On Suborbital Graphs for A Special Congruence Subgroup

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Outline

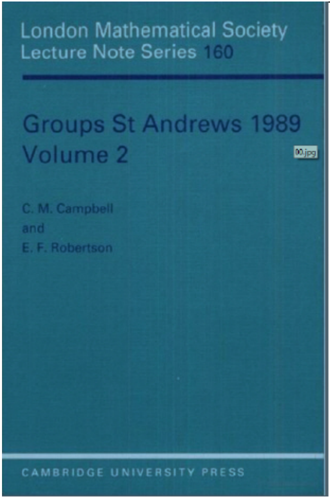
Modular Group

$$T : z \rightarrow \frac{az+b}{cz+d} : a, b, c, d \text{ are real numbers, } ad - bc = 1$$

- $PSL(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$
- $\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ $\rightarrow$ Modular Group

- $\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a \equiv d \equiv 1 \pmod{n}, b \equiv c \equiv 0 \pmod{n} \right\}$
- $\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a \equiv d \equiv 1 \pmod{n}, c \equiv 0 \pmod{n} \right\}$
- $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; c \equiv 0 \pmod{n} \right\}$
- $\Gamma(n) \leq \Gamma_1(n) \leq \Gamma_0(n) \leq \Gamma \leq PSL(2, \mathbb{R})$
- $\Gamma(n) \lhd \Gamma, \Gamma(n) \lhd \Gamma_0(n), \Gamma(n) \lhd \Gamma_1(n), \Gamma_1(n) \lhd \Gamma_0(n)$

Jones,Singerman,Wicks



THE MODULAR GROUP AND GENERALIZED FAREY GRAPHS

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1. Introduction

The *modular group*

$$\Gamma = PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$$

is the quotient of the unimodular group $SL(2,\mathbb{Z})$ by its centre $\{\pm I\}$. Thus the elements of Γ are the pairs of matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (a,b,c,d \in \mathbb{Z},\ ad - bc = 1); \tag{1.1}$$

we will omit the symbol $\pm$, and identify each matrix with its negative.

It is both traditional and useful to represent Γ as a group of Möbius transformations of the upper half-plane

$$\mathcal{U} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\},$$

with the element (1.1) acting by

$$z \mapsto \frac{az+b}{cz+d}. \tag{1.2}$$

For example, using the fact that this action of Γ is discontinuous, one can show that Γ is isomorphic to a free product $C_2 * C_3$; more specifically, Γ is generated by the elements

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \tag{1.3}$$

with defining relations

This paper is organized as follows:

- Γ acts transitively but imprimitively on $\hat{\mathbb{Q}}$
- They summarize Sims' theory
- They obtained a graph $G_{u,n}$ on which Γ acts
- They examine the Farey graph $(G_{1,1})$ as a simplest case
- They focus a subgraph $F_{u,n}$
- They found edge and circuit conditions respectively
- They give a conjecture that $G_{u,n}$ is forest iff it contains no triangles

Jones, G.A., Singerman D. and Wicks, K.
The modular group and generalized Farey graphs.
LMS Lect. Note Ser., 60 (1991), 316-338.

Jones, Singerman, and Wicks used the notion of the imprimitive action for a Γ- invariant equivalence relation induced on $\hat{\mathbb{Q}}$ by the congruence subgroup $\Gamma_0(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{n}\}$ to obtain some suborbital graphs and examined their connectedness and forest properties. They left the forest problem as a conjecture, which was settled down by Akbaş.

We introduce a different Γ-invariant equivalence relation by using the congruence subgroup $\Gamma_0^3(n)$ instead of $\Gamma_0(n)$ and obtain some results for the newly constructed subgraphs $F^3_{u,n}$.

Some papers concerning suborbitals graphs

Akbaş, M.
On suborbital graphs for the modular group.
Bull. London Math. Soc., 33 (2001), 647-652.

Keskin, R.
Suborbital graphs for some Hecke groups.
Discrete Mathematics, 9(3) (2001), 589-602.

Sims Theory (Suborbital Graphs)

Let (G, Δ) be transitive permutation group.
G acts on $\Delta \times \Delta$ by $(g \in G; \alpha, \beta \in \Delta)$

$$g(\alpha, \beta) = (g(\alpha), g(\beta))$$

$*$ The orbits of this action are called suborbitals of G
$*$ The orbit containing (α, β) is denoted by $O(\alpha, \beta)$
From $O(\alpha, \beta)$ we can form a suborbital graph $G(\alpha, \beta)$

Sims, C.C.
Graphs and finite permutation groups.
Math.Z., 95 (1967), 76-86.

