Data Analysis: A 3-D Perspective

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FSTA 2014 - 30, January

भारतीय प्रौद्योगिकी संस्थान हैदराबाद

Data Analysis: A 3-D Perspective

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Dimension,

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Dimension, **D**istance and

Data Analysis: A 3-D Perspective

Dimension, Distance and Distribution.

Data Analysis: A 3-D Perspective

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Who is your nearest neighbour?

Who is your closest friend?

Who is your closest friend?

Does it even make sense?

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3-D's in Data Analysis

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1. Dimensions

High Dimensional Data

High Dimensional Data

Occurence

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- Images -

High Dimensional Data

Occurence

- Images - 256×256 resolution -

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Dimension

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Distribution

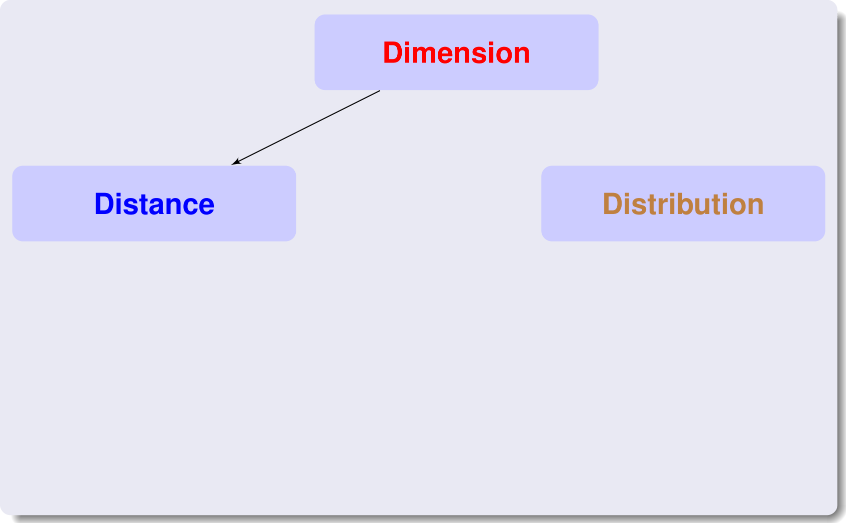
This talk: At a Glance!

Dimension

Distance

Distribution

This talk: At a Glance!



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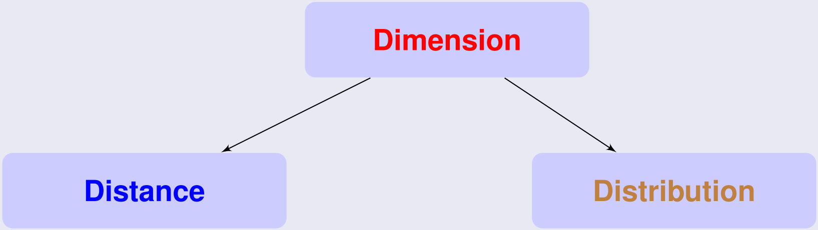

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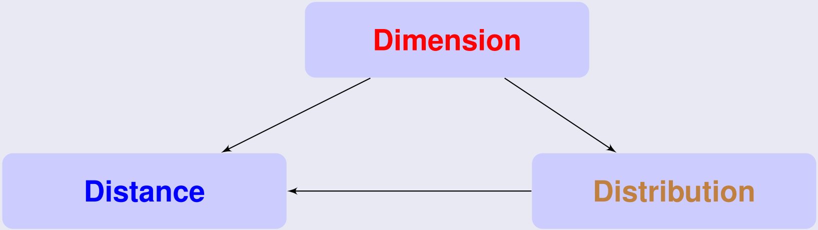

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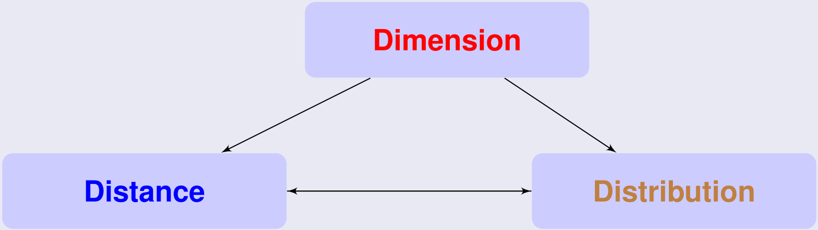

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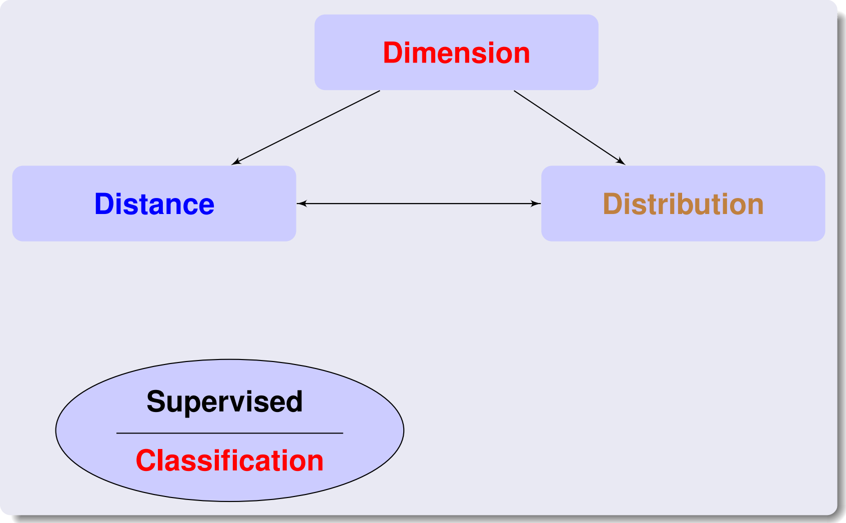

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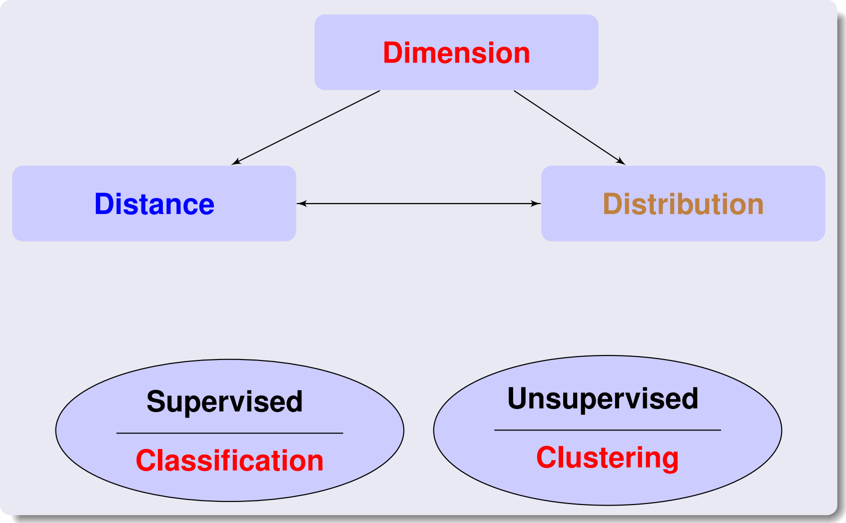

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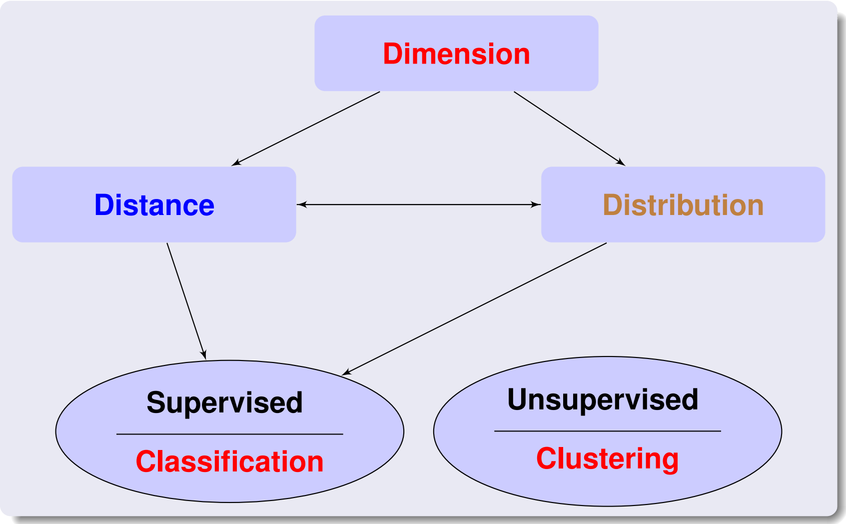

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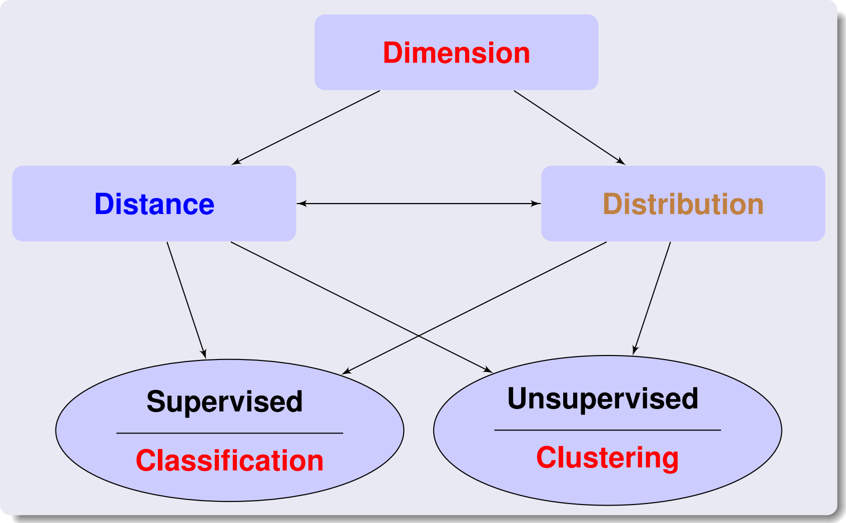

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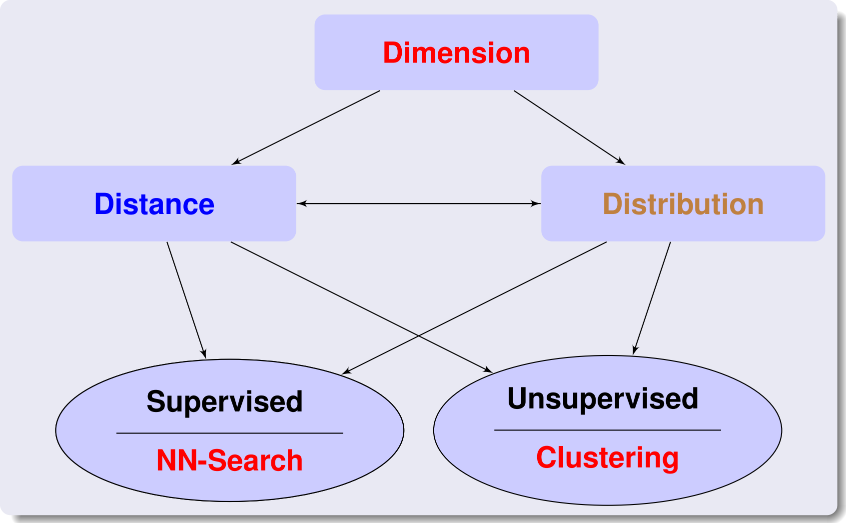

