Construction methods for bivariate copulas

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Introduction

Farlie–Gumbel–Morgenstern copulas

$$C_\lambda^{FGM}(x,y) = x \cdot y + \lambda x \cdot y \cdot (1-x) \cdot (1-y), \tag{1}$$

where $\lambda \in [-1,1]$

For a given copula $C : [0,1]^2 \to [0,1]$, we will look for constraints on the noise $H : [0,1]^2 \to \Re$ so that the function $C_H : [0,1]^2 \to [0,1]$ given by

$$C_H(u,v) = \max\left(0,\ C(u,v) + H(u,v)\right) \tag{2}$$

is also a copula. Obviously FGM copulas given by (1) are linked to $C = \Pi$ and $H_\alpha(u,v) = \alpha * u * (1-u) * v * (1-v)$ (observe that in this case, no truncation is necessary).

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As another example, consider the comonotonicity copula M, $M(u,v) = \min(u,v)$, and a parametric noise family $H_\alpha : [0,1]^2 \to [0,1],\ \alpha \in [0,1]$ given by $H_\alpha(u,v) = \alpha\,(\max(u,v) - 1)$. Then

$$M_{H_\alpha}(u,v) = \max\left(0, (1-\alpha) * M(u,v) + \alpha * (u+v-1)\right)$$

defines a singular copula with support on 3 segments connecting the point $\left(\frac{\alpha}{1+\alpha}, \frac{\alpha}{1+\alpha}\right)$ with vertices $(0,1),(1,1)$ and (1,0) (if $\alpha = 0,\ M_{H_0} = M$; if $\alpha = 1,\ M_{H_1} = W$ is the countermonotonicity copula given by $W(u,v) = \max\left(0, u+v-1\right)$).

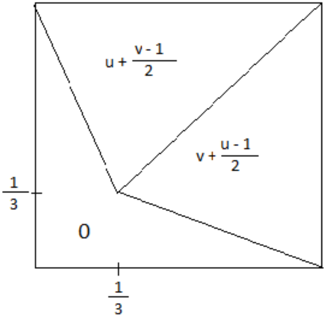

Figure: Formulae for the copula $M_{H_{\frac{1}{2}}}(u,v)$

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A general formula for constructing bivariate copulas

If C is a singular copula, the function $H \neq 0$ cannot be absolutely continuous. Similarly, if C is an absolutely continuous copula, H cannot be singular. Therefore, as a special case of the modification (2), one can deal with modification related to functions $f, g : [0,1] \to [0,1]$ and constant $\lambda \in \Re$

$$C_{\lambda,f,g}(u,v) = \max\left(0,\ C(u,v) + \lambda * C(f(u), f(v))\right). \tag{3}$$

Obviously, FGM family given in (1) can be seen as a special case of construction (3), considering $C = \Pi,\ \lambda \in [-1, 1]$ and $f = g$ given by $f(x) = x - x^2$. Note that as a necessary condition to ensure that $e = 1$ is a neutral element of $C_{\lambda,f,g}$, one should consider $f(1) = g(1) = 0$. On the other hand, if $\lambda \leq 0$, then $C_{\lambda,f,g}$ is always grounded. However, if $\lambda > 0$, then one should consider $f(0) = g(0) = 0$.

Proposition 1

Let $f, g : [0,1] \to [0,1]$ be Lipschitz functions with Lipschitz Constants *c* and *d*, respectively, and let $f(0) = g(0) = f(1) = g(1) = 0$. Then the function $\Pi_{\lambda,f,g} : [0,1]^2 \to [0,1]$ given by

$$\Pi_{\lambda,f,g}(u,v) = \max\left(0, u * v + \lambda * f(u) * g(v)\right)$$

is a copula whenever $|\lambda * c * d| \leq 1$.

