Fuzzy Relational Inference based on **Generalised Operators**

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Known Facts..!!

$T - I_T$ Residual Pair

- T - a left continuous t-norm.
- $I_T(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\}$
- I_T- a fuzzy implication.

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- Generalise T to some operator C...
- ... to get $I_C \in \mathcal{FI}$

Generalised Operators

The Known Classes

Generalised Operators

Residual

- $C\colon [0,1]^2 \to [0,1]$ be an arbitrary function,
- $I_C\colon [0,1]^2 \to [0,1]$, defined as ...
- $I_C(x,y) = \sup\{t \in [0,1] | C(x,t) \leq y\}$

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- I_C is increasing in the second variable.
- I_C need not be a fuzzy implication.
- $C(x,y) = x$, then $I_C = I_{RS}$, Rescher implication.
- $C(x,y) = y$, then $I_C(x,y) = y$, not a fuzzy implication.

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Fuzzy Conjunction, [Krol A., 2011]

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What kind of studies have been done..

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- $C-$ a left-continuous semicopula.

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- $C-$ a left-continuous semicopula.

- I_C has (OP);
- $I_C(1, y) = y$ for all $y \in [0, 1]$;
- I_C is decreasing in the first variable;
- I_C is increasing in the second variable;
- I_C is left-continuous in its first variable;
- I_C is right-continuous in its second variable.

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[Krol A., 2011]

- C has left neutral element 1 $\Longleftrightarrow$ I_C has (NP);
- C fulfills (EP) $\Longleftrightarrow$ I_C has (EP) ;
- C fulfils $C(x,1) \leq x$, $x \in [0,1]$ $\Longleftrightarrow$ I_C has (IP) ;
- C has right neutral element 1 $\Longleftrightarrow$ I_C has (OP) .

Motivation

Minor

Find the most generalised C so that I_C is a fuzzy implication.

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Find the most generalised C so that I_C is a fuzzy implication.

Major

Applicability of $(C, I_C)-$ pair to Fuzzy Relational Inference.

Fuzzy Relational Inference

The Mechanism

FRI - The Procedure

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SISO Rule Base

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If $\tilde{x}$ is A_i Then $\tilde{y}$ is B_i, $i = 1, 2, \ldots, n$.

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Relation Representation of Rules

- Relate the antecedents and consequents ...
- ... by a fuzzy relation $R \in \mathcal{F}(X \times Y)$
- $R_i : X \times Y \to [0, 1]$ represents each of the rules.

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Commonly Employed Relations R

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y))$$

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))$$

FRI - The Procedure

Output from Composition

- Let $A' \in \mathcal{F}(X)$ be the given input.
- Compose A' with R to get the B' : $B' = A' @ R$

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Typical Compositions

- **Compositional Rule of Inference**: **CRI**

$$B'(y) = \bigvee_{x \in X} (A'(x) * R(x, y))$$

- **Bandler-Kohout Subproduct**: **BKS**

$$B'(y) = \bigwedge_{x \in X} (A'(x) \rightarrow R(x, y))$$

FRI With Generalised Operators

A Generalization of CRI - $\circ_c$

- $* = T$, a t-norm
- $* = C$, the generalised operator.

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CRI With C-Operator

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CRI With C-Operator

- **(CRI-C)**

$$B' = A' \circ_c R = \bigvee_{x \in X} C(A'(x), R(x, y)). \tag{1}$$

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- $\circ_c$: $\sup - C$ composition.
- $C \in \mathscr{C}$: Generalised C - Operator .

FRI With Generalised Operators

A Generalization of BKS - $\lhd_c$

- $\longrightarrow = I_T$, the residual of a left-continuous t-norm T.
- $\longrightarrow = I_c$. the residual of the generalised operator.

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- $\lhd_c$: inf $-I_c$ composition.
- I_c : $C \in \mathscr{C}$, Residual of the C - Operator.

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Is mere substitution enough?

