underline{d} = \vee \underline{d}_{\Phi_3}, \quad \overline{d} = \wedge \overline{d}_{\Phi_3}$
  (iv) $\underline{d} = \wedge \underline{d}_{\Phi_4}, \quad \overline{d} = \vee \overline{d}_{\Phi_4},$

  and for all $A \subset \mathbb{N}$

$$\{\underline{d}_\phi(A) \mid \phi \in \Phi_4\} = \{\overline{d}_\phi(A) \mid \phi \in \Phi_4\} = [\underline{d}(A), \overline{d}(A)].$$

- The following theorem shows that the well-known and frequently used densities form natural bounds of d_ϕ densities for ϕ belonging to above defined classes.
- **Theorem.**
  (i) $\underline{b} = \vee \underline{d}_{\Phi_1}, \quad \overline{b} = \wedge \overline{d}_{\Phi_1}.$
  (ii) $\underline{b} = \wedge \underline{d}_{\Phi_2}, \quad \overline{b} = \vee \overline{d}_{\Phi_2}, \quad \underline{p} = \vee \underline{d}_{\Phi_2}, \quad \overline{p} = \wedge \overline{d}_{\Phi_2}$
  (ii) $\underline{p} = \wedge \underline{d}_{\Phi_3}, \quad \overline{p} = \vee \overline{d}_{\Phi_3} \quad \underline{d} = \vee \underline{d}_{\Phi_3}, \quad \overline{d} = \wedge \overline{d}_{\Phi_3}$
  (iv) $\underline{d} = \wedge \underline{d}_{\Phi_4}, \quad \overline{d} = \vee \overline{d}_{\Phi_4},$

  and for all $A \subset \mathbb{N}$

$$\{\underline{d}_\phi(A) \mid \phi \in \Phi_4\} = \{\overline{d}_\phi(A) \mid \phi \in \Phi_4\} = [\underline{d}(A), \overline{d}(A)].$$

- The following theorem shows that the well-known and frequently used densities form natural bounds of d_ϕ densities for ϕ belonging to above defined classes.
- **Theorem.**

(i) $\underline{b} = \vee \underline{d}_{\Phi_1}, \quad \overline{b} = \wedge \overline{d}_{\Phi_1}.$

(ii) $\underline{b} = \wedge \underline{d}_{\Phi_2}, \quad \overline{b} = \vee \overline{d}_{\Phi_2}, \quad \underline{p} = \vee \underline{d}_{\Phi_2}, \quad \overline{p} = \wedge \overline{d}_{\Phi_2}$

(ii) $\underline{p} = \wedge \underline{d}_{\Phi_3}, \quad \overline{p} = \vee \overline{d}_{\Phi_3} \quad \underline{d} = \vee \underline{d}_{\Phi_3}, \quad \overline{d} = \wedge \overline{d}_{\Phi_3}$

(iv) $\underline{d} = \wedge \underline{d}_{\Phi_4}, \quad \overline{d} = \vee \overline{d}_{\Phi_4},$

and for all $A \subset \mathbb{N}$

$$\{\underline{d}_\phi(A) \mid \phi \in \Phi_4\} = \{\overline{d}_\phi(A) \mid \phi \in \Phi_4\} = [\underline{d}(A), \overline{d}(A)].$$