Proposition 2

Let $N : [0,1] \to [0,1]$ be an involutive decreasing bijection (i.e., a strong negation) such that it is 1–Lipschitz on the interval $[k, 1]$, where *k* is the fixed point of *N*, $N(k) = k$. Then the function $M_{-1,N,N} : [0,1]^2 \to [0,1]$ given by

$$M_{-1,N,N}(u, v) = \max\left(0, \min(u, v) - \min(N(u), N(v))\right)$$

is a copula.

Proposition 3

Let $C : [0,1]^2 \to [0,1]$ be a copula and $H : [0,1]^2 \to [0,1]$ be a function so that $C + H \geq 0$ and C_H is copula, i.e., $C_H = C + H$ is a copula. Then also $C_{\lambda * H} = C + \lambda * H$ is a copula for each $\lambda \in [0,1]$.

Proposition 4

Under the constraints of Proposition 3, the function $\hat{C}_{\bar{H}}$ is a copula, where $\hat{C} : [0,1]^2 \to [0,1]$ is the survival copula related to C,

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v),$$

and $\bar{H} : [0,1]^2 \to [0,1]$ is given by

$$\bar{H}(u,v) = H(1-u, 1-v).$$

Theorem 1

Let $C : [0,1]^2 \to [0,1]$ be a copula and define $H_\lambda^C : [0,1]^2 \to [0,1], \lambda \in [0,1]$ by

$$H_\lambda^C(u,v) = \lambda * (u - C(u,v)) * (v - C(u,v)).$$

Then $C_{H_\lambda^C} : [0,1]^2 \to [0,1]$ given by

$$C_{H_\lambda^C}(u,v) = C(u,v) + \lambda * (u - C(u,v)) * (v - C(u,v)) \quad (4)$$

is a copula for each $\lambda \in [0,1]$ and any copula *C*.

Remark 1

There is an interesting relation between the modification method (4) and *M*–based ordinal sums. Indeed, consider $C = M - (\langle a_k, b_k, C_k \rangle \, | k \in \mathcal{K})$. Then

$$C_{H_\lambda^C} = M - \left(\left\langle a_k, b_k, C_{H^{C_k}_{\lambda(b_k - a_k)}} \right\rangle | k \in \mathcal{K} \right).$$

So, for example, if $C = M - \left(\left\langle 0, \frac{1}{2}, W \right\rangle, \left\langle \frac{1}{2}, 1, W \right\rangle\right)$, then $C_{H_1^C} = M - \left(\left\langle 0, \frac{1}{2}, \frac{\Pi + W}{2} \right\rangle, \left\langle 0, \frac{1}{2}, \frac{\Pi + W}{2} \right\rangle\right)$, because of $C^W_{H_{0.5}} = \frac{\Pi + W}{2}$.

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Modifications of the product copula Π

Observe that each polynomial p is Lipschitz on $[0, 1]$, and considering the constraints $p(0) = p(1) = 0$, clearly $p : [0, 1] \to \Re$ is given by $p(x) = x * (1 - x) * q(x)$, where q is some polynomial (possibly a constant, as a polynomial of degree 0). Hence for any polynomials q, h there is non–zero constant λ so that for $H_\lambda : [0, 1]^2 \to \Re$ given by

$$H_\lambda(u, v) = \lambda * u * (1 - u) * v * (1 - v) * q(u) * h(v)$$

it holds $\Pi_{H_\lambda} = \Pi + H_\lambda$, and Π_{H_λ} is a copula. Obviously, the set of all such constants λ is a closed interval $[\alpha, \beta]$ such that $\alpha < 0 < \beta$.

Theorem 2

Let $N : [0,1] \to [0,1]$ be a convex strong negation. Then the function $\Pi_N : [0,1]^2 \to [0,1]$ given by

$$\Pi_N(u,v) = \max\left(0,\ u * v - N(u) * N(v)\right) \tag{5}$$

is a negative quadrant dependent copula.