Desirable Properties of an inference mechanism

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Interpolativity

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- BKS: Štěpnička & Jayaram, IEEE TFS (2010)

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Robustness

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Robustness

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Our Work

Generalised Operators

The class $\mathscr{C}^+$, [Demirli K., De Baets B., 1999]

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The class $\mathscr{C}^-$

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The class $\mathscr{C}^0$

$C : [0,1]^2 \to [0,1]$ be a function satisfying

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$$\mathscr{C}^0 = \mathscr{C}^- \cup \mathscr{C}^+.$$

Generalised operator *a lá* Implication

The class $\mathscr{I}$

$I : [0,1]^2 \to [0,1]$ be a function satisfying

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$$\mathscr{I}^* = \mathscr{I} \cap \mathcal{FI}$$

Class $\mathscr{C}^0$ and Properties of I_C

Properties

Theorem

$$C \in \mathscr{C}^0 \Longrightarrow I_C \in \mathscr{I}^* (= \mathcal{FI} \cap \mathscr{I})$$

Properties

Theorem

$$C \in \mathscr{C}^0 \implies I_C \in \mathscr{I}^*(= \mathcal{FI} \cap \mathscr{I})$$

Theorem

$\mathscr{I}_{\mathscr{C}^0} = \{I_C | C \in \mathscr{C}^0\}$ is such that...

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$$\mathscr{I}_{\mathscr{C}^0} = \mathscr{I}^*.$$

At a Glance!

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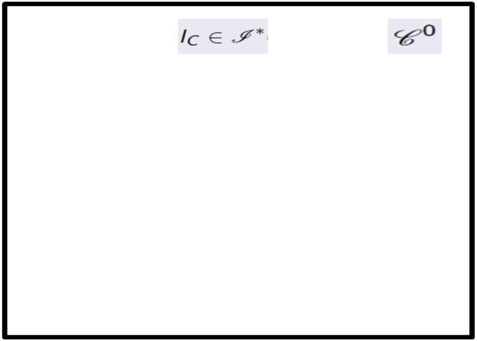

Class $\mathscr{C}^0$ and Solvability of FREs

Applicability of C and I_C where $C \in \mathscr{C}^1$

Fuzzy Relational Equations

Generalised Operators

Recall: The class $\mathscr{C}^0$

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The class $\mathscr{C}^1$

$C : [0,1]^2 \to [0,1]$ be a function satisfying

- C is increasing in **both** the variables,
- $C(0,1) = 0$,
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Fuzzy Relational Equations where $C \in \mathscr{C}^1$

Generalised Composition - C- composition

$$Q \circ_c P = S$$

$$Q \circ_c P(x,z) = \sup_{y \in Y} C\,(Q(x,y), P(y,z))$$

Generalised Composition - I_c- composition

$$Q \triangleleft_c P = S$$

$$Q \triangleleft_c P(x,z) = \inf_{y \in Y} I_c\,(Q(x,y), P(y,z))$$

Fuzzy Relational Equations where $C \in \mathscr{C}^1$

Solvability

Solvability of FRE : $C \in \mathscr{C}^1$

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$$Q \circ_c P = S \iff P \subseteq Q^{-1} \lhd_c S, \quad Q^{-1}(x,y) = Q(y,x) \ .$$

Proposition

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Solvability Of $Q \circ_c P = S$ for P

$\hat{P} = Q^{-1} \lhd_c S$ is the **largest** solution.

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Similar to what happens with $(T, I_T)-$ pair.

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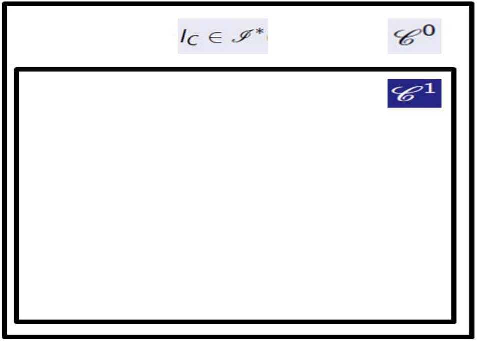

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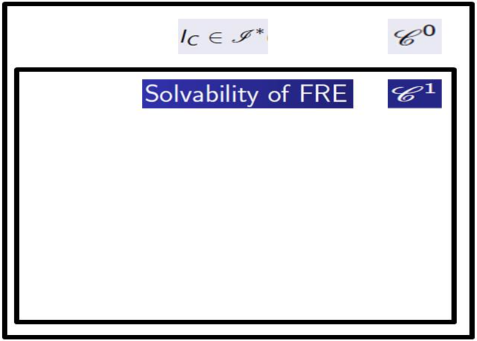