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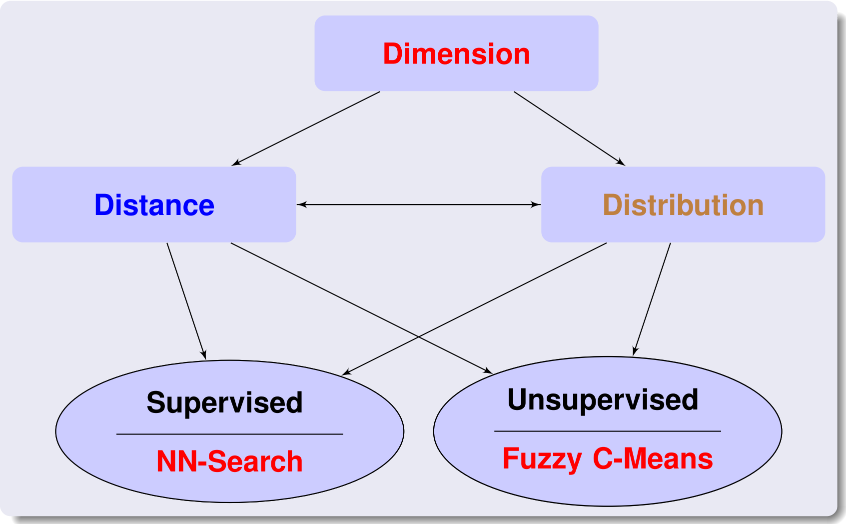

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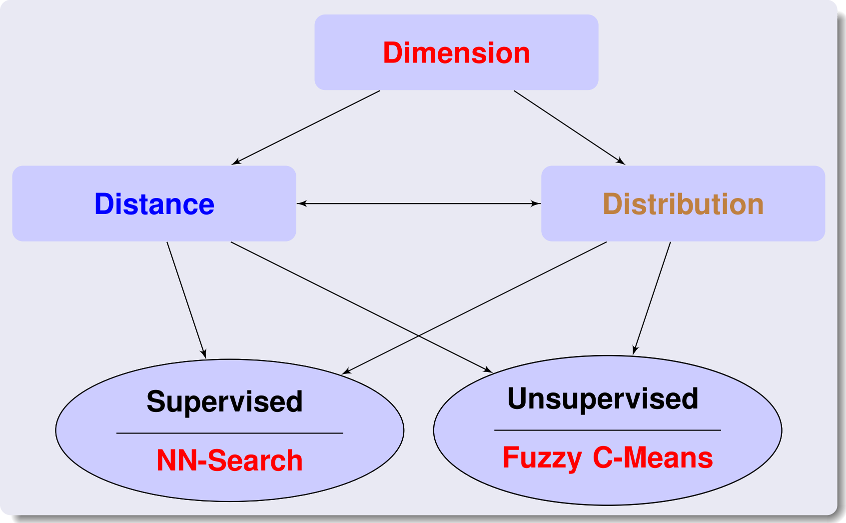


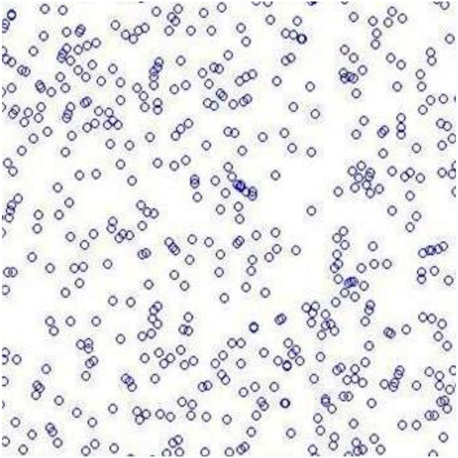
Algorithms in Data Analysis

Algorithms in Data Analysis

Query Searching & Clustering

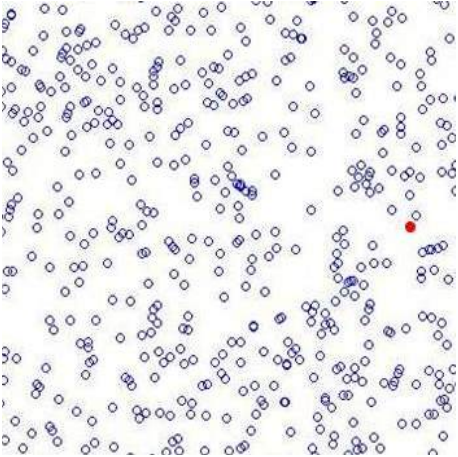
Nearest Neighbour Search - A Pictorial Example

Consider the data points in the figure.



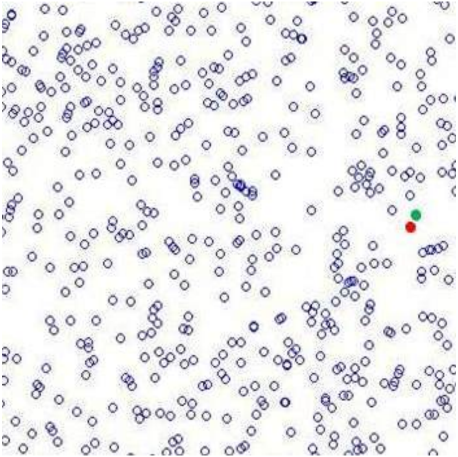
Nearest Neighbour Search - A Pictorial Example

Consider a query point in **red**.



Nearest Neighbour Search - A Pictorial Example

Its nearest neighbour is the point in **green**



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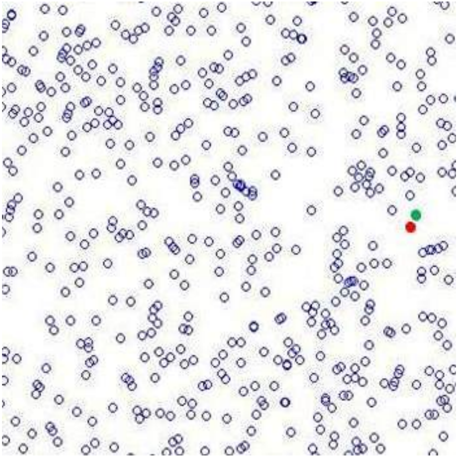
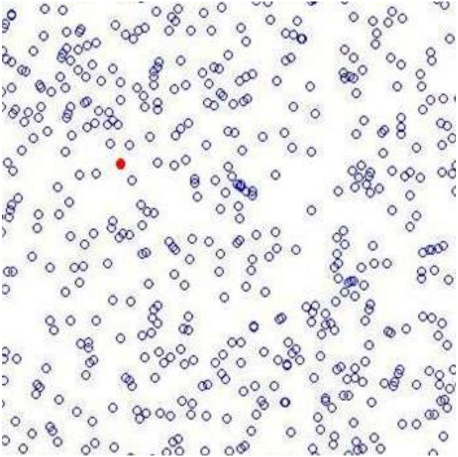
$$NN_1(\bar{x}) = \underset{\bar{x}_i \in \mathcal{X}}{\arg\min}\{\|\bar{x} - \bar{x}_i\|\}.$$



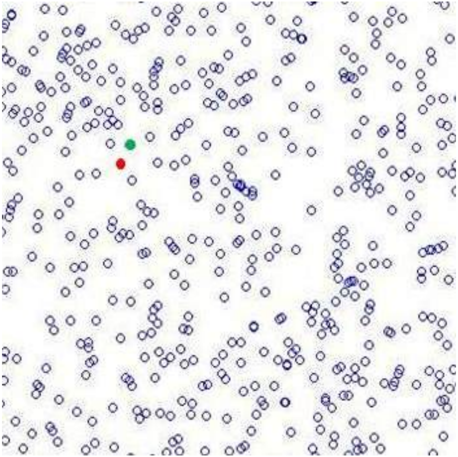
Nearest Neighbour Search - A Pictorial Example

Consider another query point in **red**.



Nearest Neighbour Search - A Pictorial Example

Who is his nearest neighbour?



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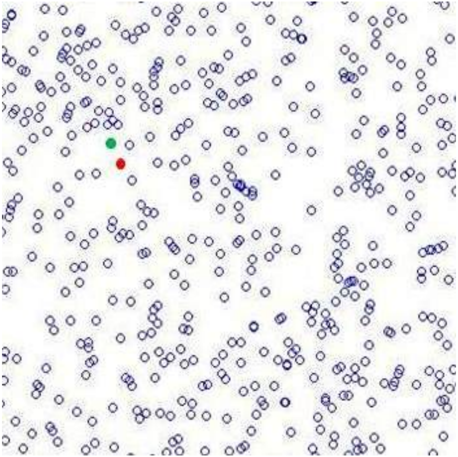

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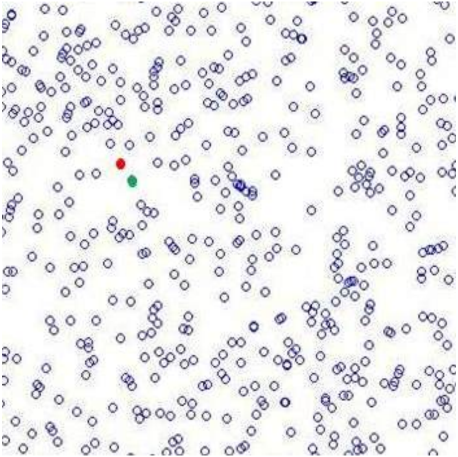
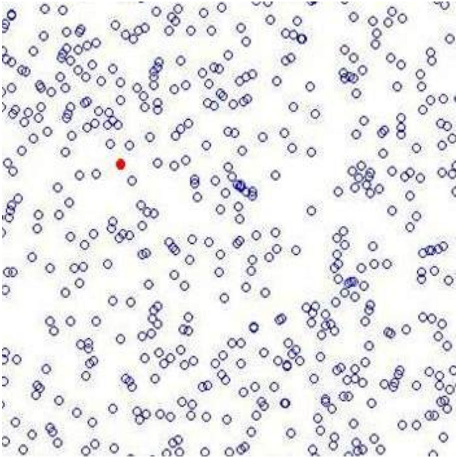


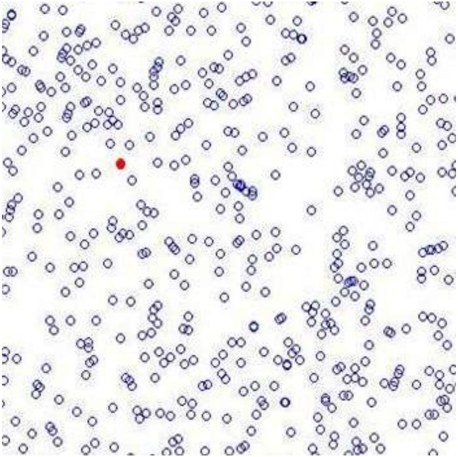
k-NN Search - A Pictorial Example

Find k-nearest neighbours of the query point in **red**.



