The Order Generated by Implications

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Outline

- Fuzzy implications are one of the most important operations in fuzzy logic having a significant role in many applications, viz., approximate reasoning, fuzzy control, fuzzy image processing, etc.
- They generalize the classical implication, which takes values in $\{0, 1\}$, to fuzzy logic, where the truth values belong to the unit interval $[0, 1]$. In general situation, since $[0, 1]$ is a bounded lattice, like in the case of other logical operators, the problem of introducing implications on a bounded lattice laid bare and Ma and Wu, Logical operators on complete lattices, have introduced them at first. Several authors have investigated the implications on a bounded lattice and their relations to the other logical operators [11, 16, 17, 20, 21, 22].

In this paper:

- We introduce an order by means of an implication possessing some special properties on a lattice and discuss some of its properties.
- We determine the relationship between the order induced by an implication and the order on the lattice. Giving example, we show that a bounded lattice needs not be a lattice with respect to the order induced by an implication.
- Also, we give an example for an implication making the unit interval $[0, 1]$ a lattice with respect to the order induced by it.

- Moreover, we obtain that such a generating method of an order is independent from the order induced by an adjoint t-norm (T-partial order)[10].
- We prove that under the conditions required to define implication based order, the considered implication must be an S-implication, and so we obtain that the order induced by an implication coincides with the order which is generated in a similar way from a t-conorm.
- Consequently, we obtain that an implication on the unit interval $[0, 1]$ is continuous if and only if the implication based order and the dual of the natural order on $[0, 1]$ coincide.

T-norm and T-conorm

Definition (De Baets and Mesiar, 1999)

Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operation T (S) on L is called a t-norm (t-conorm) if it satisfies the following conditions:
(1) $T(T(a,b),c) = T(a,T(b,c))$ (associative law),
(2) $T(a,b) = T(b,a)$ (commutative law),
(3) $b \leq c \Rightarrow T(a,b) \leq T(a,c)$ (monotonicity),
(4) $T(a,1) = a$ $(S(a,0) = a)$ (boundary condition),
where a, b and c are any elements of L.

- The four basic t-norms on $[0,1]$ are the minimum T_M, which is the largest t-norm, the product T_P, the Łukasiewicz t-norm T_L and the drastic product T_D, which is the smallest t-norm, given by, respectively,
  $T_M(x,y) = min(x,y)$,
  $T_P(x,y) = xy$,
  $T_L(x,y) = max(0, x+y-1)$ and
  $$T_D(x,y) = \begin{cases} x & if \quad y = 1, \\ y & if \quad x = 1, \\ 0 & \text{otherwise.} \end{cases}$$
- Also, t-norms on a bounded lattice $(L, \leq, 0, 1)$ are defined in similar way, and then extremal t-norms T_D as well as $T_\wedge$ on L are defined similarly as T_D and T_M on $[0,1]$.

Negation

Definition (Ma and Wu, 1991)

Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \rightarrow L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation N on L is called strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in L$.

- On each bounded lattice L we have two extremal negations $N^{+}, N^{-} : L \to L$ given by

$$N^{-}(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$N^{+}(x) = \begin{cases} 0 & if \quad x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

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- Obviously, for any negation $N : L \to L$, it holds $N^{-} \leq N \leq N^{+}$.

Implication

Definition (Baczynski and Jayaram, Fuzzy implications, 2008)

Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operator $I : L^2 \to L$ is said to be an implication function, shortly an implication, if it satisfies

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(I2) For every elements a, b with $a \leq b$, $I(x, a) \leq I(x, b)$ for all $x \in L$.

(I3) $I(1, 1) = 1$, $I(0, 0) = 1$ and $I(1, 0) = 0$.

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  $I(x, I(y, z)) = I(y, I(x, z))$ for all $x, y, z \in L$

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Obviously, for a strong negation N, the left contrapositive symmetry and the contrapositive symmetry coincide.

Natural negation

Definition (Baczynski and Jayaram, Fuzzy implications, 2008)

Let $(L, \leq, 0, 1)$ be a lattice and I be an implication on L. The function $N_I : L \longrightarrow L$ given by

$$N_I = I(x, 0) \text{ for all } x \in L$$

is a negation and it is called the natural negation of I.