Corollary 1

Let $N : [0,1] \to [0,1]$ be a convex strong negation, and let $\lambda \in]0,1]$. Define $f_\lambda : [0,1] \to [0,1]$ by

$$f_\lambda(x) = \lambda * N\left(\min\left(1, \frac{x}{\lambda}\right)\right).$$

Then $\Pi_{f_\lambda} : [0,1]^2 \to [0,1]$ given by

$$\Pi_{f_\lambda}(u,v) = \max\left(0, u * v - \lambda^2 * N\left(\min\left(1, \frac{u}{\lambda}\right)\right) * N\left(\min\left(1, \frac{v}{\lambda}\right)\right)\right)$$

is a negative quadrant dependent copula.

Example 1

Consider the standard negation given by $N(x) = 1 - x$. Observe that for any $u, v \in]0,1[,\ N'(u) = N'(v) = -1$, and thus $1 \geq \lambda^2 * N'(u) * N'(v) = \lambda^2$ for any $\lambda \in]0,1]$. Then
1) $\Pi_{g_\lambda} : [0,1]^2 \to [0,1]$ given by

$$\Pi_{g_\lambda}(u,v) = \max\big(0, u * v - \lambda^2 * (1-u) * (1-v)\big) =$$

$$= \max\big(0, (1-\lambda^2) * u * v + \lambda^2 * (u + v - 1)\big)$$

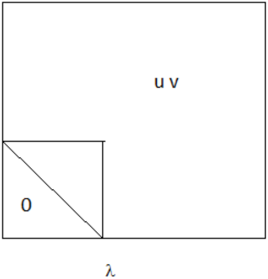
is a Sugeno–Weber copula for any $\lambda \in]0,1]$;

Example 1a

2) $\Pi_{f_\lambda} : [0,1]^2 \to [0,1]$ given by

$$\Pi_{f_\lambda}(u,v) = \begin{cases} \max(0, \lambda * u + \lambda * v - \lambda^2) & \text{if } (u,v) \in [0,\lambda]^2, \\ u * v & else \end{cases}$$

is a copula for each $\lambda \in]0,1]$. Observe that Π_{f_λ} has a singular part with mass λ^2 uniformly distributed over the segment connecting points $(0,\lambda)$ and $(\lambda,0)$, and its absolutely continuous part has density 1 on the domain $[0,1]^2 \setminus [0,\lambda]^2$. Observe that each copula Π_{f_λ} can be seen as a Π–ordinal sum, $\Pi_{f_\lambda} = \Pi - (\langle 0, \lambda, W \rangle)$.



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Theorem 3

Let $f : [0,1] \to [0,1]$ be a non–increasing function such that $f(1) = 0,\ f(k) = k$ for some $k \in]0,1[$, and f is 1–Lipschitz on the interval $[k,1]$. Then the function $M_{-1,f,f} : [0,1]^2 \to [0,1]$ given by

$$M_{-1,f,f}(u,v) = \max\left(0, \min(u,v) - \min(f(u), f(v))\right)$$

is a copula.

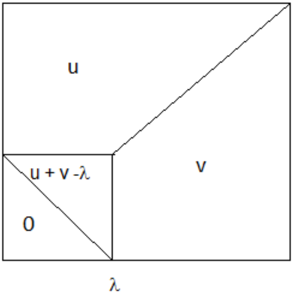
Example 2

For $\lambda \in [0,1]$, let $g_\lambda : [0,1] \to [0,1]$ be given by $g_\lambda(x) = \max(\lambda - x, 0)$. Then $M_{-1,g_\lambda,g_\lambda}$ is a singular copula given by

$$M_{-1,g_\lambda,g_\lambda}(u,v) = \max\left(0, \min(u,v) - \min\left(\max(\lambda - u, 0), \max(\lambda - v, 0)\right)\right).$$

Observe that $M_{-1,g_\lambda,g_\lambda}$ is the *M*–based ordinal sum,

$$M_{-1,g_\lambda,g_\lambda} = M - (\langle 0, \lambda, W \rangle).$$



Theorem 4

Let $f : [0, 1] \to [0, 1]$ be a unimodal 1–Lipschitz function such that $f(0) = f(1) = 0$. Then $f(x) \leq x$ for all $x \in [0, 1]$ and the function $M_{-1,f,f} : [0, 1]^2 \to [0, 1]$ given by

$$M_{-1,f,f}(u, v) = \min(u, v) - \min(f(u), f(v))$$

is a singular copula.