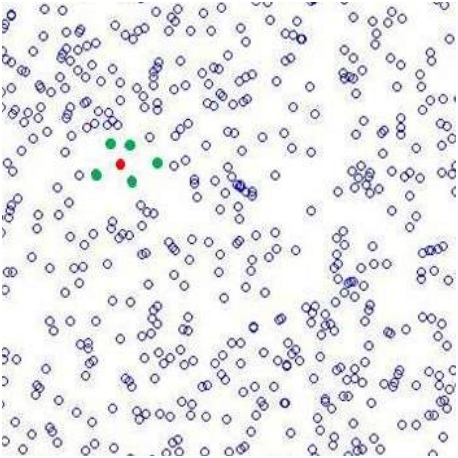
k-NN Search - A Pictorial Example

$NN_k(\bar{x})$ - k-nearest neighbours of $\bar{x}$.



k-NN Search - A Pictorial Example

Find 5-nearest neighbours of the query point in **red**.



ε-Neighbourhood - A Pictorial Example

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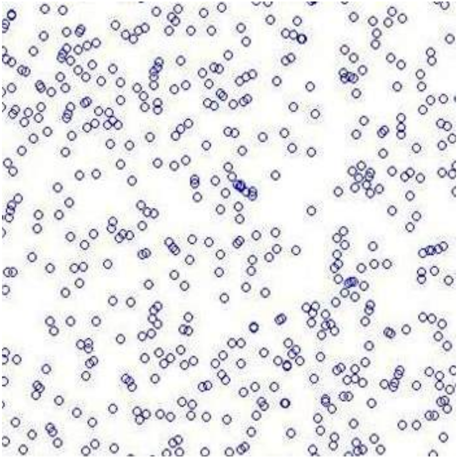

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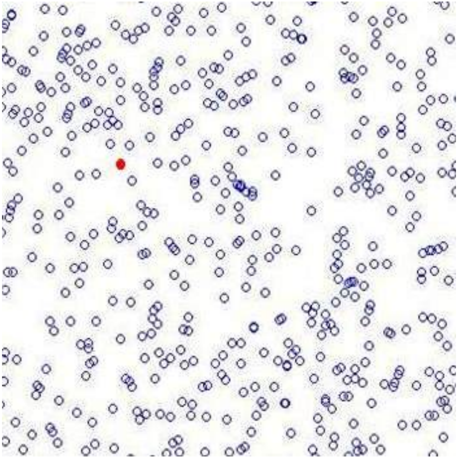

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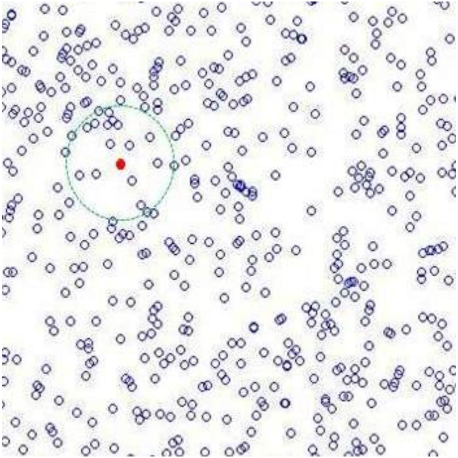

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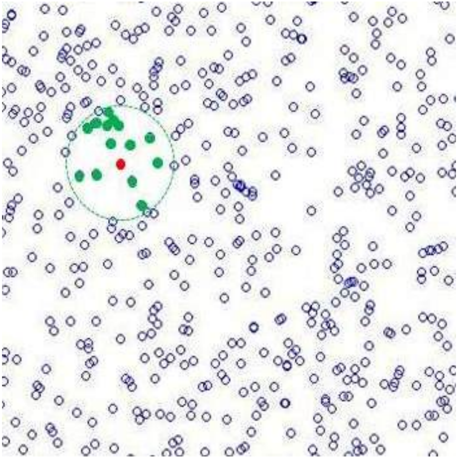

Data Clustering

Clustering - General Idea

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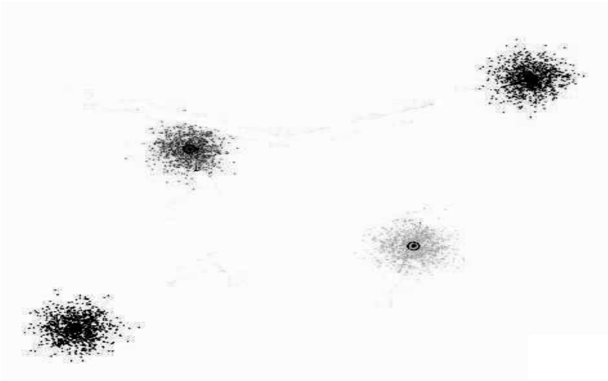
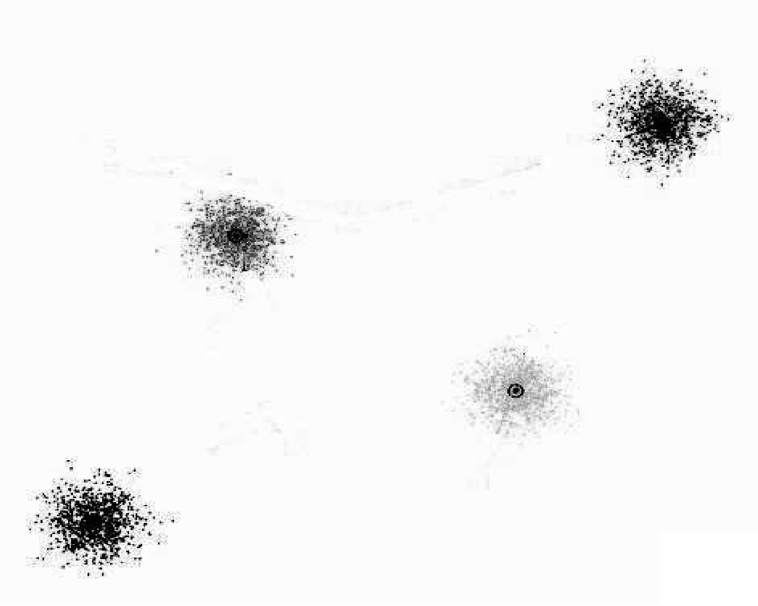
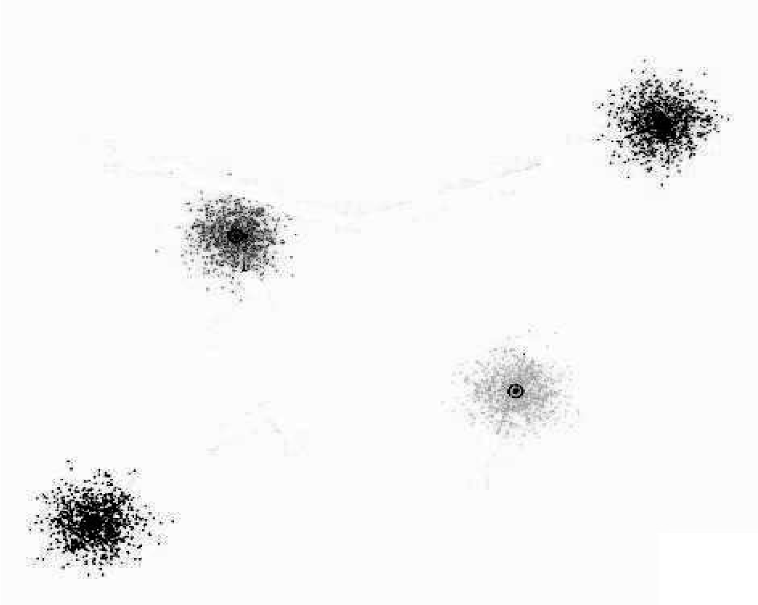


Data to be clustered



Fuzzy C-Means



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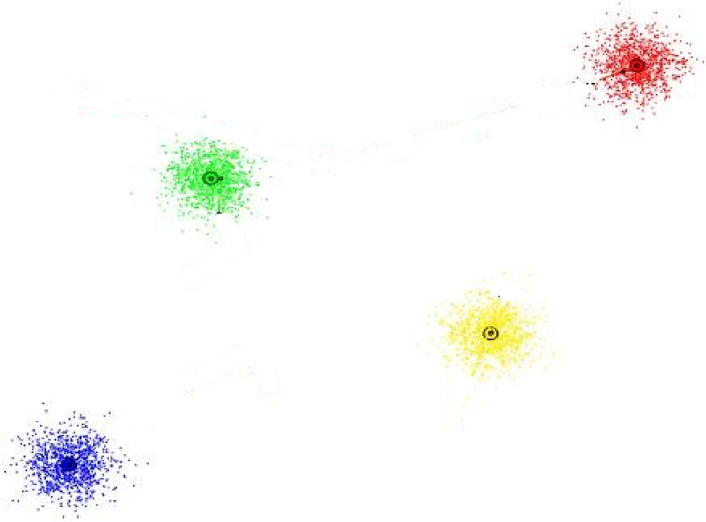
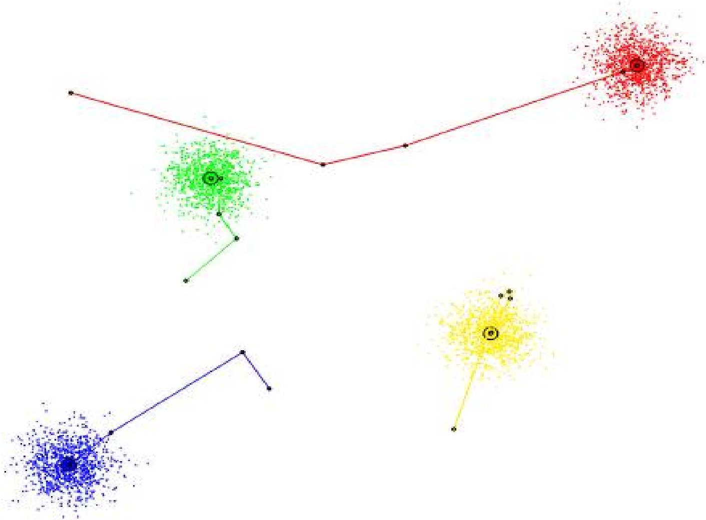