S-implication

Definition (F. Karaçal, 2006)

Let $(L, \leq, 0, 1)$ be a lattice. An implication $I : L^2 \rightarrow L$ is called an S-implication if there exists a t-conorm S and a strong negation N such that for every $x, y \in L$

$$I(x, y) = S(N(x), y).$$

T-partial order

Definition (Karaçal and Kesicioğlu, 2011)

Let L be a bounded lattice, T be a t-norm on L. The order defined as following is called a T- partial order (triangular order) for t-norm T

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

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Proposition (Karaçal and Kesicioğlu, 2011)

Let T be a t-norm on a bounded lattice $(L, \leq, 0, 1)$. Then, if $x \preceq_T y$ necessarily we have also $x \leq y$.

Definition (Klement, Mesiar, Pap, Triangular Norms, 2000)

Let $T : [0,1]^2 \to [0,1]$ be a left-continuous t-norm. The function $I_T : [0,1]^2 \to [0,1]$ given by

$$I_T(x,y) = \sup\{z \in [0,1] | T(x,z) \leq y\} \tag{1}$$

is an implication and it is called as a residual implication.

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is an implication and it is called as a residual implication.

- Observe that the definition (1) can be applied to any t-norm $T : L^2 \to L$ acting on a complete lattice L, and the resulting function $I_T : L^2 \to L$ is an implication on L.

Definition

Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ be an implication. Define the relation $\preceq_I$ on L as follows:
For every $x, y \in L$

$$y \preceq_I x :\Leftrightarrow \exists \ell \in L \quad such \quad that \quad I(\ell, x) = y. \tag{2}$$

Proposition

The relation $\preceq_I$ is a partial order on L, whenever $I : L^2 \to L$ is an implication satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I.

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We will call such an order defined in (2) as the *ordering based on the implication I*.

Observe that the converse of Proposition does not hold.

Example

Consider Goguen implication $I : [0,1]^2 \to [0,1]$ given by

$$I(x,y) = \begin{cases} 1 & if \quad x \leq y, \\ y/x & \text{otherwise.} \end{cases}$$

Obviously, its natural negation is the Gödel negation,

$$N^{-}(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is not involutive. Therefore, I can not satisfy the contrapositive symmetry.

On the other side,

$$\preceq_I = \{(x,y) | 0 < y \leq x \leq 1\} \cup \{(0,0),(1,0)\} \tag{3}$$

is a partial order on $[0,1]$, whose Hasse diagram is depicted on Figure 1.

Figure: Hasse diagram of $\preceq_I$ given by (3)

Proposition

Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \rightarrow L$ be an implication satisfying (EP) and (CP) with respect to the strong natural negation N_I. If $(x, y) \in \preceq_I$, then $(y, x) \in \leq$.

Remark

Let $(L, \leq, 0, 1)$ *be a bounded lattice and* I *be an implication satisfying (EP) and (CP-*N_I*).*

- *It is clear that* 0 *and* 1 *are the greatest and the least element with respect to* $\preceq_I$*, respectively.*

Remark

Let $(L, \leq, 0, 1)$ be a bounded lattice and I be an implication satisfying (EP) and (CP-N_I).

- *It is clear that 0 and 1 are the greatest and the least element with respect to $\preceq_I$, respectively.*
- *The converse of the previous Proposition may not be satisfied.*

For example: Consider the lattice $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ whose lattice diagram is displayed in Figure 2:

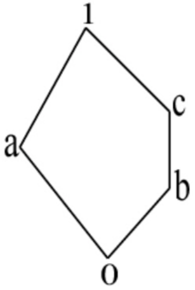

Figure: $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$

Define the function $I : L^2 \to L$ as follows:

I	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	c	1	1	1	1
c	b	1	1	1	1
1	0	a	b	c	1

Table: The implication I on L

Obviously, I is an implication on L satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I defined as

$$N_I(x) = \begin{cases} a & if \quad x = a, \\ c & if \quad x = b, \\ b & if \quad x = c, \\ 1 & if \quad x = 0, \\ 0 & if \quad x = 1. \end{cases}$$

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It is clear that $b \leq c$, but $c \not\preceq_I b$. The order $\preceq_I$ on L has its Hasse diagram as follows:

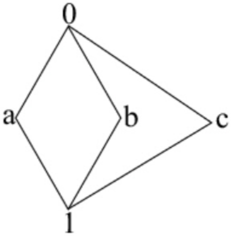

Figure: $(L = \{0, a, b, c, 1\}, \preceq_I, 0, 1)$

- Even if $(L, \leq, 0, 1)$ is a chain, the partially ordered set $(L, \preceq_I)$ may not be a chain.