Class $\mathscr{C}^0$ and Interpolativity of FRI

BKS with I_c-implications

Interpolativity

Generalised Operators

The class $\mathscr{C}^2$

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- **C is commutative.**

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Weak Law of Importation [S.Massanet, J.Torrens, 2009]

$$I_c(x, I_c(y,z)) = I_c(C(x,y),z) \qquad \text{(WLI)}$$

Proposition [S.Massanet, J.Torrens, 2009]

$$C \in \mathscr{C}^2 \implies I_c \text{ satisfies (WLI)}$$

Interpolativity FRI

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$$A = A_i$$

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$$A = A_i \implies$$

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- Can R be any fuzzy relation $\mathcal{F}(X \times Y)$??

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- What is a **correct** model R of the given rule base for $\triangleleft_c$?, ...
- ... i.e., an R such that ...

$$A_i \triangleleft_c R = B_i \ .$$

- Can R be any fuzzy relation $\mathcal{F}(X \times Y)$??
- $\hat{R}_c$ is the maximal solution of $\triangleleft_c$ compositions.

Interpolativity of $\triangleleft_c$, $C \in \mathscr{C}^2$: A Sufficient Condition

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A possible relation for R : $\hat{R}_c$

$$\hat{R}_c(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_c B_i(y)) \ .$$

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Theorem

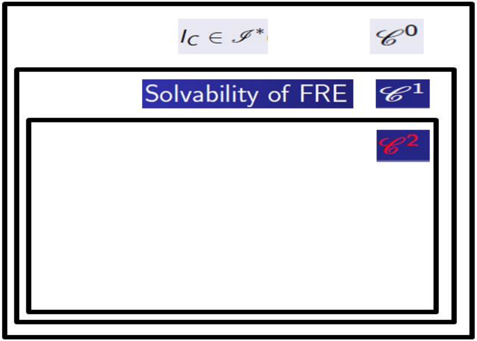
Let A_i for $i = 1, 2, \ldots n$ be normal.

- $\hat{R}_c$ *is a* **correct** *model of the rule base for* $\lhd_c$ **if**....
- ...*for any* $i, j \in \{1 \ldots n\}$,

$$\bigvee_{x \in X} C\left(A_i(x), A_j(x)\right) \leq \bigwedge_{y \in Y} \left(B_i(y) \longleftrightarrow_{\mathbf{c}} B_j(y)\right) \ ,$$

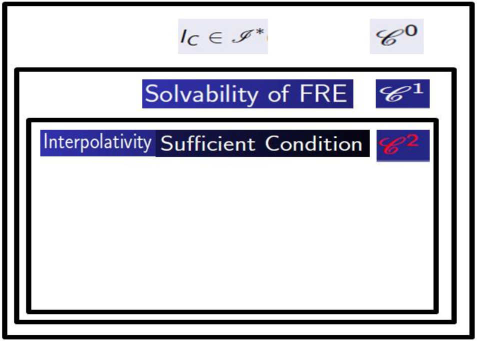
- $\longleftrightarrow_{\mathbf{c}}$ *is bi-implication,*
- $C \in \mathscr{C}^2$.

At a Glance!


$I_C \in \mathscr{I}^*$
$\mathscr{C}^0$
Solvability of FRE
$\mathscr{C}^1$
$\mathscr{C}^2$

Generalised Operators

At a Glance!



The class $\mathscr{C}^3$

$C : [0,1]^2 \to [0,1]$ be a function satisfying

- C is increasing in both the variables,
- $C(0,1) = 0$,
- $C(1,y) = y$ for all y.
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Note

$$\mathscr{C}^3 \subsetneq \mathscr{C}^2 \subsetneq \mathscr{C}^1 \subsetneq \mathscr{C}^0$$

Interpolativity of $\lhd_c$, $C \in \mathscr{C}^3$: An Equivalence Condition

Recall: $R = \hat{R}_c$

$$\hat{R}_c(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_c B_i(y)) \; .$$

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Theorem

Let A_i for $i = 1, 2, \ldots n$ be normal. Then TFAE,

1. $\hat{R}_c$ *is a* **correct** *model of the rule base for* $\lhd_c$,
2. *For any* $i, j \in \{1 \ldots n\}$,

$$\bigvee_{x \in X} C\left(A_i(x), A_j(x)\right) \leq \bigwedge_{y \in Y} \left(B_i(y) \longleftrightarrow_{\mathbf{c}} B_j(y)\right) \ ,$$