Fuzzy C-Means - In Low Dimensions



Issues with Fuzzy C-Means

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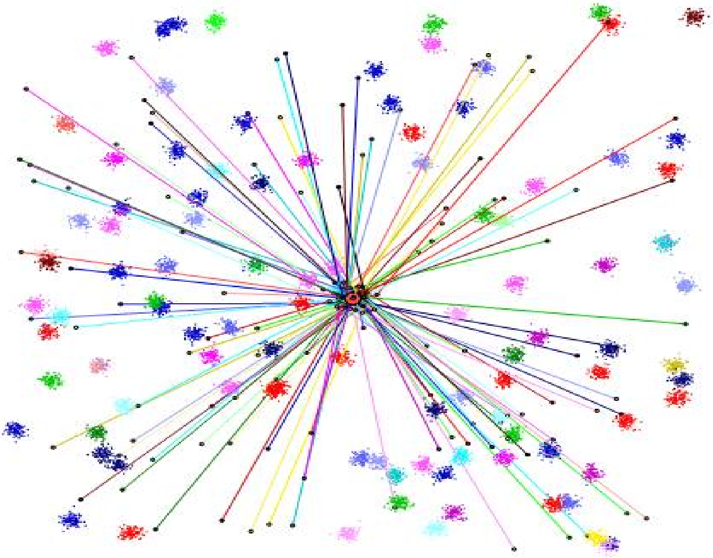

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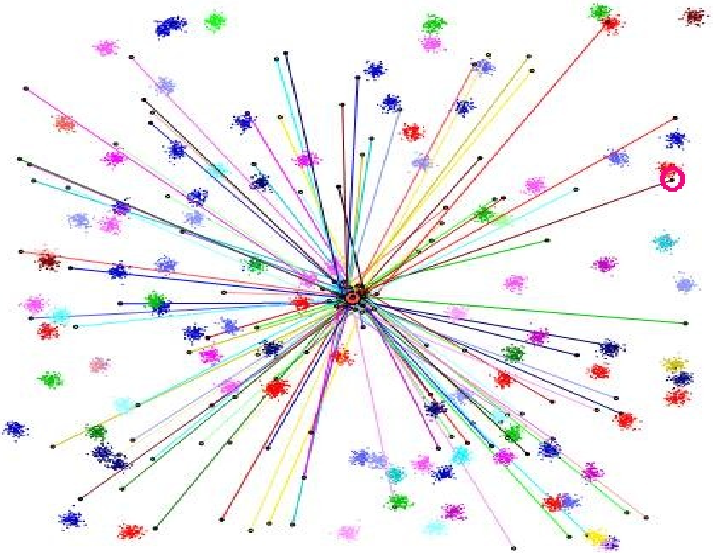

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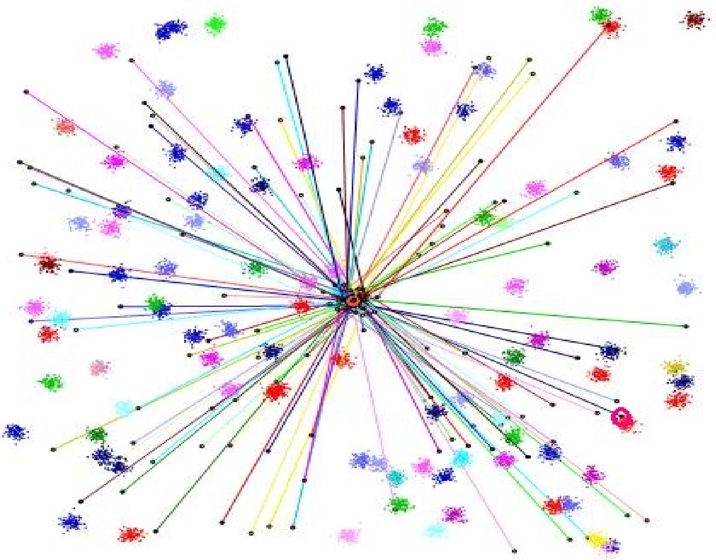

Data Analysis - A Math Challenge!!

Issues in High Dimensional Data

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Curse of Dimensionality

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Curse of Dimensionality

1. Combinatorial Explosion in Search Space

Curse of Dimensionality - Aspect 1

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What does it affect?

Increases Computational Complexity.

Curse of Dimensionality

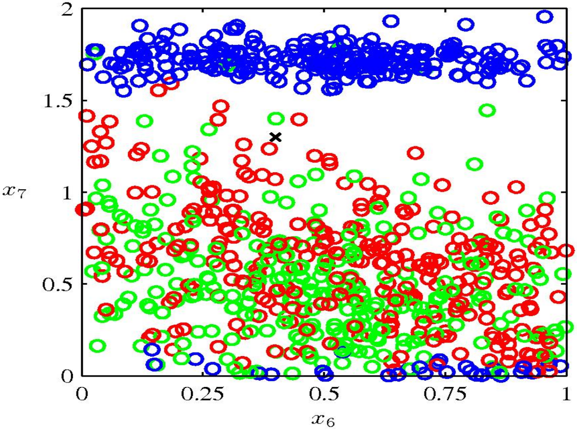
2.

Curse of Dimensionality

2. Need for Greed

Curse of Dimensionality - Aspect 2

Classify the point marked x.



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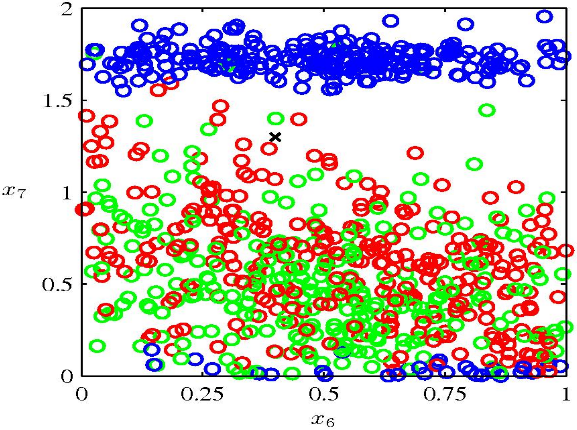
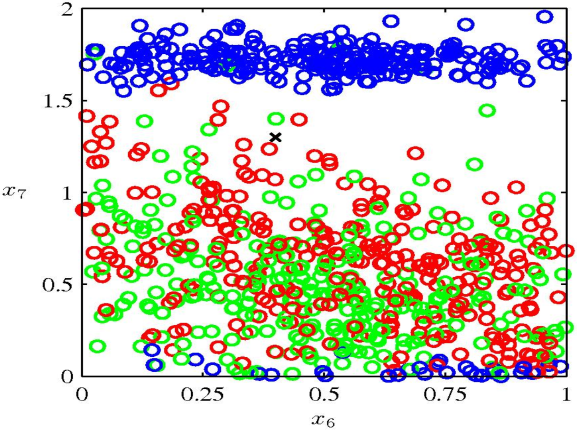


Curse of Dimensionality - Aspect 2

Partition the space into grids.



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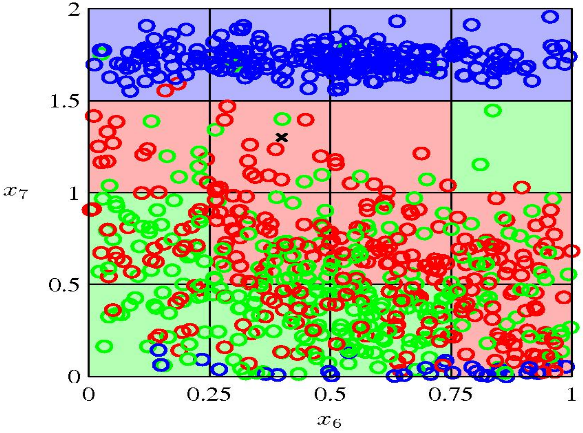

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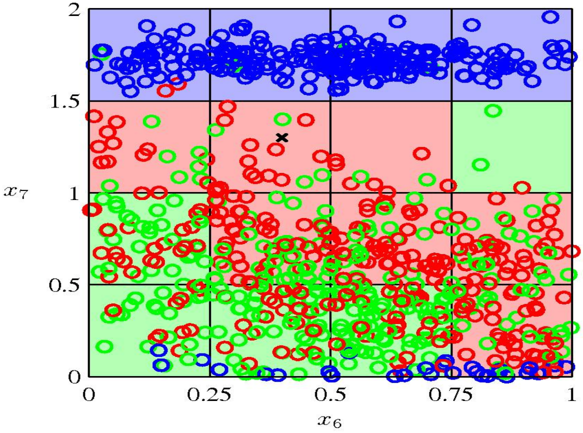
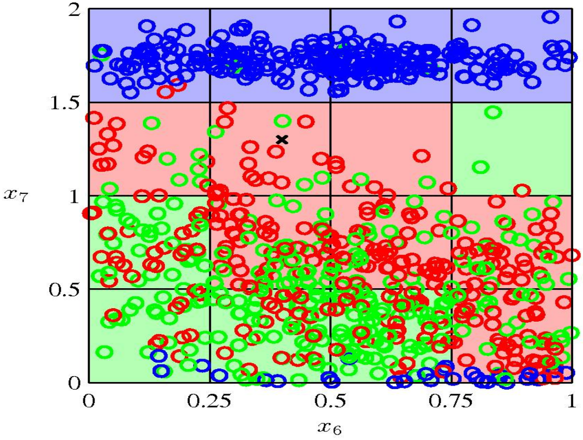
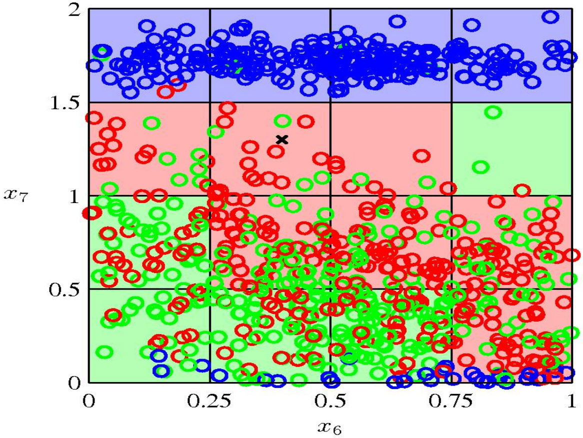


Count the no. of points in each class in the grid containing x.



Curse of Dimensionality - Aspect 2

Assign x to the class having most points in the same grid.