$*$ its vertices are the elements of Δ
$*$ there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$ means that

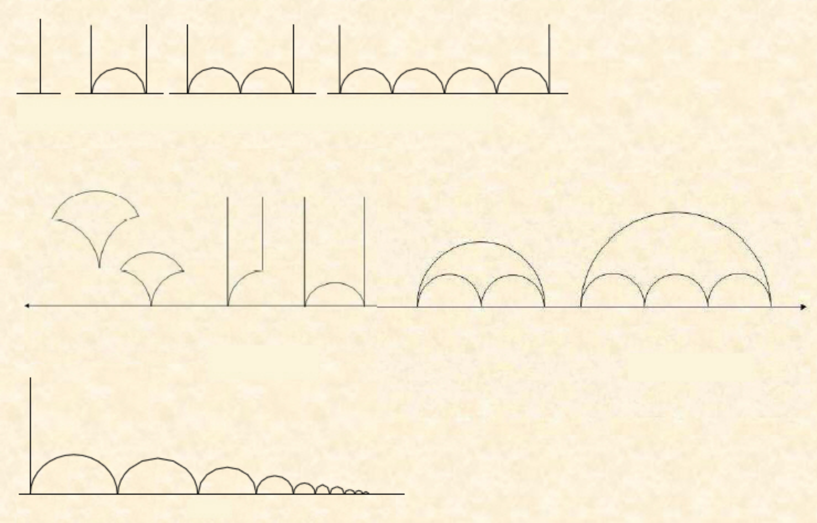
$$\gamma \rightarrow \delta :\Leftrightarrow T \in G \text{ such that } \begin{array}{l} T(\alpha) = \delta \\ T(\beta) = \gamma \end{array}$$

If $\alpha = \beta$, the corresponding suborbital graph $G(\alpha, \beta)$, called the trivial suborbital graph, is self-paired: it consist of a loop based at each vertex $\alpha \in \Delta$.

By a directed circuit of length m (or a closed edge path) we mean a sequence $v_1 \to v_2 \to \ldots \to v_m \to v_1$ of different vertices where $m \geq 3$; an anti-directed circuit will denote a configuration like the above with at least an arrow (not all) reversed.

- If $m = 2$, the circuit $v_1 \to v_2 \to v_1$ will be called a $2 - gon$.
- If $m = 3$ or 4 then the circuit is called a triangle or rectangle.

A graph is called a forest if it contains no circuits other than 2-gons.



Imprimitive Action

Let us give a general discussion of primitivity of permutation groups. Let (G, Δ) be a transitive permutation group, consisting of a group G acting on a set Δ transitively. An equivalence relation $\approx$ on Δ is called G-invariant if, whenever

$\alpha, \beta \in \Delta$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$.

The equivalence classes are called blocks, and the block containing α is denoted $[\alpha]$.
We call (G, Δ) imprimitive if Δ admits some G-invariant equivalence relation different from

- the identity relation, $\alpha \approx \beta$ iff $\alpha = \beta$.
- the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Delta$.

Otherwise (G, Δ) is called primitive. These two relations are supposed to be trivial relations.

Theorem

Let (G, Δ) be a transitive permutation group. (G, Δ) is primitive if and only if G_α, the stabilizer of $\alpha \in \Delta$, is a maximal subgroup of G for each $\alpha \in \Delta$.

* We suppose that $G_\alpha < H < G$. Since G acts transitively, for $g, h \in G$

$$g(\alpha) \approx h(\beta) \text{ if and only if } g^{-1}h \in H$$

is an imprimitive G-invariant equivalence relation.

Bigg, N.L.;White, A.T.,
Permutation groups and combinatorial structures.
London Mathematical Society Lecture Note Series, 33 (1979)

Subgraph $F^3_{u,n}$

We apply these ideas to the case:

$$G_\alpha = \Gamma^3_\infty, H = \Gamma^3_0(n), G = \Gamma^3$$
$$\Delta = \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$$

$$\Gamma^3 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ab + cd \equiv 0(mod3)\}$$
$$\Gamma^3_0(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^3 : c \equiv 0(mod\, n)\}$$
$$\Gamma^3_\infty = \langle\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\rangle$$
$\Delta = \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ →extended rational numbers

$$\Gamma^3_\infty < \Gamma^3_0(n) \leqslant \Gamma^3$$

Γ^3 must be one of the three types

$$\begin{pmatrix} 3a & b \\ c & 3d \end{pmatrix}, \begin{pmatrix} a & 3b \\ 3c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c,$ and $d,$ are rational integers and $a, b, c, d \not\equiv 0(mod3)$ in the third matrix.