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  For example: consider $L = [0, 1]$ and take the Fodor implication $I = I_{FD}$ defined as

$$I_{FD}(x, y) = \begin{cases} 1 & if \quad x \leq y, \\ \max(1 - x, y) & if \quad x > y. \end{cases} \tag{4}$$

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$$I_{FD}(x, y) = \begin{cases} 1 & if \quad x \leq y, \\ \max(1 - x, y) & if \quad x > y. \end{cases} \tag{4}$$

It is clear that I_{FD} satisfies the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation $N_{I_{FD}} = N_C$, $N_C(x) = 1 - x$.
Obviously, $1/2$ *and* $3/4$ *are not comparable* with respect to $\preceq_{I_{FD}}$.

Remark

Let T be a left continuous t-norm on $[0,1]$ and I_T be the corresponding residual implication. Then, the implication based ordering and the T-partial order are **independent**. *For example: consider the nilpotent minimum t-norm T^{nM} given by*

$$T^{nM}(x,y) = \begin{cases} 0 & x+y \leq 1, \\ \min(x,y) & \textit{otherwise.} \end{cases}$$

Then, its corresponding residual implication is the Fodor implication I_{FD}, see (4).
It is clear that $1/2 \preceq_{I_{FD}} 1/8$, but $1/8 \npreceq_{T^{nM}} 1/2$ and conversely, $1/2 \preceq_{T^{nM}} 3/4$, but $3/4 \npreceq_{I_{FD}} 1/2$.

Notations

Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \rightarrow L$ be an implication satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I. For $X \subseteq L$, we denote the set of the upper (lower) bounds of X w.r.t. $\preceq_I$ on L by $\overline{X}_I$ ($\underline{X}_I$). We denote the least upper bound (the greatest lower bound) w.r.t. $\preceq_I$ by $\vee_I$ ($\wedge_I$).

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- For any incomparable elements $x, y \in [0, 1]$, since $x \wedge_{I_{FD}} y = 1$, $(L, \preceq_{I_{FD}})$ is a meet-semi lattice.

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- For any incomparable elements $x, y \in [0, 1]$, since $x \wedge_{I_{FD}} y = 1$, $(L, \preceq_{I_{FD}})$ is a meet-semi lattice.
- There does not exist the least element of the upper bound $\overline{\{1/2, 3/4\}}_{I_{FD}} = [0, 1/4)$.

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- There does not exist the least element of the upper bound $\overline{\{1/2, 3/4\}}_{I_{FD}} = [0, 1/4)$.
  So, $(L, \preceq_{I_{FD}})$ is not a lattice.

Proposition

For every implication I satisfying (EP) and the contrapositive symmetry (CP) with respect to the natural strong negation N_I, there exists a t-conorm S such that

$$I(x, y) = S(N_I(x), y),$$

that is, I is an S-implication.

Corollary

Let $I : L^2 \to L$ be an implication satisfying (EP) and the contrapositive symmetry (CP) with respect to the natural strong negation N_I. Then, for any $a, b \in L$

$$a \preceq_I b \text{ if and only if } N_I(a) \preceq_T N_I(b),$$

where $T : L^2 \to L$ is a t-norm given by $T(x, y) = N_I(I(x, N_I(y)))$.

Theorem

Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication satisfying (EP) and the contrapositive symmetry (CP) with respect to the natural strong negation N_I and $\preceq_I$ be the order linked to the implication I. Then, I is continuous if and only if $\preceq_I = \geq$.

Proposition (Baczynski, 2010)

Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \rightarrow L$ a function defined as

$$I(x, y) = N(x) \vee y, \quad \forall x, y \in L, \tag{5}$$

where $N : L \rightarrow L$ is a strong negation on L. Then, I is an implication on L satisfying the exchange principle (EP) and the strong negation N is its natural negation. Moreover, I satisfies the contrapositive symmetry (CP) w.r.t. the natural negation N.

Proposition

Let $(L, \leq, 0, 1)$ *be a bounded lattice and let* I *be defined as (5). Then, the order obtained from the implication* I *is equal to the dual of the order on* L*, that is,* $\preceq_I = \geq$.

Remark

For a general bounded lattice, a strong negation on L need not be existing (see [11], example 2). If $(L, \leq, \wedge, \vee)$ is a Boolean algebra, it can be found always a strong negation on L defined as $N(x) = x'$. So, the previous Proposition is always true for Boolean algebras.

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Proposition

Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ an implication on L defined as $I(x, y) = 1$ when $x \neq 1$ and $y \neq 0$, satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I, that is the implication $I : L^2 \to L$ determined by

$$I(x,y) = \begin{cases} y & x = 1, \\ N_I(x) & y = 0, \\ 1 & \textit{otherwise.} \end{cases}$$

Then, $(L, \preceq_I)$ is a lattice.

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