2. Need for Greed - $(N \gg m)$

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$D = 1$

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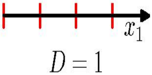

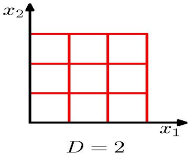

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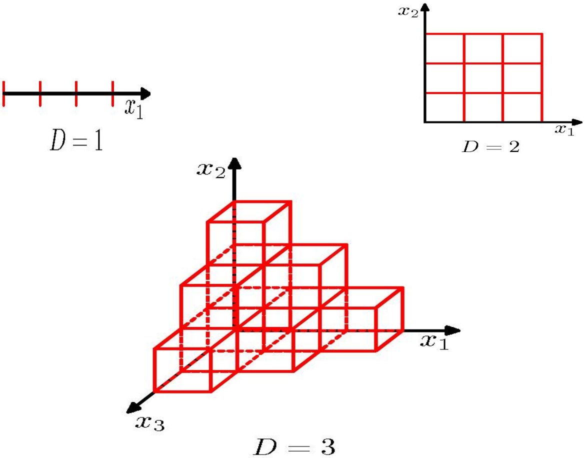

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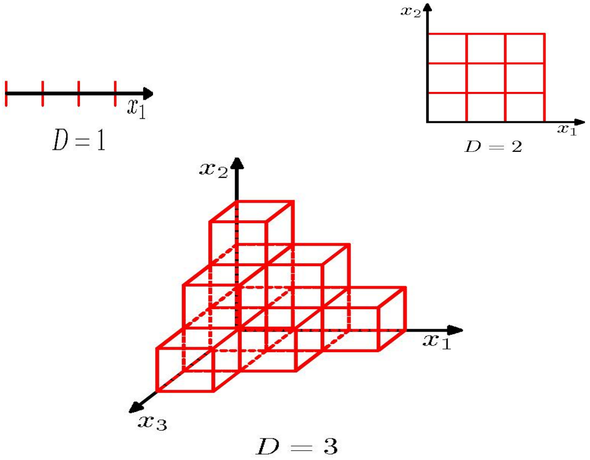

As m increases, the no. of points N also should increase!

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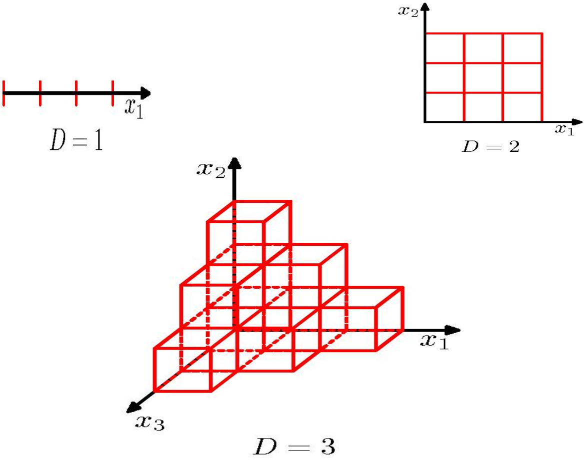

Increases Storage Complexity.

Curse of Dimensionality

3.

Curse of Dimensionality

3. The Empty Space Phenomenon

Curse of Dimensionality - Aspect 3

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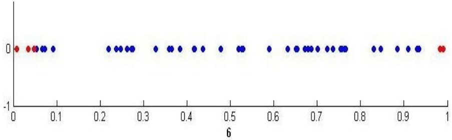

Curse of Dimensionality - Aspect 3

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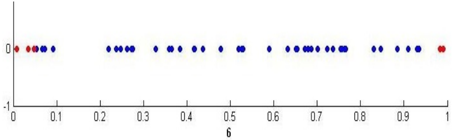

Roughly 10% of the data are in the edges.

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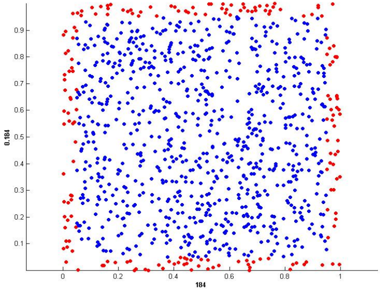

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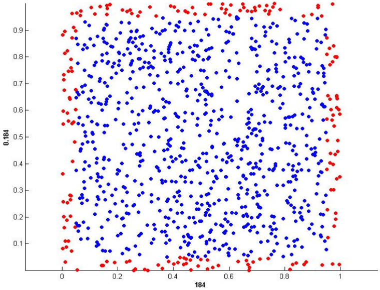

Roughly 19% of the data are in the edges.

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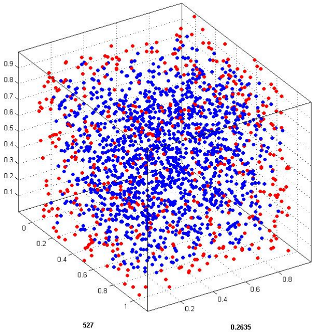

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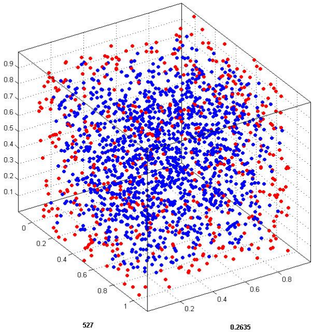

Roughly 27% of the data are in the edges.

3. The Empty Space Phenomenon

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3. The Empty Space Phenomenon

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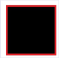

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3. The Empty Space Phenomenon

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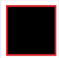
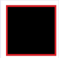


- In 2-D: $1 - (0.9)^2 = 0.19 = 19\%$
- In 3-D: $1 - (0.9)^3 = 0.271 = 27.1\%$

3. The Empty Space Phenomenon

- How many data points are close to the edges?
- In 1-D: $1 - 0.9 = 0.1 = \color{blue}{10\%}$



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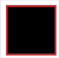

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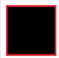

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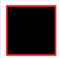

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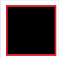

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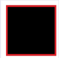

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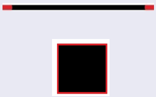

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The Empty Space Phenomenon !!

Curse of Dimensionality

4.

Curse of Dimensionality

4. Relations among the Dimensions

Curse of Dimensionality - Aspect 4

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Correlation among the Dimensions

Curse of Dimensionality - Aspect 4

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Data may lie in a low-dimensional manifold, i.e., $\ell \ll m$.

4. Intrinsic dimn far less than Embedding dimn - $\ell \ll m$

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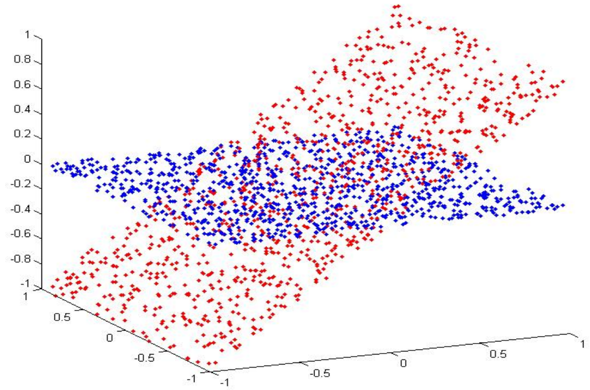
Linear Manifolds - $\ell = 2 < m = 3$



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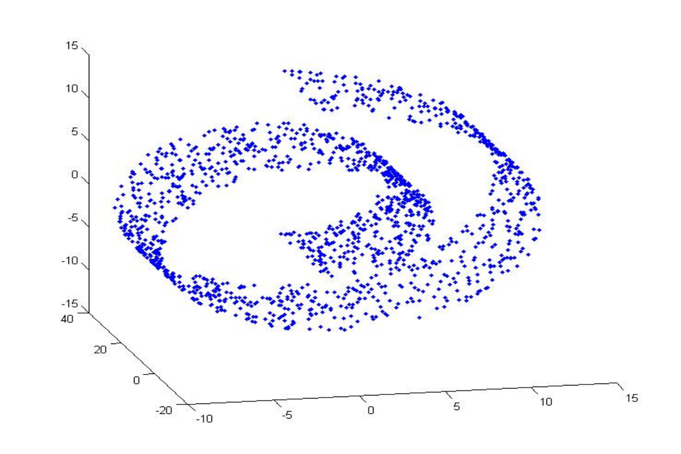

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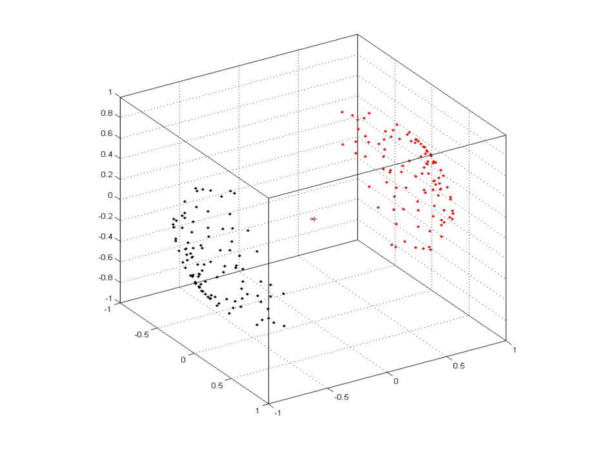

The story so far ...

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Dimension

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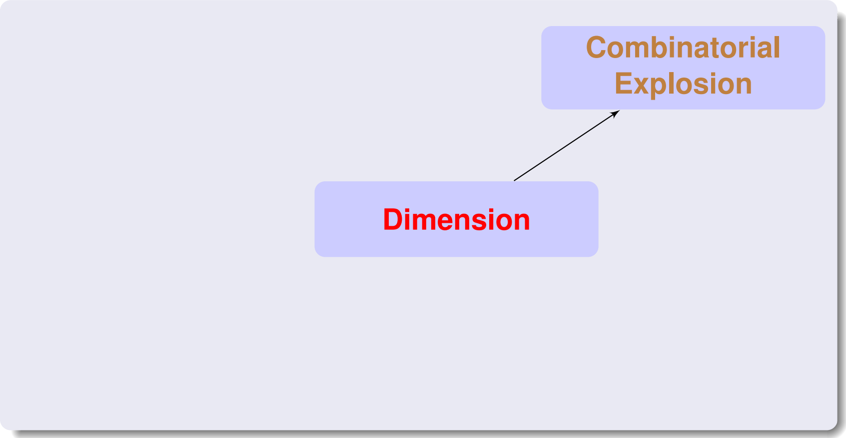

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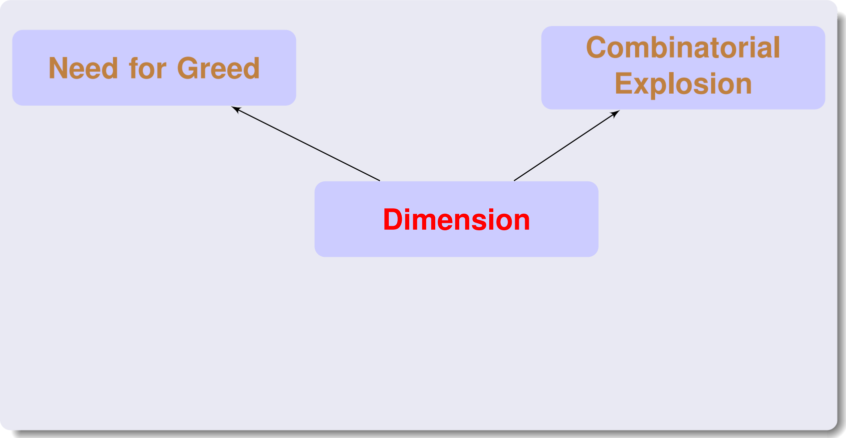