Example 3

For any Archimedean copula C generated by an additive generator $h : [0,1] \to [0,\infty]$ (h is convex, strictly decreasing and $h(1) = 0$),

$$C(u,v) = h^{-1}\left(\min(h(0), h(u) + h(v))\right),$$

its opposite diagonal section

$$f : [0,1] \to [0,1],\ f(x) = h^{-1}\left(\min(h(0), h(x) + h(1-x))\right),$$

satisfies the constraints of Theorem 4 with modal argument $m = \frac{1}{2}$, and $\varphi(x) = 1 - x$.

Example 3a

Consider, e.g., the product copula Π. Then $f(x) = \Pi(x, 1-x) = x * (1-x)$, and for each $\lambda \in [0,1]$,

$$M_{\lambda,f,f}(u,v) = \begin{cases} (1-\lambda)u + \lambda * u^2 & \text{if } u \le v \le 1-u, \\ (1-\lambda)v + \lambda * v^2 & \text{if } v \le u \le 1-v, \\ v + \lambda * (u^2 - u) & \text{if } 1-u \le v \le u, \\ u + \lambda * (v^2 - v) & otherwise, \end{cases}$$

determines a singular copula.

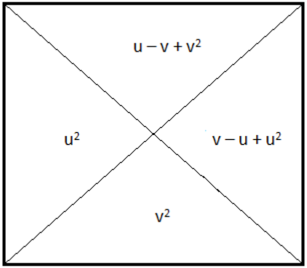


Figure: Formulae for the copula $M_{1,f,f}$

Example 3b

Consider the opposite diagonal section of an ordinal sum copula $C = M - \left(\left\langle 0, \frac{1}{2}, W\right\rangle, \left\langle \frac{1}{2}, 1, W\right\rangle\right)$ given by

$$f(x) = C(x, 1-x) = \begin{cases} \min(x, \frac{1}{2} - x) & \text{if } x \in [0, \frac{1}{2}], \\ \min(x - \frac{1}{2}, 1 - x) & otherwise. \end{cases}$$

Then f is 1–Lipschitz, $f(0) = f(1) = 0$, but f is not unimodal (two modal arguments are $m_1 = \frac{1}{4}$ and $m_3 = \frac{3}{4}$).
Then the section $M_{-1,f,f}(x, \frac{3}{4}) = 2x - 1$ if $x \in [\frac{1}{4}, \frac{1}{2}]$, i.e., $M_{-1,f,f}$ is not 1–Lipschitz and thus not a copula.

Remark 2

Observe that each function $f : [0,1] \to [0,1]$ satisfying the constraints of Theorem 4 which is symmetric wrt. point 0.5, i.e., $f(x) = f(1-x)$ for all $x \in [0,1]$, can be seen as an opposite diagonal section of some symmetric copula. Then also $M_{-1,f,f}$ is a symmetric copula (with support on the main and opposite diagonals) and its opposite diagonal section $g : [0,1] \to [0,1]$ is given by

$$g(x) = M_{-1,f,f}(x, 1-x) = \min(x - f(x), 1 - x - f(1-x)).$$

Thus also $M_{-1,g,g}$ is a symmetric copula and

$$M_{-1,g,g}(x, 1-x) = f(x)$$

for all $x \in [0,1]$.

