Effect algebras with a state operator

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FSTA, Liptovský Ján, 27.1.2014

State MV-algebras

A state MV-algebra is a structure $(M, \boxplus, ', \sigma)$, where

- $(M, \boxplus, ')$ is an MV-algebra
- $\sigma : M \to M$ is a state operator:
  - $\sigma(0) = 0$,
  - $\sigma(x') = \sigma(x)'$,
  - $\sigma(x \boxplus y) = \sigma(x) \boxplus \sigma(y \boxminus (x \boxdot y))$, $x, y \in M$,
  - $\sigma(\sigma(x) \boxplus \sigma(y)) = \sigma(x) \boxplus \sigma(y)$, $x, y \in M$.

The range $\sigma(M)$ is a sub-MV-algebra of M.

(Flaminio, Montagna, 2009)

State effect algebras

Let E be an effect algebra. A state operator on E is mapping $\tau : E \to E$ such that

- $\tau(1) = 1$,
- $\tau(e \oplus f) = \tau(e) \oplus \tau(f)$, $e \perp f$,
- $\tau(\tau(e)) = \tau(e)$, $e \in E$.

The range $\tau(E)$ of τ is a sub-effect algebra of fixed points of τ.

(Buhagiar, Chetcuti, Dvurečenskij, 2011)

Strong state operators

Let E be an effect algebra, $\tau : E \to E$ a state operator. Then τ is strong if for $e, f \in E$

$$\exists\, \tau(e) \wedge \tau(f) \implies \tau(\tau(e) \wedge \tau(f)) = \tau(e) \wedge \tau(f)$$

- τ is strong $\iff$ $\tau(E)$ is closed under $\wedge$.
- If E is an MV-effect algebra, then τ is an MV-algebra state operator if and only if τ is strong.
- If τ is faithful ($\tau(e) = 0$ implies $e = 0$), then τ is strong.

(Buhagiar, Chetcuti, Dvurečenskij, 2011)

Convex effect algebras

A convex structure on E is a bimorphism $[0,1] \times E \to E$, $(\alpha, e) \mapsto \alpha e$, such that

- $\alpha(\beta e) = (\alpha\beta)e$,
- $1e = e$.

Then E is called convex. Any convex effect algebra is affinely isomorphic to an (essentially unique) algebra of the following form:

Example

Let (V, K) be an ordered real linear space and let $0 \neq u \in K$ be such that $K = \mathbb{R}^+[0, u]$. For $x, y \in [0, u]$, define $x \oplus y = x + y$ if $x + y \leq_K u$. Then $([0, u], \oplus, 0, u)$ is a convex effect algebra.

(Gudder, Pulmannová 1998)

Effect algebras and convex effect algebras

Let E be an effect algebra. Suppose E admits a state.

- The tensor product $[0,1] \otimes E$ exists and is convex.
- E embeds into $\tilde{E} := [0,1] \otimes E$ as $e \mapsto 1 \otimes e$.
- Any morphism $\phi : E \to F$ with F convex uniquely extends to a morphism $\tilde{\phi} : \tilde{E} \to F$.
- Any state operator τ on E uniquely extends to a state operator $\tilde{\tau} : \alpha \otimes e \mapsto \alpha \otimes \tau(e)$ on $[0,1] \otimes E$.

State operators on convex effect algebras

Let E be convex, $E \simeq [0, u]$ a generating interval in an ordered real vector space (V, K). Let $\tau : E \to E$ be a state operator. Then τ uniquely extends to a map $p : V \to V$, which is

- linear,
- $p(K) \subseteq K$,
- $p^2 = p$.

If τ is strong, then $p(p(x) \wedge p(y)) = p(x) \wedge p(y)$ whenever $p(x) \wedge p(y)$ exists.

We study such maps in some special cases and show their relation to conditional expectation.