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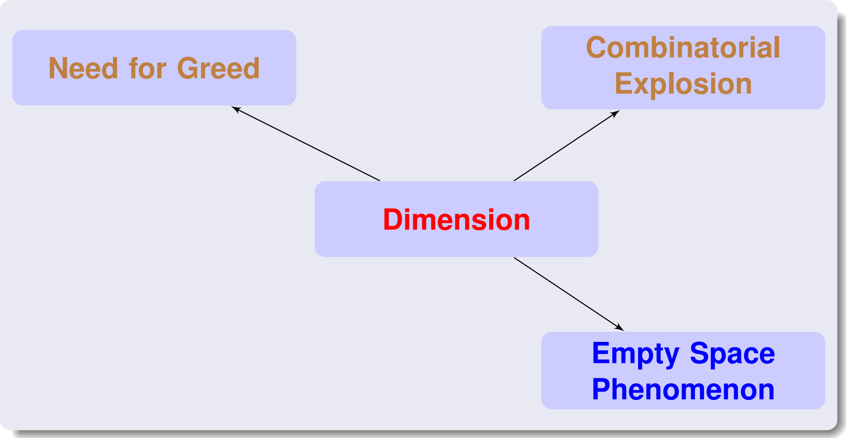

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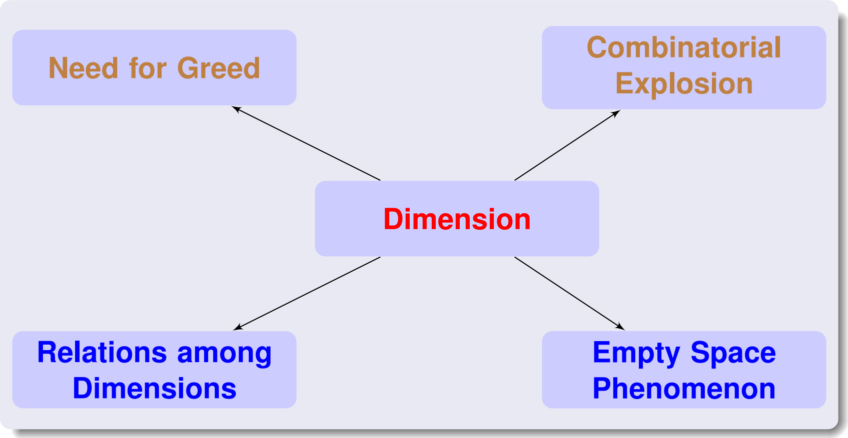

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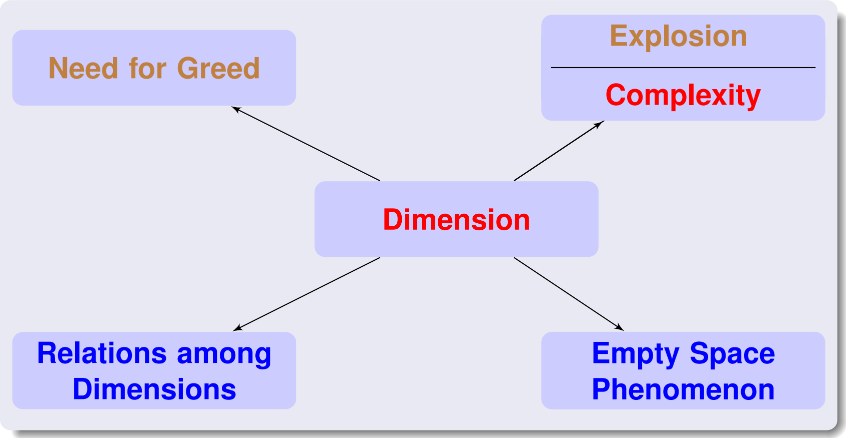

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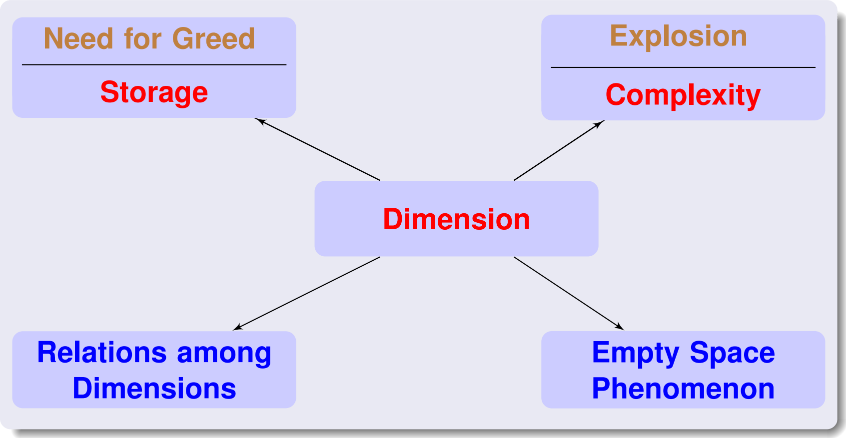

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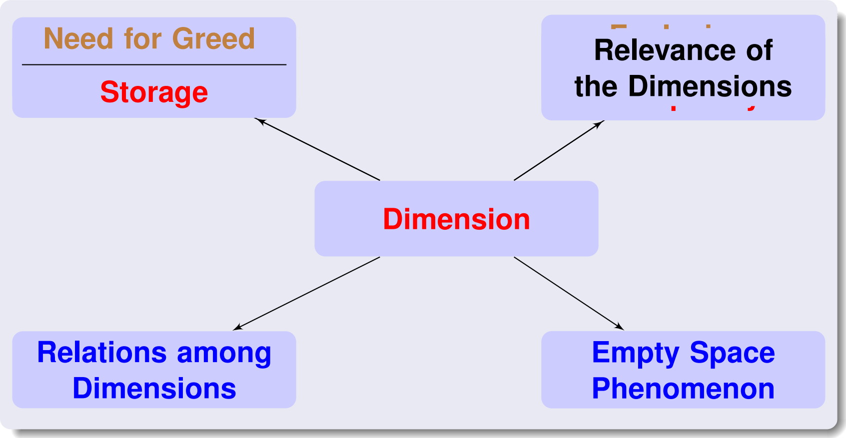

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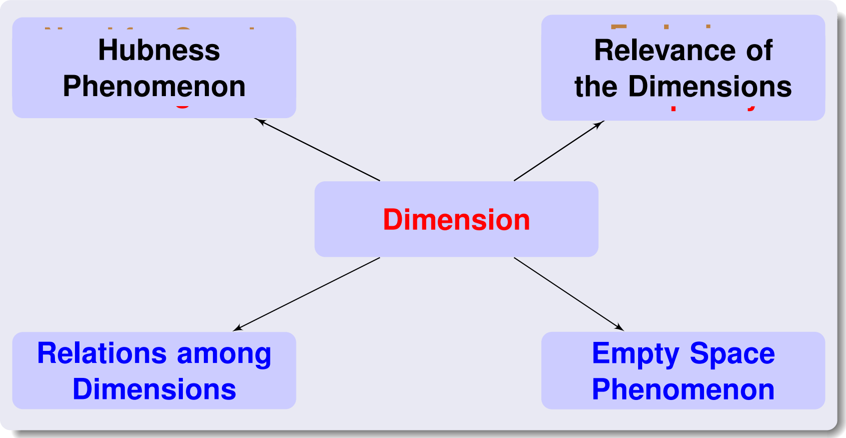

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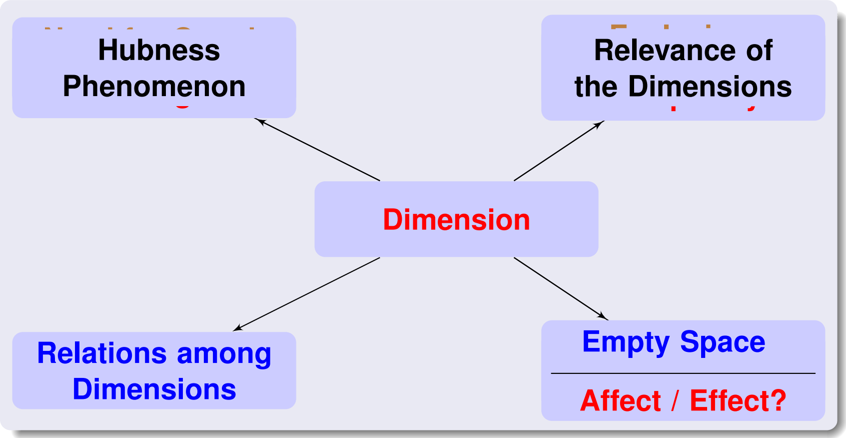

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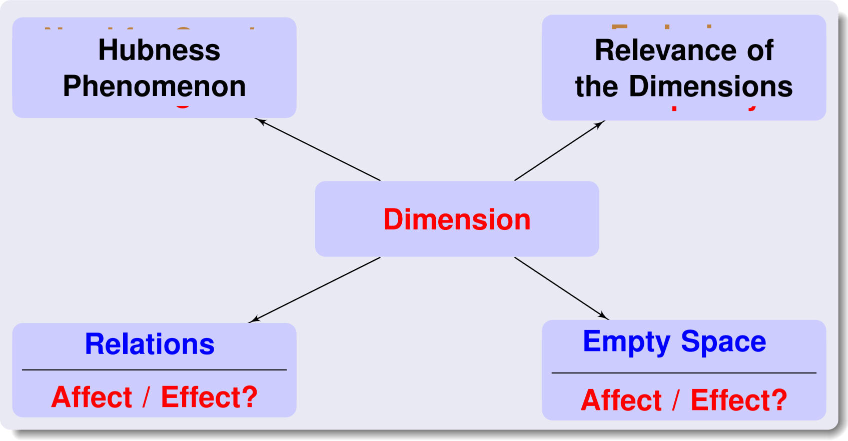

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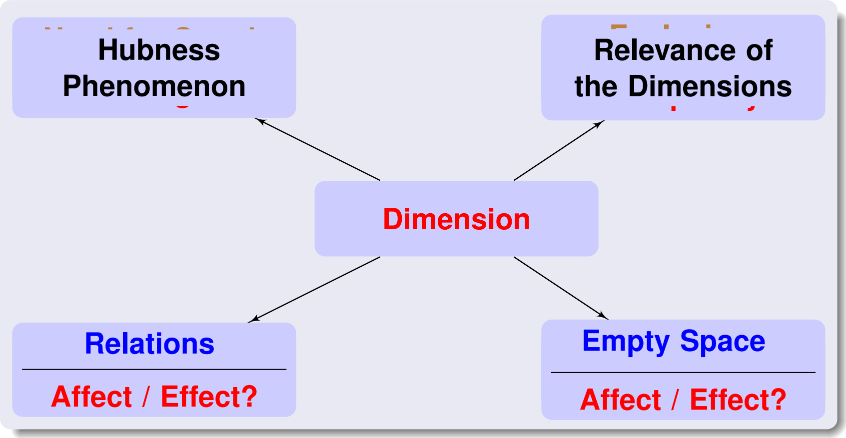

The story ahead ...