Remark 2 - continued

Thus the repeated application of Theorem 4 can be seen as a method of constructing a symmetric copula with a given opposite diagonal section. So, for example, consider $f(x) = x * (1 - x)$, the opposite diagonal section of the product copula Π. Then $g(x) = \min(x^2, (1 - x)^2)$, and

$$M_{-1,g,g}(u,v) = \begin{cases} u - u^2 & \text{if } u \leq v \leq 1 - u, \\ v - v^2 & \text{if } v \leq u \leq 1 - v, \\ v - (1 - u)^2 & \text{if } 1 - u \leq v \leq u, \\ u - (1 - v)^2 & otherwise, \end{cases}$$

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Theorem 5

Let $C : [0,1]^2 \to [0,1]$ be a copula. Then the following are equivalent:

i) C is radially symmetric (C and its survival copula $\hat{C}$ coincide), i.e.,

$$C(u,v) = \hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

ii) C_{-1,N_s,N_s} is a copula;

iii) $C_{-1,N_s,N_s} = W$.

Corollary 2

Let C be a radially symmetric copula and let $\lambda \in [0, 1]$. Then the function $C_{-\lambda, N_s, N_s} : [0, 1]^2 \to [0, 1]$ given by

$$C_{-\lambda, N_s, N_s}(u, v) = \max\left(0, C(u, v) - \lambda * C(1 - u, 1 - v)\right)$$

is a copula.

Example 4

Consider the ordinal sum copula $C = M - \left(\left\langle 0, \frac{1}{2}, W \right\rangle, \left\langle \frac{1}{2}, 1, W \right\rangle\right)$. Then $C = \hat{C}$, and $C_{-\lambda, N_s, N_s}$, $\lambda \in [0,1]$, is a singular copula depicted in Figure.

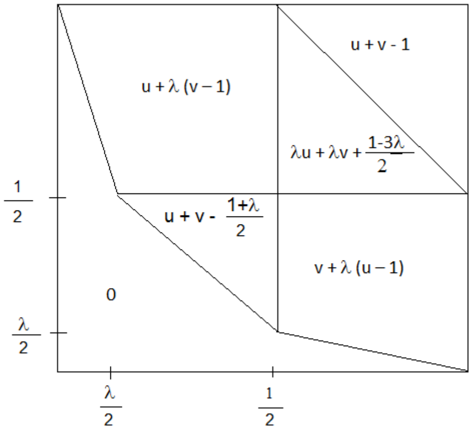

Figure: Description of copula $C_{-\lambda, N_s, N_s}$

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Concluding remarks

We have introduced several methods for modifications of binary copulas, resulting to new construction methods for copulas. As an important topic for the further investigation we open the problem of modification of copulas of higher dimensions. Some extensions of results recalled or introduced in this paper are obvious. For example, considering the approach linked to FGM family it is not difficult to check that for each 1–Lipschitz functions $f_i : [0,1] \to [0,1]$ such that $f_i(0) = f_i(1) = 0,\ i = 1, \cdots, n$, the function $D_\lambda : [0,1]^n \to [0,1]$ given by

$$D_\lambda(u_1, \cdots, u_n) = \prod_{i=1}^{n} u_i + \lambda \prod_{i=1}^{n} f_i(u_i)$$

is an n–ary copula for each $\lambda \in [-1, 1]$.

Concluding remarks

On the other side, we cannot directly generalize Theorem 4. Considering the function $D : [0,1]^n \to [0,1]$ given by

$$D(u_1, \cdots, u_n) = \max\left(0, \min(u_1, \cdots, u_n) - \min(1-u_1, \cdots, 1-u_n)\right),$$

it is not difficult to check that for each 2–dimensional marginal function $D_{i,j}$ with fixed $u_k = 1$ whenever $k \notin \{i, j\}$ it holds

$$D_{i,j}(u_i, u_j) = \max\left(0, \min(u_i, u_j) - 0\right) = \min(u_i, u_j).$$

Supposing D is a copula, then $D_{i,j} = M$ for each $i, j \in \{1, \cdots, n\}$, $i \neq j$, if and only if $D(u_1, \cdots, u_n) = \min(u_1, \cdots, u_n)$ for all $(u_1, \cdots, u_n) \in [0,1]^n$, what is a contradiction whenever $n > 2$.

Thanks for your attention