EXTREME VALUE ANALYSIS BASED ON COPULAS AND THEIR DIAGONALS

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The problem of risk accumulation

- September 11^{th}2001 terrorist attacks;
- the explosion of the space shuttle Challenger on January 28, 1983: the exceptionally low temperature (15 degrees F lower than the coldest previous launch) the night before launching led to failure of the O-rings which caused the disaster;
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Generali collaboration: Solvency II and the dependence between extreme events

Danish Fire Insurance Data (1980–2002)

Gaussian copula

Traditional, but underestimation of the risk of joint downside movements.

Gumbel copula

Overestimation of the risk.

t-Student copula

Dependence in the tails, but not in the center.

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Elliptic copulas

Gaussian copula

t-Student copula

The Normal copula is the dependence function

$$C_n^{\Phi}(\mathbf{u};\Omega) = \Phi_n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n);\Omega), \qquad (1)$$

where Φ_n is the cdf for the n-variate standard normal distribution with correlation matrix Ω.

$$C_n^{\Psi}(\mathbf{u};\Omega,\nu) = \Psi_n(\Psi^{-1}(u_1,\nu),\ldots,\Psi^{-1}(u_n;\nu);\Omega,\nu), \qquad (2)$$

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Archimedean copula

Gumbel copula: an extreme copula

$$C_n^G(\mathbf{u}; a) = \exp\left(-\left(\sum_{i=1}^{n}(-\log u_i)^a\right)^{\frac{1}{a}}\right), \qquad (3)$$

with $a \geq 1$, where $a = 1$ implies independence.
Upper tail dependence but lower tail independence.

Extreme copula

Definition: MEV copulas

An extreme copula satifies

$$C(u_1^t, \ldots, u_n^t, \ldots, u_N^t) = C^t(u_1, \ldots, u_n, \ldots, u_N)\ \forall t > 0.$$

MEV copulas are easily recognized from

$$A(\mathbf{x}) = -\log G(\mathbf{x}),$$

being homogeneous of order 1, i.e., $A(t\mathbf{x}) = tA(\mathbf{x})$, for all $t > 0$, with $\bar{G}(\mathbf{x}) = C(e^{-x_1}, \ldots, e^{-x_m})$.

Remark

Gaussian copula is not an extreme value copula.

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Extreme copulas

Topological properties

The set $\mathcal{C}$ of 2-copulas is compact with any of the following topologies, equivalent on $\mathcal{C}$: punctual convergence, uniform convergence on $[0,1]^2$, weak convergence of the associated probability measure.

Let $\mathcal{E}\mathrm{x}(C)$ be the set of the extreme points of $\mathcal{C}$. Then Choquet's representation of $\mathcal{C}$ similar to the Birkhoff's theorem:

$$\mathcal{C} \text{ is the convex hull of } \mathcal{E}\mathrm{x}(C).$$

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Extreme copulas

The determination of the extreme points of $\mathcal{C}$ is an open problem.

Theorem Any element of $\mathcal{C}$ that possesses a left or right inverse is extreme.

Examples Ordinal sums of $C^-(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ and $C^+(u_1, u_2) = \min(u_1, u_2)$ are extreme points of $\mathcal{C}$.

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Link between extreme value copulas and the multivariate extreme value theory

Denote $\chi^+_{n,m} = \max(X_{n,1}, \ldots, X_{n,k}, \ldots, X_{n,m})$ with $\{X_{n,k}\}$, k i.i.d. random variables with the same distribution. Let G_n be the marginal distribution of the univariate extreme $\chi^+_{n,m}$. Then, the joint limit distribution G of $(\chi^+_{1,m}, \ldots, \chi^+_{n,m}, \ldots, \chi^+_{N,m})$ is such that

$$G(\chi^+_1, .., \chi^+_n, .., \chi^+_N) = C(G_1(\chi^+_1), .., G_n(\chi^+_n), .., G_N(\chi^+_N)),$$

where C is an extreme value copula and G_n a non-degenerate univariate extreme value distribution.