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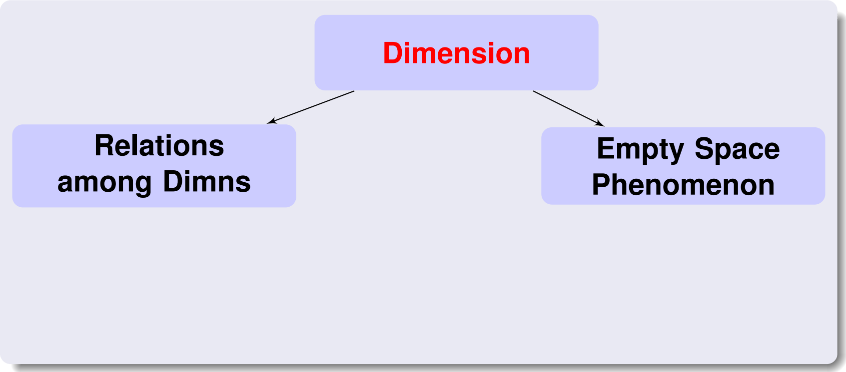

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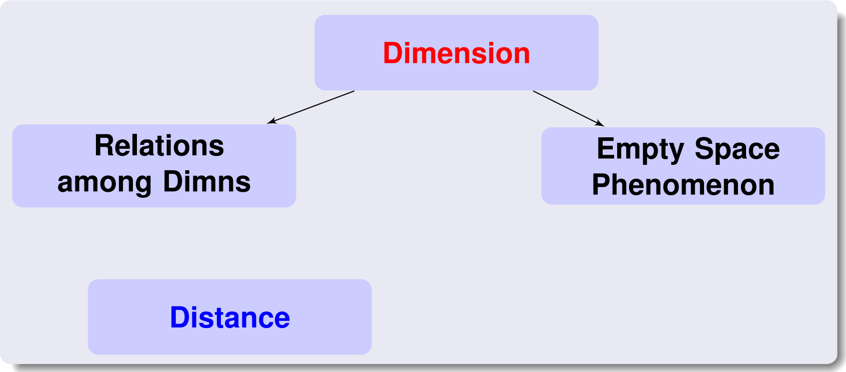

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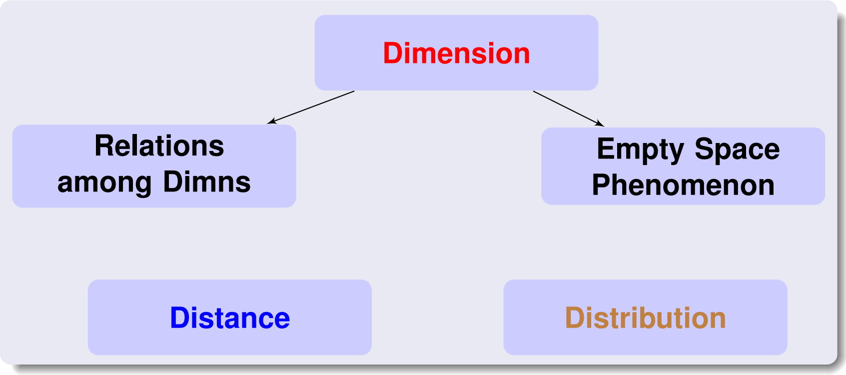

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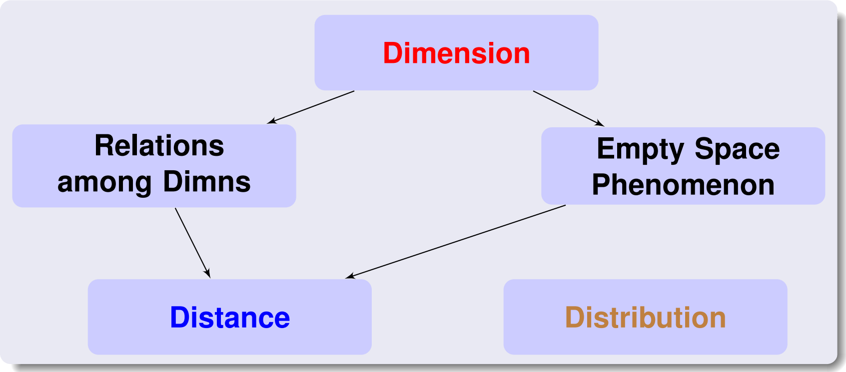

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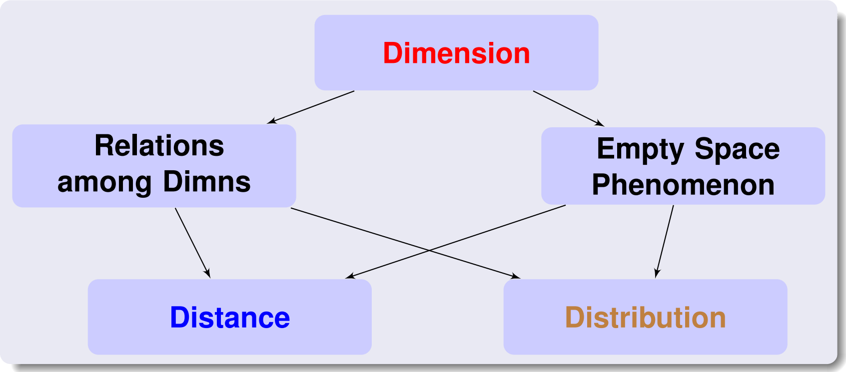

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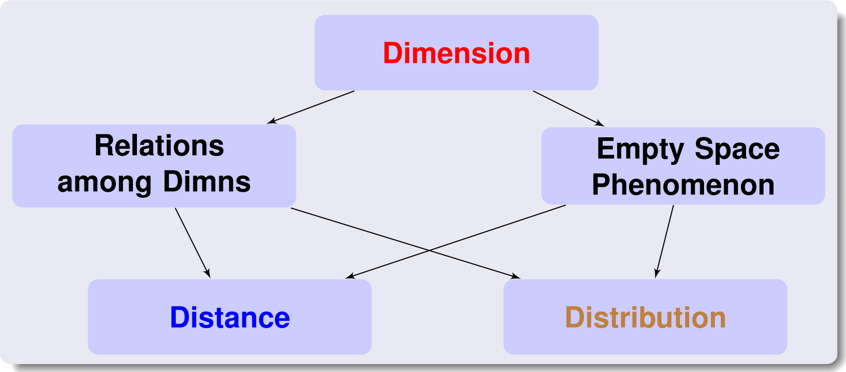

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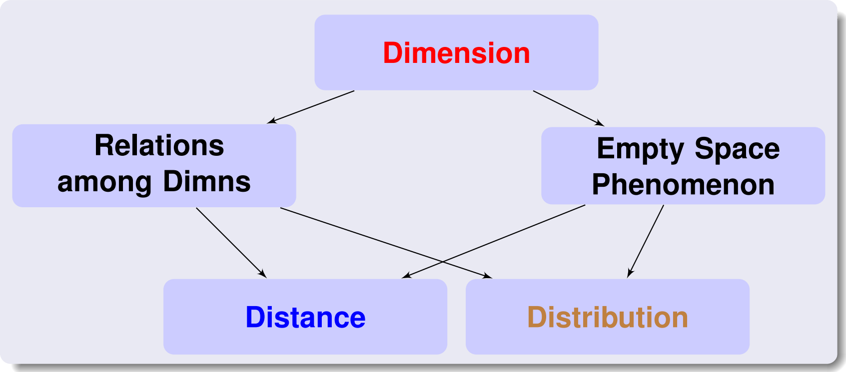

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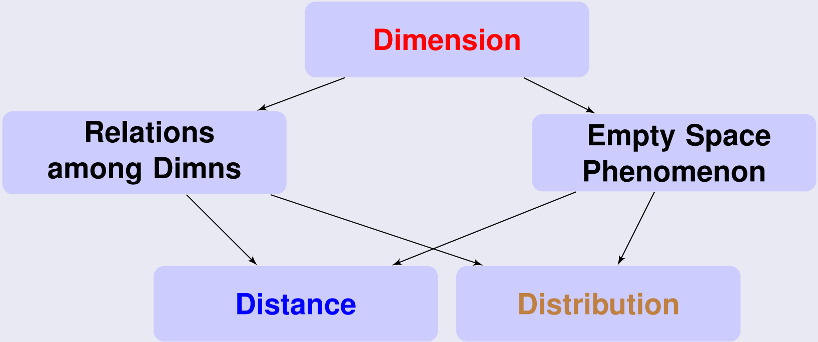
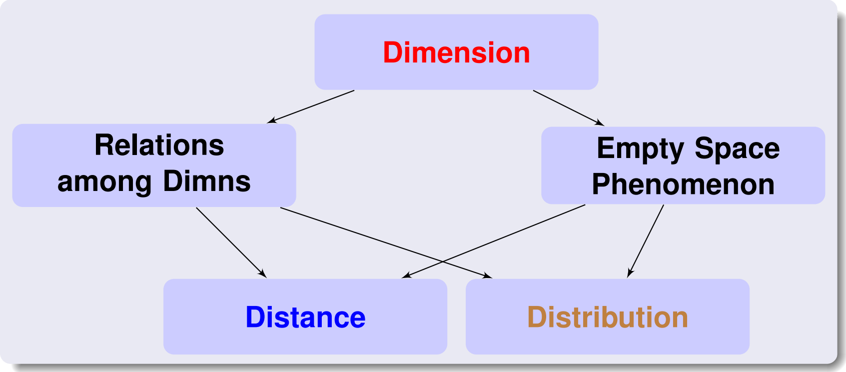


- How **D** & **D** are affected by the above issues.

The story ahead ...



