DIAGONAL COPULAS AND QUASI-COPULAS

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FSTA 2014, January 28, 2014

Contents

Diagonal section of n-dimensional copula

Definition 1.
For an n-dimensional copula

$$C : [0,1]^n \to [0,1]\,, n \geq 2,$$

its diagonal section $\delta_C(x)$ is defined by

$$\delta_C\,(x) = C\,(x, ..., x)\,.$$

We will discus the reverse problem, i.e., how to find for an a priori given diagonal section $\delta : [0,1]^n \to [0,1]$ (of some unknown copula) an n-dimensional copula $C : [0,1]^n \to [0,1]$ so that $\delta = \delta_C$.

Let for a fixed $n \in \{2, 3, ...\}$, $\mathcal{C}_n$ be the class of all n-dimensional copulas and $\mathcal{D}_n$ be the class of all diagonal sections of copulas from $\mathcal{C}_n$.

If the function $d : [0,1] \longrightarrow [0,1]$ is an element of $\mathcal{D}_n$ then it satisfies the next conditions:

$(D1)$ d is non-decreasing,

$(D2)$ $d \leq id_{[0,1]}$,

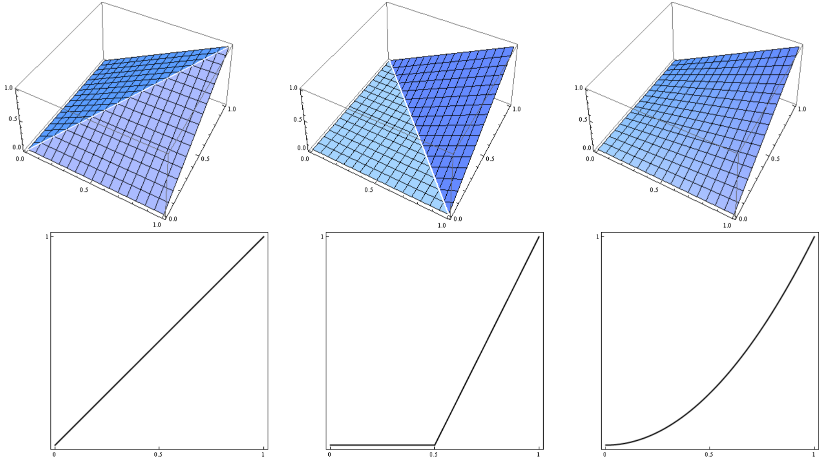
$(D3)$ $d(1) = 1$,

$(D4)$ d is n-Lipschitz, i.e., $|d(x) - d(y)| \leq n|x - y|$ for all $x, y \in [0,1]$.

Proposition 1.

Let $d : [0, 1] \to [0, 1]$ be a function and $n \in \{2, 3, ...\}$ be a fixed dimension. Then d is a diagonal section of some n-dimensional copula, i.e., $d \in \mathcal{D}_n$ if and only if d satisfies conditions (D1) - (D4).

Copulas $M(x,y), W(x,y), \Pi(x,y)$ and their diagonal sections δ_M, δ_W and δ_Π

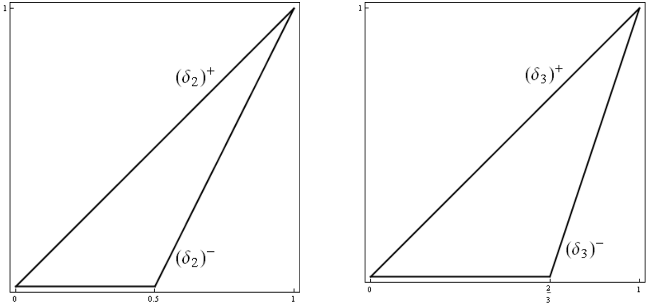


- The classes $\mathcal{C}_n$ and $\mathcal{D}_n$ are convex.
- $\mathcal{D}_n$ is closed under suprema (infima).
- The smallest element of $\mathcal{D}_n$ is given by

$$d_n^-(x) = max(0, nx - n + 1),$$

while its greatest element is given by $d_n^+(x) = x$.
- The class $\mathcal{C}_n$ is not closed under suprema (infima).
- The greatest element of $\mathcal{C}_n$ is the comonotonicity copula M, $M(x_1, ..., x_n) = min(x_1, ..., x_n)$.
- The smallest element in $\mathcal{C}_n,\ n > 2$ does not exist.
- In the case of $\mathcal{C}_2$, the smallest element is the countermonotonicity copula W, $W(x_1, x_2) = max(0, x_1 + x_2 - 1)$.