Example: von Neumann-Lüders conditional expectation

Let $\mathcal{H}$ be a Hilbert space, $E = E(\mathcal{H})$ effects: $0 \leq a \leq I$

Let $\{p_i\}$ be projections, $\sum_i p_i = 1$, $\tau(a) = \sum_i p_i a p_i$

- E is convex, $V = B_{sa}(\mathcal{H})$, $K = B(\mathcal{H})^+$, $u = I$
- τ is a state operator
- τ is faithful, hence strong
- the extension is the von Neumann-Lüders conditional expectation
- let $b = \sum \lambda_i p_i$, ρ a state: $a \mapsto \rho(\tau(a))$ is the state after measurement of b in the initial state ρ

JC-effect algebras

- JC-algebra: a norm-closed real vector subspace $\mathcal{J} \subseteq B_{sa}(\mathcal{H})$ closed under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Suppose $I \in \mathcal{J}$.
- JC-effect algebra: unit interval in $\mathcal{J}$: $E(\mathcal{J}) = E(\mathcal{H}) \cap \mathcal{J}$
- a state operator τ on $E(\mathcal{J})$ extends to a positive unital idempotent map $p : \mathcal{J} \to \mathcal{J}$

$p(\mathcal{J})$ is a JC-algebra with product $p(a) * p(b) = p(p(a) \circ p(b))$. If p is faithful, then $p(\mathcal{J})$ is a Jordan subalgebra of $\mathcal{J}$.

(Effros, Störmer 1979)

Conditional expectations and Jordan operators

Kadison inequality: Let $p : \mathcal{J} \to \mathcal{J}$ be positive and unital. Then

$$p(a)^2 \leq p(a^2), \qquad a \in \mathcal{J}$$

If p is also idempotent:

$$p(a)^2 \leq p(p(a)^2) \leq p(a^2), \qquad a \in \mathcal{J}$$

Let $\tau : E(\mathcal{J}) \to E(\mathcal{J})$ be additive and unital. Then τ is a

- conditional expectation if $\tau(a)^2 = \tau(\tau(a)^2)$
- Jordan operator if $\tau(\tau(a)^2) = \tau(a^2)$

Conditional expectations and state operators on $E(\mathcal{J})$

Let $\tau : E(\mathcal{J}) \to E(\mathcal{J})$ be a conditional expectation. Then

- τ is a state operator
- $\tau(E(\mathcal{J}))$ is a sub-JC-effect algebra in $E(\mathcal{J})$
- the extension $p : \mathcal{J} \to \mathcal{J}$ of τ is a conditional expectation onto $p(\mathcal{J})$ in the algebraic sense:

$$p(p(a) \circ b) = p(a) \circ p(b), \quad a, b \in \mathcal{J}$$

If $\tau : E(\mathcal{J}) \to E(\mathcal{J})$ is a *faithful* state operator, then τ is a conditional expectation.

Strong state operators?

If $\tau(E(\mathcal{J}))$ is commutative, then τ is a strong state operator if and only if τ is a conditional expectation.

Jordan operators and state operators on $E(\mathcal{J})$

Let $\tau : E(\mathcal{J}) \to E(\mathcal{J})$ be a Jordan operator. Then

- τ is a state operator.
- if $p : \mathcal{J} \to \mathcal{J}$ is the extension of τ, then

  $$\mathcal{I}_\tau := \{a \in \mathcal{J}, p(a^2) = 0\}$$

  is a Jordan ideal and $[a]_{\mathcal{I}_\tau} \mapsto p(a)$ defines an isometric Jordan isomorphism $\mathcal{J}|_{\mathcal{I}_\tau}$ onto $p(\mathcal{J})$.
- if τ is faithful then p is a Jordan isomorphism

Decomposition of state operators: JW-algebras

JW-effect algebra: $E(\mathcal{J})$, $\mathcal{J}$ closed in the weak topology

$\tau : E(\mathcal{J}) \to E(\mathcal{J})$ is normal if it is completely additive

Theorem

Let τ be a normal state operator on a JW-effect algebra $E(\mathcal{J})$. Then there is a

- *faithful normal conditional expectation μ on $E(\mathcal{J})$*
- *normal Jordan operator ϕ on the range of μ*

such that $\tau = \phi \circ \mu$.