Lemma

The action of Γ^3 on $\hat{\mathbb{Q}}$ is transitive.

- We get the following imprimitive Γ^3- invariant equivalence relation on $\hat{\mathbb{Q}}$ by $\Gamma_0^3(n)$ as

$$\frac{r}{s} \approx \frac{x}{y} \quad \text{if and only if} \quad g^{-1}h \in \Gamma_0^3(n)$$

where $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ and h is similar.

- From the above we can easily verify that

$$\frac{r}{s} \approx \frac{x}{y} \text{ if and only if } ry - sx \equiv 0 \,(\text{mod } n).$$

- The equivalence classes are called blocks and the block containing $\frac{x}{y}$ is denoted by $[\frac{x}{y}]$.

- Since Γ acts transitively on $\hat{\mathbb{Q}}$, every suborbital $O(\alpha,\beta)$ contains a pair $(\infty, \frac{u}{n})$ for $\frac{u}{n} \in \hat{\mathbb{Q}}, n \geq 0, (u,n) = 1$. In this case, we denote the suborbital graph by $G_{u,n}$ for short.
- As Γ permutes the blocks transitively, all subgraphs corresponding to blocks are isomorphic. Therefore we will only consider the subgraph $F^3_{u,n}$ of $G_{u,n}$ whose vertices form the block $[\infty] = [\frac{1}{0}]$, which is the set

$$\{\frac{x}{y} \in \hat{\mathbb{Q}} \mid y \equiv 0 \,(\text{mod } n)\}$$

.

Edge Conditions

Theorem

$\frac{r}{s} \longrightarrow \frac{x}{y}$ *in* $F^3_{u,n}$ *if and only if*

1. *If* $r \equiv 0(mod3)$, *then* $x \equiv \pm ur(modn), y \equiv \pm us(mod3n)$ *and* $ry - sx = \pm n$ *or,*
2. *If* $s \equiv 0(mod3)$, *then* $x \equiv \pm ur(mod3n), y \equiv \pm us(modn)$ *and* $ry - sx = \pm n$ *or,*
3. *If* $r, s \not\equiv 0(mod3)$, *then* $x \equiv \pm ur(modn), x \not\equiv \pm ur(mod3n)$, $y \equiv \pm us(modn), y \not\equiv \pm us(mod3n)$ *and* $ry - sx = \pm n$.

Theorem

$\Gamma_0^3(n)$ *permutes the vertices and edges of* $F_{u,n}^3$ *transitively.*

Theorem

$F_{u,n}^3$ *contains no directed triangles.*

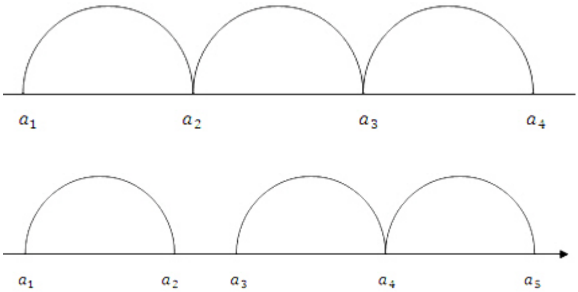
Theorem

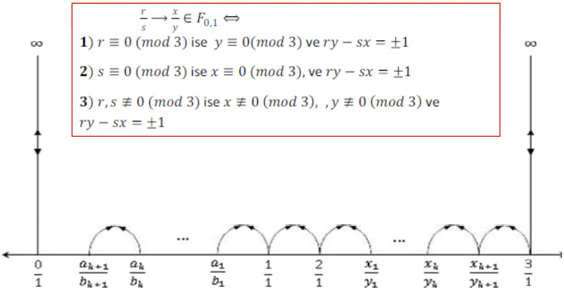
$F_{u,n}^3$ *contains a 2-gon if and only if* $n \not\equiv 0 (mod 3)$ *and* $u^2 \equiv -1 (mod n)$.

Connectedness

Definition

A subgraph K of $F^3_{u,n}$ is called connected if any pair of its vertices can be joined by a path in K.





Theorem

The subgraph $F^3_{0,1}$ is not connected.