The smallest and the greatest elements of $\mathcal{D}_n$ for $n = 2$ and $n = 3$



Bertino copulas

Bertino copulas [a]

For any $d \in \mathcal{D}_2$, the function $B_d : [0,1]^2 \to [0,1]$ given by

$$B_d(x,y) = \bigvee_{t \in [x \wedge y, x \vee y]} \left(d(t) - (t-x)^+ - (t-y)^+ \right)^+, \quad (1)$$

where $u^+ = max(u,0)$ for $u \in R$, is a copula. B_d is the smallest copula with diagonal section d, and it is simultaneously the smallest quasi-copula possessing diagonal section d.

[a] (Bertino, S., 1977, Fredricks, G.A., Nelsen, R.B., 2002)

Diagonal copulas

Diagonal copulas[a]

For any $d \in \mathcal{D}_2$, the function $K_d : [0,1]^2 \to [0,1]$ given by

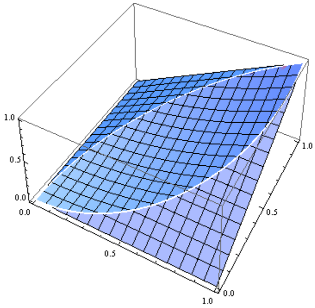
$$K_d(x,y) = min\left(x, y, \frac{d(x)+d(y)}{2}\right) \tag{2}$$

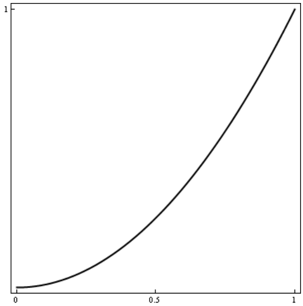
is a copula.

K_d is the greatest symmetric copula with diagonal section d, but not necessarily the greatest one.

[a](Fredricks, G.A., Nelsen, R.B., 1997)

Copula $K_d(x, y)$ for $d(x) = x^2$





There are several other constructions of a copula with an a priori given diagonal section d, however, these methods are not universal, they can be applied to diagonal sections from some special subdomains of $\mathcal{D}_2$.

This is, for example, the case of semilinear copulas, biconic copulas, or the construction methods based on patchwork techniques.

Proposition 2.
Let A and B be symmetric copulas from $\mathcal{C}_2$ with the same diagonal section $d \in \mathcal{D}_2$. Then the function $C_{A,B} : [0,1]^2 \to [0,1]$ given by

$$C_{A,B}(x,y) = \begin{cases} A(x,y) & if \quad x \leq y, \\ B(x,y) & else, \end{cases} \tag{3}$$

is a copula from $\mathcal{C}_2$, and $d_{C_{A,B}} = d_A = d_B = d$.

The proposition allow to introduce for any $d \in \mathcal{D}_2$ two copulas C_{B_d,K_d} and C_{K_d,B_d} with diagonal section d.
For any $d \in \mathcal{D}_2,\ d \neq d^+$, $card\{B_d, K_d, C_{B_d,K_d}, C_{K_d,B_d}\} = 4$.

n-dimensional diagonal copulas

Proposition 3.

For a fixed $n \in \{2, 3, ...\}$, let $d \in \mathcal{D}_n$. Then the function

$$J_d : [0,1]^n \to [0,1]$$

given by

$$J_d\left(x_1, ..., x_n\right) = \frac{1}{n} \sum_{i=1}^{n} min\left(f\left(x_{i+1}\right), ..., f\left(x_{i+n-1}\right), d\left(x_{i+n}\right)\right) \tag{4}$$

where $f : [0,1] \to [0,1]$ is given by

$$f\left(x\right) = \frac{nx - d\left(x\right)}{n-1}$$

and $x_j = x_{j-n}$ for $j \in \{n+1, ..., 2n\}$, is a copula, $J_d \in \mathcal{C}_n$.

For $n = 2$, $f(x) = 2x - d(x)$, and

$$J_d(x_1, x_2) = \frac{1}{2}\left(min\left(2x_2 - d(x_2), d(x_1)\right) + min\left(2x_1 - d(x_1), d(x_2)\right)\right) =$$

$$= min\left(x_1, x_2, \frac{d(x_1) + d(x_2)}{2}\right) = K_d(x_1, x_2),$$

i.e., copula introduced by Jaworski coincide with diagonal copula K_d.