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Generalised Operators

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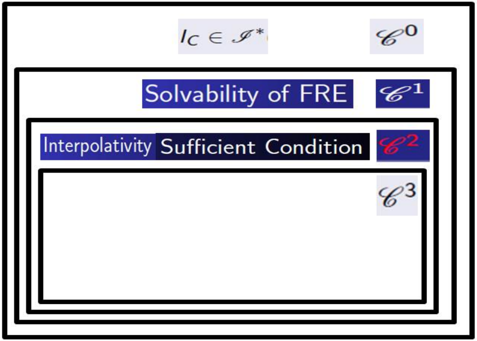

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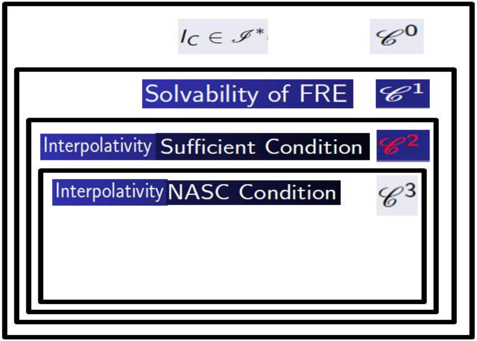

$I_C \in \mathscr{I}^*$
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Solvability of FRE
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Class $\mathscr{C}^0$ and Continuity

BKS with I_c-implications

Continuity

Continuity of $\lhd_c$, $C \in \mathscr{C}^3$

Continuity of $\lhd_c$, $C \in \mathscr{C}^3$

Definition

- Let $R \in \mathcal{F}(X \times Y)$ be a fuzzy relation.
- R is said to be a **Continuous** model of the rule base for $\lhd_c$...
- ... if for each $i \in I$ and for any $A \in \mathcal{F}(X)$ ...

$$\bigwedge_{y \in Y} [B_i(y) \longleftrightarrow_{\mathbf{C}} (A \lhd_c R)(y)] \geq \bigwedge_{x \in X} [A_i(x) \longleftrightarrow_{\mathbf{C}} A(x)] \ .$$

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Is $\hat{R}_c$ a continuous model for $\lhd_c$?

Continuity of $\lhd_c$, $C \in \mathscr{C}^3$

Continuity of $\lhd_c$, $C \in \mathscr{C}^3$

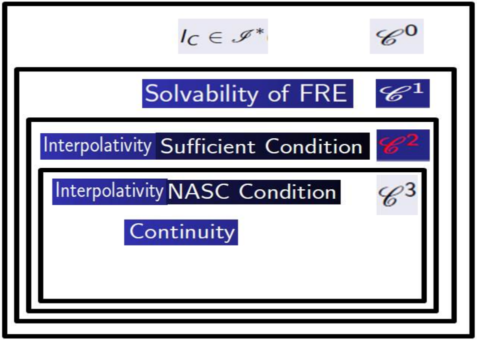
Theorem

Let a SISO rule base be given. The following are equivalent:

1. $\hat{R}_c$ *is a* **Continuous** *model for* $\lhd_c$.
2. $\hat{R}_c$ *is a* **Correct** *model for* $\lhd_c$.

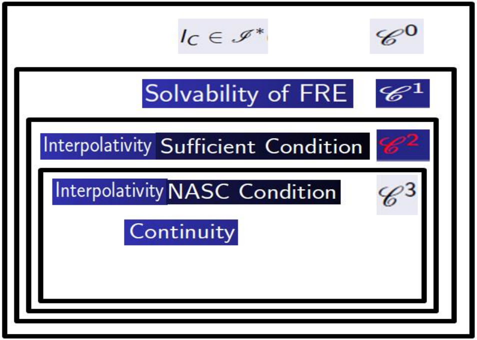
Continuity

At a Glance!



Continuity

At a Glance!



$C \in \mathscr{C}^3$ + C is associative = t-norm.

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- Discussed **BKS** with I_c-implications.
- Illustrated that some of the desirable properties of **BKS** are available.
- Specifically, **Interpolativity, Continuity**.

Thanks for your patient listening !!!