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Topics on extreme value theory

The univariate case

Let us first consider m independent random variables $X_1, \ldots, X_k, \ldots, X_m$ with the same probability function F. The distribution of the extremes $\chi_m^+ = (\wedge_{k=1}^m X_k)$ is also given by Fisher-Tippet theorem:

Theorem

If there exist some constants a_m and b_m and a non-degenerate limit distribution G such that

$$\lim_{m\to\infty} P\left\{\frac{\chi_m^+ - b_m}{a_m} \leq x\right\} = G(x) \quad \forall x \in \mathbb{R}$$

then G is one of the following distributions:

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Distributions

Fréchet

Weibull

Gumbel

$$G(x) = \Upsilon_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-x^{-\alpha}) & x > 0 \end{cases}$$

$$G(x) = \Psi_\alpha(x) = \begin{cases} \exp(-(-x^{\alpha})) & x \leq 0 \\ 1 & x > 0 \end{cases}$$

$$G(x) = \Lambda(x) = \exp(-e^{-x}) \qquad x \in \mathbb{R}$$

In this case, we say that F belongs to the maximum domain of attraction of G, $F \in MDA(G)$.

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Max-stable distribution

Definition A non-degenerate rv X (the corresponding distribution or df) is called *max-stable* if it satisfies the identity in law

$$\max(X_1, \ldots, X_n) \stackrel{d}{=} c_n X + d_n$$

for i.i.d. $X, X_1, \ldots, X_n$, appropriate constants $c_n > 0$, $d_n \in \mathbb{R}$ and every $n \geq 2$.

Proposition The class of multivariate extreme value distributions is the class of max-stable distribution functions with nondegenerate marginals.

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Our problem

General copula

Kendall's tau: measure of association

Generic choice of copulas also depending on Kendall's τ (which is also in the a parameter of the Gumbel through the link $a = \frac{1}{1-\tau}$). Therefore, we have the following situation:

$$\tau_C = 4 \int \int_{I^2} C(u,v) dC(u,v) - 1 =$$
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My proposal: Diagonals

Definition

Let $C : [0,1] \to [0,1]$ be an n-dimensional copula, $n \geq 2$. The function

$\delta : [0,1] \to [0,1]$, $\delta(t) = C(t, \ldots, t)$ is called a *diagonal section* or *diagonal* for short.

Kendall's τ in connection with the general copula C.

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Tail dependence

Upper tail dependence

If a bivariate copula C is such that

$$\lim_{u \to 1} \frac{\bar{C}(u,u)}{1-u} = \lambda_U$$

exists, then C has upper tail dependence for $\lambda_U \in (0,1]$ and no upper tail dependence for $\lambda_U = 0$.

Lower tail dependence

If a bivariate copula C is such that

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Tail dependence: some examples

Gumbel family

The Gumbel family has upper tail dependence, with

$$\lambda_U = 2 - 2^{\frac{1}{\alpha}}$$

Clayton family

The Clayton family has lower tail dependence for $\alpha > 0$, since

$$\lambda_L = 2^{-\frac{1}{\alpha}}$$

Frank family

The Frank family has neither lower nor upper tail dependence.

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Diagonal section of C

$$\delta_C(t) = C(t,t)$$

λ_U in connection with the general copula C by:

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1-t} = 2 - \delta_C'(1^-).$$

$$\lambda_L = \lim_{t \to 0^+} \frac{\delta(t)}{t}$$

Remark

The measure λ is extensively used in extreme value theory. It is the probability that one variable is extreme given that the other is extreme.

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Order statistics

Probabilistic interpretation

Let $U = (U_1, \ldots, U_n)$ be an n-variate random variable with uniform margins, $U_i \sim \mathcal{U}(0,1)$, C its distribution function (hence a copula) and δ the diagonal section of C. Then δ is a distribution function of the random variable $\max\{U_1, \ldots, U_n\} = U_{n:n}$.

Application to two assets option pricing

Multivariate RNDs and copulas

Let $\mathbb{Q}_n$ and $\mathbb{Q}$ be the risk-neutral probability distributions of $S_n(T)$ and $\mathbf{S}(T) = (S_1(T) \ldots S_N(T))^\top$. With arbitrage theory, we can show that

$$\mathbb{Q}(+\infty, \ldots, +\infty, S_n(T), +\infty, \ldots, +\infty) = \mathbb{Q}_n(S_n(T)).$$

$\Rightarrow$ The margins of $\mathbb{Q}$ are RNDs $\mathbb{Q}_n$ of Vanilla options.

Remark

European option prices permit to caracterize the probability distribution of $S_n(T)$

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Bivariate case

For a call max option $\Phi(T, K)$ is the diagonal section of the copula

$$\Phi(T, K) = C(\mathbb{Q}_1(K), \mathbb{Q}_2(K))$$

For a spread option, we have

$$\Phi(T, K) = \int_0^{+\infty} \partial_1 C(\mathbb{Q}_1(x), \mathbb{Q}_2(x + K)) d\mathbb{Q}_1(x).$$

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Concluding remarks

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THANK YOU FOR YOUR ATTENTION!