The generalization of Bertino copula B_d for $n > 2, d \in \mathcal{D}_n$ is not a universal method for n-dimensional copulas. $B_{d^-}(x_1, x_2, ..., x_n) = W(x_1, x_2, ..., x_n) = max\left(0, \sum_{i=1}^{n} x_i - (n-1)\right)$ is not a copula.

Similarly, the generalization of diagonal copulas K_d for fixed $d \in \mathcal{D}_n,\ n > 2$, given by

$$K_d(x_1, x_2, ..., x_n) = min\left(x_1, x_2, ..., x_n, \frac{d(x_1) + ... + d(x_n)}{n}\right)$$

is not a universal method for n-dimensional copulas. K_d is a symmetric quasi-copula for any $d \in \mathcal{D}_n$.

Due to ordinal sum representation of copulas, we can introduce a notion of the ordinal sums of diagonal sections,

$$d = \left(\langle a_k, b_k, d_k\rangle \left| k \in \mathcal{K}\right.\right),$$

where $\mathcal{K}$ is an index system, $(]a_k, b_k[)_{k\in\mathcal{K}}$ is a disjoint system of open subintervals of $[0,1]$, and $d_k \in \mathcal{D}_n$ for each $k \in \mathcal{K}$. Then

$d : [0,1] \to [0,1]$ is given by

$$d(x) = \begin{cases} a_k + (b_k - a_k)\, d_k\left(\frac{x-a_k}{b_k-a_k}\right) & \text{if } x \in]a_k, b_k[\text{ for some } k \in \mathcal{K}, \\ x & \text{else.} \end{cases}$$

The corresponding function $f : [0,1] \to [0,1]$ given by

$$f(x) = \frac{nx - d(x)}{n-1}$$

can be written in the form

$$f(x) = \begin{cases} a_k + (b_k - a_k)\, f_k\left(\frac{x-a_k}{b_k-a_k}\right) & \text{if } x \in \,]a_k, b_k[\quad \text{for some } k \in \mathcal{K}, \\ x & \text{else.} \end{cases}$$

Proposition 4.

For a fixed $n \in \{2, 3, ...\}$, let $d \in \mathcal{D}_n$ be an ordinal sum,

$$d = (\langle a_k, b_k, d_k \rangle \, | k \in \mathcal{K}) \, .$$

Then J_d is an ordinal sum copula $J_d = (\langle a_k, b_k, J_{d_k} \rangle \, | k \in \mathcal{K}) \, .$

Construction (4) and ordinal sum constructions commute, construction (4) does not commute with convex sums construction. The only elements of $\mathcal{D}_n$ which do not admit a non-trivial convex sum decomposition are the ordinal sums of type $(\langle a_k, b_k, d^- \rangle \, | k \in \mathcal{K})$. We denote their class by $\mathcal{E}_n$.

Proposition 5.

For a fixed $n \in \{2, 3, ...\}$, let $d \in \mathcal{D}_n \setminus \mathcal{E}_n$, i.e.,

$$d = \lambda d_1 + (1 - \lambda)\, d_2$$

for some $d_1, d_2 \in \mathcal{D}_n,\ d_1 \neq d_2,\ \lambda \in \,]0, 1[\,$. Then

$$J_{\lambda, d_1, d_2} = \lambda J_{d_1} + (1 - \lambda)\, J_{d_2}$$

is a copula from $\mathcal{C}_n$ with diagonal section d, and $J_{\lambda, d_1, d_2} \neq J_d$, in general.

For $n = 2$, any construction of a binary copula from an a priori given diagonal section $d \in \mathcal{D}_2$ can be "dualized", using the notion of a survival diagonal section.

Example 1. Consider the weakest diagonal section $d^- \in \mathcal{D}_3$. Then J_{d^-} and K_{d^-} are described in Table 1.