Since $\infty \to \frac{0}{1}$ is an edge in $F^3_{0,1}$ and $F^3_{0,1}$ is periodic with period 3, we can do calculation only, in the strip $\frac{0}{1} \leq Rez \leq \frac{3}{1}$. It is clear that ∞ is adjacent to $\frac{0}{1}$ and $\frac{3}{1}$ in $F^3_{0,1}$, but to no intermediate vertices. We shall show that the edge $\frac{1}{1} \to \frac{2}{1} \in F^3_{0,1}$ is not adjacent to ∞. We assume that $\frac{1}{1}$ can be joined to ∞ by a path D in $F^3_{0,1}$. We may assume that D has the form

$\frac{1}{1} \longrightarrow \frac{a_1}{b_1} \longrightarrow \ldots \longrightarrow \frac{a_k}{b_k} \longrightarrow \frac{0}{1} \longrightarrow \frac{1}{0}$, where some arrows may be reversed. From the above edge conditions, we easily see that $\frac{1}{1} \leftrightarrows \frac{a_1}{b_1}$ if and only if $a_1, b_1 \not\equiv 0(mod3)$. Then $\frac{a_1}{b_1} \leftrightarrows \frac{a_2}{b_2}$ if and only if $a_2, b_2 \not\equiv 0(mod3)$. If we proceed in this way, we obtain that $\frac{a_k}{b_k} \leftrightarrows \frac{a_{k+1}}{b_{k+1}}$ if and only if $a_{k+1}, b_{k+1} \not\equiv 0(mod3)$. This contradicts to $\frac{a_{k+1}}{b_{k+1}} \leftrightarrows \frac{0}{1}$ since we get $0 \not\equiv 0(mod3)$ from the edge conditions. This shows that there is no path of the form D. Similarly, we can show that there is no path of the form $\frac{2}{1} \longrightarrow \frac{x_1}{y_1} \longrightarrow \ldots \longrightarrow \frac{x_k}{y_k} \longrightarrow \frac{x_{k+1}}{y_{k+1}} \longrightarrow \frac{3}{1}$. Consequently $F^3_{0,1}$ is not connected.

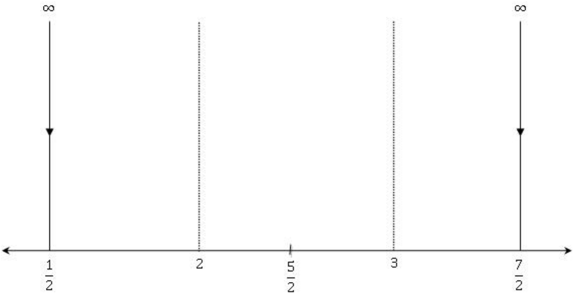


Figure: The subgraph $F^{3}_{1,2}$

Theorem

The subgraphs $F^{3}_{1,2}, F^{3}_{3,2}$ and $F^{3}_{5,2}$ are not connected.

Corollary

All subgraphs $F^{3}_{u,2}$ are not connected.

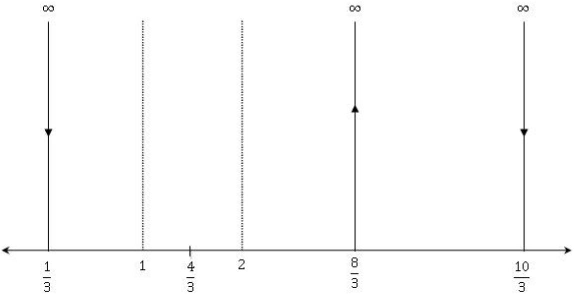


Figure: The subgraph $F^3_{1,3}$

Theorem

The subgraph $F^3_{1,3}, F^3_{2,3}, F^3_{4,3}, F^3_{5,3}, F^3_{7,3}$ and $F^3_{8,3}$ are not connected.

Corollary

All subgraphs $F^3_{u,3}$ are not connected.

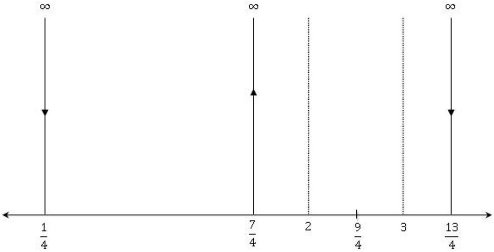


Figure: The subgraph $F^3_{1,4}$

Theorem

The subgraph $F^3_{1,4}, F^3_{3,4}, F^3_{5,4}, F^3_{7,4}, F^3_{9,3}$ *and* $F^3_{11,3}$ *are not connected.*

Corollary

All subgraphs $F^3_{u,4}$ *are not connected.*

Theorem

If $n \geqslant 5$, then $F^3_{u,n}$ is not connected.

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THANKS