D-posets and the Kalmbach monad

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Adjoint pair of functors

Definition

Let $\mathbf{C}, \mathbf{D}$ be categories, $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. Then F is left adjoint to G if and only if for all $X \in \mathbf{C}$, $Y \in \mathbf{D}$,

$$\mathrm{Hom}_{\mathbf{D}}(F(X), Y) \simeq \mathrm{Hom}_{\mathbf{C}}(X, G(Y)),$$

where $\mathrm{Hom}(A, B)$ is the set of all morphisms $A \to B$.

The most important property

Adjoint functors come in pairs

For every functor G, there is (up to isomorphism), at most one F such that F is left adjoint to G and vice versa.

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- let $F : \mathbf{Set} \to \mathbf{D}$ be the functor that maps a set X to the free algebra generated by X.

Then,

$$\mathrm{Hom}_{\mathbf{D}}(F(X), A) \simeq \mathrm{Hom}_{\mathbf{C}}(X, G(A))$$

Monads

Let $\mathbf{C}$ be a category. A monad is a triple (T, η, μ), where

- $T : \mathbf{C} \to \mathbf{C}$ is a functor,
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- Let **Set** be the category of sets, let P be the powerset endofunctor. (What are η and μ?)

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Corollary

Orthomodular lattices do not satisfy any special lattice equation.

The Kalmbach embedding

- Let L be a bounded lattice. Let $K(L)$ be the set of all finite chains in L with even number of elements.
- Introduce a partial order on the set $K(L)$ by the following rule:

  $$[a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}] \leq [b_1 < b_2 < \cdots < b_{2n-1} < b_{2k}]$$

  if and only if for every pair $1 \leq n$ there exists $1 \leq j \leq k$ such that $b_{2j-1} \leq a_{2i-1} < a_{2i} \leq b_{2j}$.

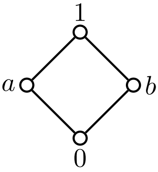
- Then $K(L)$ is a bounded lattice (boring proof).
- Moreover, it is an orthomodular lattice: the orthocomplementation is

$$(\{a_i\}_{i=1}^{2n})' := \{a_i\}_{i=1}^{2n} \Delta \{0,1\},$$

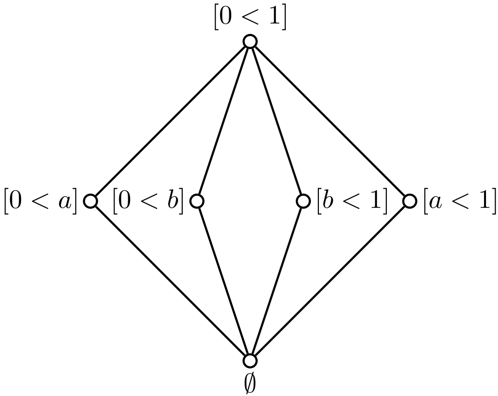
where Δ denotes the symmetric difference and
- the mapping $\eta_L : L \to K(L)$ given by $\eta_L(x) = [0 < x]$ for $x > 0$ and $\eta_L(0) = \emptyset$ is a injective morphism of lattices.

Example

L



$K(L)$



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- However, [Mayet and Navara, 1995, Harding, 2004] K can be extended to a functor from the category of bounded posets to the category of orthomodular posets;
- for $f : P \to Q$ is **BPos**, $K(f) : K(P) \to K(Q)$ is given by the rule

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- [Harding, 2004] K is left adjoint to the forgetful functor U from the category of orthomodular posets **OMP** to the category of bounded posets **BPos**.

The Kalmbach monad

Definition

The Kalmbach monad (T, η, μ) on the category **BPos** is given as follows

- T : **BPos** $\to$ **BPos** is the Kalmbach embedding K : **BPos** $\to$ **OMP** composed with the forgetful functor U : **OMP** $\to$ **BPos**, that means, $T = U \circ K$;
- $\eta_P : P \to T(P)$ is given by

$$\eta_P(x) = \begin{cases} [0 < x] & x > 0 \\ \emptyset & x = 0 \end{cases}$$

- $\mu_P : T^2(P) \to T(P)$ is given by

$$\mu_P([C_1 < C_2 < \cdots < C_{2n-1} < C_{2n}]) = C_1 \Delta C_2 \Delta \ldots \Delta C_{2n},$$

  where Δ denotes the symmetric difference of sets.

Algebras for an endofunctor

Let $T : \mathbf{C} \to \mathbf{C}$ be an endofunctor. The category of algebras of T is the category with

- Objects: arrows in $\mathbf{C}$ of the type $T(X) \to X$.
- Morphisms: let $f : T(X) \to X$ and $g : T(Y) \to Y$. An arrow $f \to g$ in the category of algebras is an arrow $h : X \to Y$ in $\mathbf{C}$ such that

$$\begin{tikzcd} T(X) \arrow[r, "T(h)"] \arrow[d, "f"'] & T(Y) \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes.

Algebras for a monad

Let (T, μ, η) be a monad on a category $\mathbf{C}$. An algebra $s : T(P) \to P$ is an algebra for that monad iff the following diagrams commute:

$$\begin{tikzcd} T^2(P) \arrow[r, "T(s)"] \arrow[d, "\mu_P"'] & T(P) \arrow[d, "s"] \\ T(P) \arrow[r, "s"'] & P \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[dr, "1_P"'] & T(P) \arrow[d, "s"] \\ & P \end{tikzcd}$$

Algebras for a monad form a category, called Eilenberg-Moore category for the monad and denoted by $\mathbf{C}^T$.

Why are algebras called algebras?

Recall, that every variety of algebras $\mathbf{D}$ gives us, via the free-forgetful adjunction, a ,,term monad" T on $\mathbf{Set}$.

Theorem

(don't know by who, maybe Beck)

$$\mathbf{D} \simeq \mathbf{Set}^T$$

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A morphism of D-posets is an isotone map preserving 0, 1 and $\ominus$.

D-posets are effect algebras

The categories of effect algebras and D-posets are isomorphic:

$$a \oplus b = c$$

is the same thing as

$$c \ominus b = a$$

Where do the D-posets come from

Theorem

The category of D-posets (and hence the category of effect algebras) is isomorphic to the Eilenberg-Moore category for the Kalmbach monad.

From effect algebras to algebras for the Kalmbach monad

- If E is an , then we define $s : T(E) \to E$

  $s([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]) = (x_2 \ominus x_1) \oplus \cdots \oplus (x_{2n-1} \ominus x_n)$.

  This is then an algebra for the Kalmbach monad.

From algebras for the Kalmbach monad to effect algebras

- If $s : T(P) \to P$ is an algebra for the Kalmbach monad, then we define, for $a \leq b$

$$b \ominus a = \begin{cases} 0 & a = b \\ s([a < b]) & a < b \end{cases}$$

$$
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Thank you for your attention

John Harding. Remarks on concrete orthomodular lattices. International Journal of Theoretical Physics, 43(10):2149–2168, 2004.

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