Table 1 Formulae for copula J_{d^-} and quasi-copula K_{d^-}, $n = 3$

domain	J_{d^-}	K_{d^-}
$\left[0, \frac{2}{3}\right]^3$	0	0
$\left[\frac{2}{3}, 1\right]^3$	$x_1 + x_2 + x_3 - 2$	$x_1 + x_2 + x_3 - 2$
$\left[0, \frac{2}{3}\right] \times \left[0, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right]$	$min\left(\frac{x_1}{2}, \frac{x_2}{2}, x_3 - \frac{2}{3}\right)$	$min\left(x_1, x_2, x_3 - \frac{2}{3}\right)$
$\left[0, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right] \times \left[0, \frac{2}{3}\right]$	$min\left(\frac{x_1}{2}, x_2 - \frac{2}{3}, \frac{x_3}{2}\right)$	$min\left(x_1, x_2 - \frac{2}{3}, x_3\right)$
$\left[\frac{2}{3}, 1\right] \times \left[0, \frac{2}{3}\right] \times \left[0, \frac{2}{3}\right]$	$min\left(x_1 - \frac{2}{3}, \frac{x_2}{2}, \frac{x_3}{2}\right)$	$min\left(x_1 - \frac{2}{3}, x_2, x_3\right)$
$\left[0, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right]$	$min\left(\frac{x_1}{2}, \frac{x_2}{2} - \frac{2}{3}\right) + min\left(\frac{x_1}{2}, x_3 - \frac{2}{3}\right)$	$min\left(x_1, x_2 + x_3 - \frac{4}{3}\right)$
$\left[\frac{2}{3}, 1\right] \times \left[0, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right]$	$min\left(\frac{x_2}{2}, x_1 - \frac{2}{3}\right) + min\left(\frac{x_2}{2}, x_3 - \frac{2}{3}\right)$	$min\left(x_2, x_1 + x_3 - \frac{4}{3}\right)$
$\left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right] \times \left[0, \frac{2}{3}\right]$	$min\left(\frac{x_3}{2}, x_1 - \frac{2}{3}\right) + min\left(\frac{x_3}{2}, x_2 - \frac{2}{3}\right)$	$min\left(x_3, x_1 + x_2 - \frac{4}{3}\right)$

$J_{d^-} \leq K_{d^-}$.

J_{d^-} is singular copula from $\mathcal{C}_3$.

Its support consists of 3 segments connecting the point $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and the mass 1 is uniformly distributed over the support of J_{d^-}.

The proper quasi-copula K_{d^-} has a negative mass $-\frac{1}{3}$ on each of rectangles

$$\left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right], \left[\frac{2}{3}, 1\right] \times \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right]$$

and

$$\left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right] \times \left[0, \frac{1}{3}\right].$$

Example 2. For the product copula $\Pi \in \mathcal{C}_n, n \geq 2$, the corresponding diagonal section $d \in \mathcal{D}_n$ is given by $d_\Pi(x) = x^n$. For $0 \leq x_1 \leq x_2 \leq ... \leq x_n \leq 1$, it holds

$$J_{d_\Pi}(x_1, ..., x_n) = \frac{1}{n}\left(x_1^n + \sum_{i=2}^{n} min\left(\frac{nx_1 - x_1^n}{n-1}, x_i^n\right)\right).$$

Consider diagonal sections $d_1, d_2 \in \mathcal{D}_3$ given by

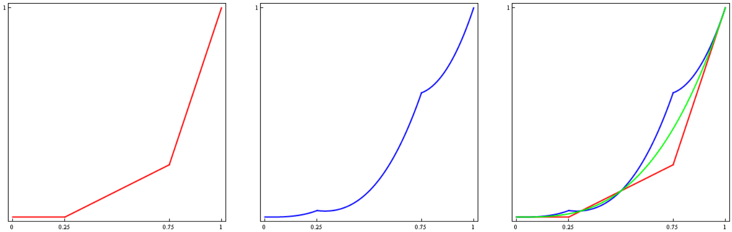
$$d_1(x) = \begin{cases} 0 & if \quad x \leq \frac{1}{4}, \\ \frac{x}{2} - \frac{1}{8} & if \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 3x - 2 & else. \end{cases}$$

and

$$d_2(x) = \begin{cases} 2x^3 & if \quad x \leq \frac{1}{4}, \\ 2x^3 - \frac{x}{2} + \frac{1}{8} & if \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 2x^3 - 3x + 2 & else. \end{cases}$$

Then $\frac{d_1+d_2}{2} = d_\Pi$ and thus the copula $\frac{1}{2}\left(J_{d_1} + J_{d_2}\right)$ has d_Π as its diagonal section.

Diagonal sections d_1, d_2 and d_Π.



Concluding remarks

- We have opened the problem of constructing n-dimensional copulas with a predescribed diagonal section, with the stress on higher dimen- sions, i.e., $n \in \{3, 4, ...\}$.
- Though there are some similarities with well developed case $n = 2$, several techniques cannot be used for higher dimensions.
- Especially, there is no universal construction leading to a smallest copula having a given diagonal section (for $n > 2$, there is no smallest copula in $\mathcal{C}_n$).
- We aim to focus on extension of particular methods known for the case $n = 2$, starting from a diagonal section $d \in \mathcal{D}_n$ with some specific properties, such as semilinear copulas or biconic copulas in the 2-dimensional case.

Thanks for attention

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