Decomposition of state operators: JC-algebras

Let $\mathcal{J}$ be a JC-algebra, $\tau : E(\mathcal{J}) \to E(\mathcal{J})$ a state operator

- $\mathcal{J}^{**}$ is a JW-algebra
- $E(\mathcal{J}^{**})$ is the strong operator closure of $E(\mathcal{J})$ in $\mathcal{J}^{**}$
- τ extends to a normal state operator on $E(\mathcal{J}^{**})$

Theorem

Let τ be a state operator on a JC-effect algebra $E(\mathcal{J})$. Then there is a

- *faithful normal conditional expectation μ on $E(\mathcal{J}^{**})$*
- *Jordan operator ϕ on the range of μ*

such that $\tau = \phi \circ \mu|_{E(\mathcal{J})}$.

Convex σ-MV-algebras

Let M be a convex σ-MV-algebra.

- Loomis-Sikorski representation: There is a tribe M^* over a compact Hausdorff space X and a σ-homomorphism η of M^* onto M.

  (Mundici 1999, Dvurečenskij 2000)
- For $a \in M$, there is a unique $a^* \in C(X)$, such that $\eta(a^*) = a$.
- The map $a \mapsto a^*$ is an MV-algebra isomorphism onto the unit interval $C_1(X)$ in $C(X)$.

State operators on convex σ-MV-algebras

Let τ be a state operator on M.

- $\tau^*(a^*) = \tau(a)^*$ defines a state operator τ^* on $C_1(X)$
- τ is strong $\iff$ τ^* is strong $\iff$ τ^* is a conditional expectation

$$\tau^*(f\tau^*(g)) = \tau^*(f)\tau^*(g), \qquad f, g \in C_1(X)$$

- $\exists\ K \subset X$ closed, $\mu : C_1(K) \to C_1(K)$ faithful conditional expectation, $\phi : C_1(K) \to C_1(X)$ positive unital extension ($\phi(f)(x) = f(x)$, $x \in K$), such that

$$\tau^*(f) = \phi \circ \mu(f|_K)$$

MV-conditional expectations

Let M be a σ-MV-algebra, $N \subseteq M$ a σ-MV-subalgebra, m a σ-additive state. Let

- $\mathcal{B}(M)$–boolean σ-algebra of idempotents in M,
- $\mathcal{B}(M^*)$–characteristic functions in M^*,
- $m^* := m \circ \eta$, $P^* := m^*|\mathcal{B}(M^*)$,
- $P^*_{b^*}(a^*) = P^*(b^* \wedge a^*), b \in \mathcal{B}(N), a \in \mathcal{B}(M)$

An MV-conditional expectation of $a \in M$ given N in the state m is a $\mathcal{B}(N^*)$-measurable function $m(a|N) : X \to \mathbb{R}$ such that for any $b \in \mathcal{B}(N)$

$$\int_X m(a|N)(\omega)dP^*_{b^*}(\omega) = m(a \wedge b)$$

(Dvurečenskij, Pulmannová 2005)

σ-additive state operators and MV-conditional expectations

- Let τ be a σ-additive strong state operator on M. Then there is a convex σ-MV-subalgebra $N \subseteq M$ and a σ-additive state m such that $\tau(a) = \eta(m(a|N))$ for some MV-conditional expectation of a with respect to N in m.
- Let $m(a|N)$ be an MV-conditional expectation with respect to a convex σ-MV-subalgebra N. Let $\tilde{M} := M|_{I_m}$, $I_m = \{a \in M, m(a) = 0\}$. Then $\tau([a]) := [\eta(m(a|N))]$ defines a σ-additive strong state operator on $\tilde{M}$.