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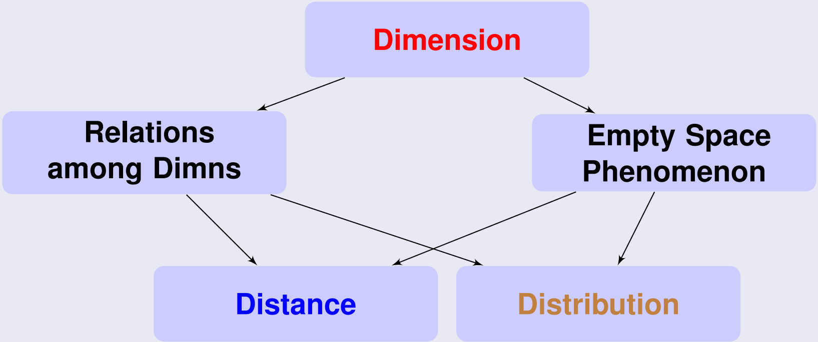

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3-D's in Data Analysis

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2. Distance

Importance of Distance Measures

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In Similarity Searches

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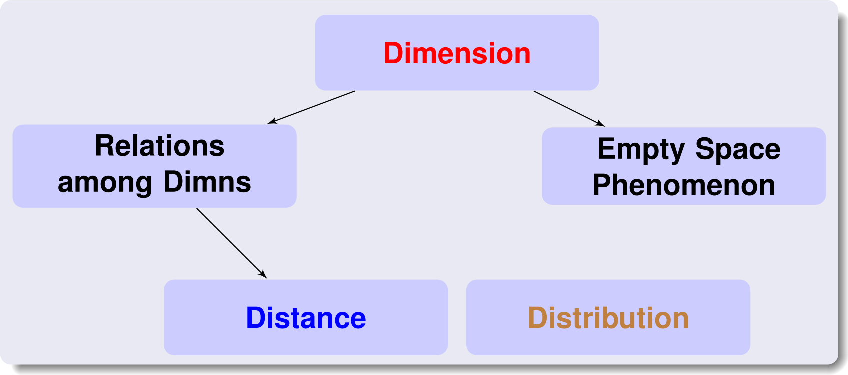
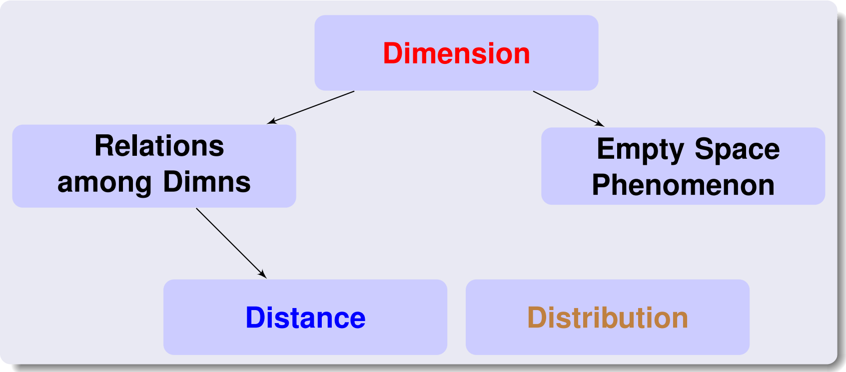

Dimension
Relations among Dimns
Empty Space Phenomenon
Distance
Distribution



Appropriateness of Distance Functions

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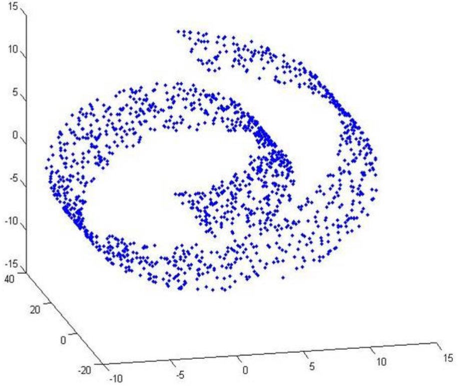
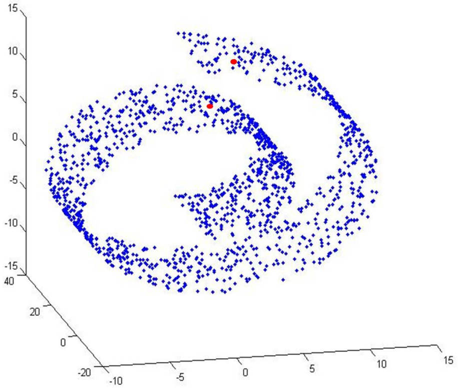
Consider the Swiss Roll data set.



Appropriateness of Distance Functions

Are these two points close or far?



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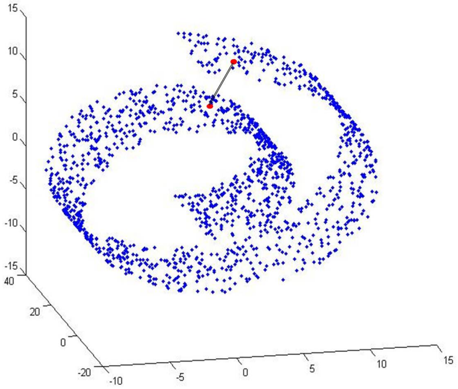
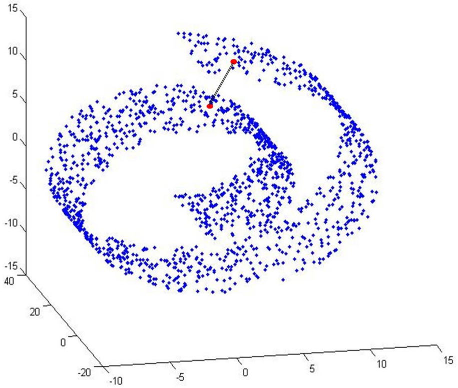


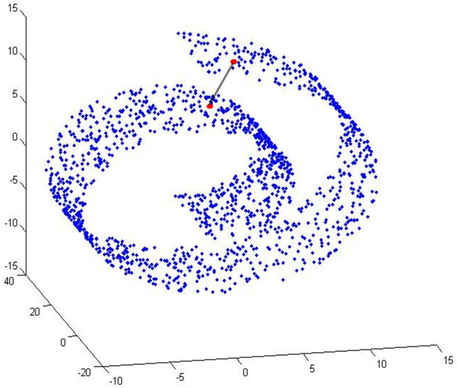
Appropriateness of Distance Functions

Intrinsic dimension $\ell = 2$, while Embedding dimension $m = 3$.



Appropriateness of Distance Functions

Distance in which dimension - Intrinsic or Embedding ?



Relation between Dimensions - Revisited

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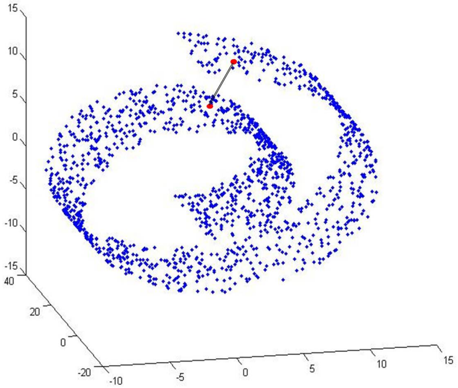
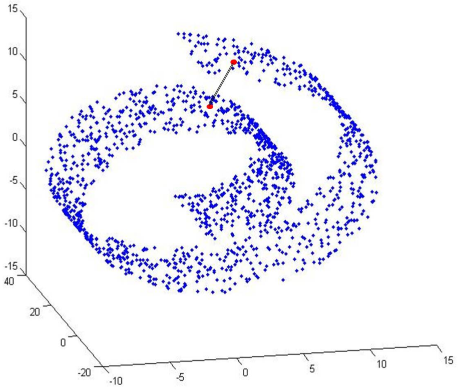


Relation between Dimensions - Revisited

Distance along which manifold ?



Appropriateness of Distance Functions

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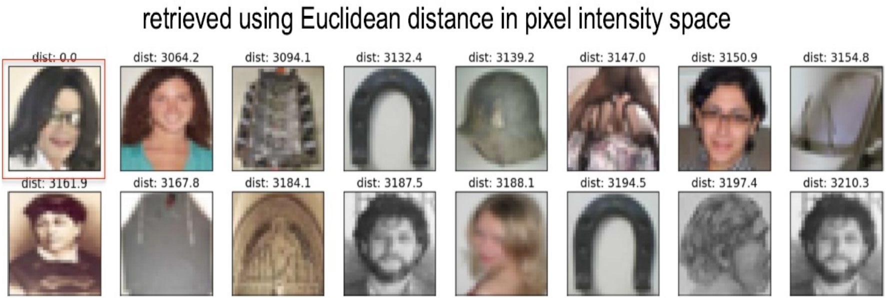
Euclidean distance in pixel intensity space

Appropriateness of Distance Functions

retrieved using Euclidean distance in pixel intensity space

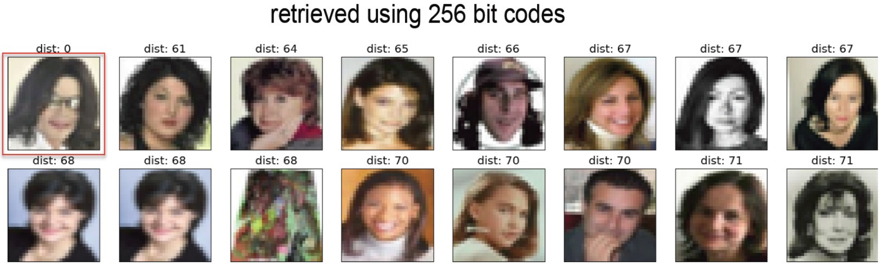
Appropriateness of Distance Functions



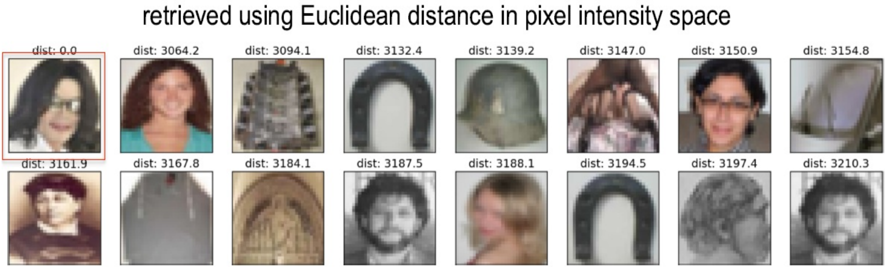
Appropriateness of Distance Functions

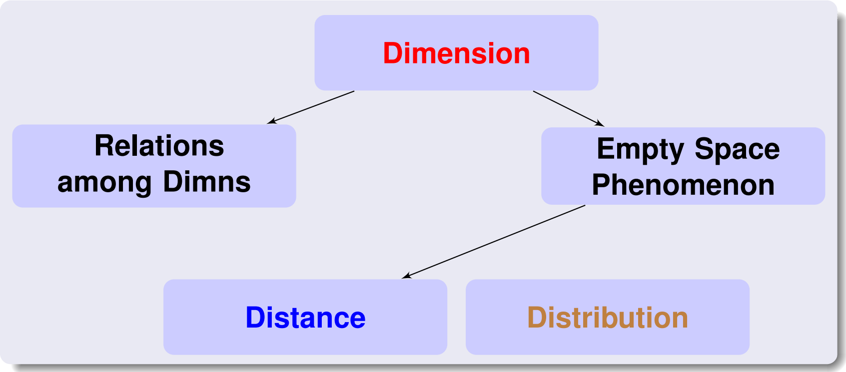
retrieved using 256 bit codes

Appropriateness of Distance Functions



Appropriateness of Distance Functions

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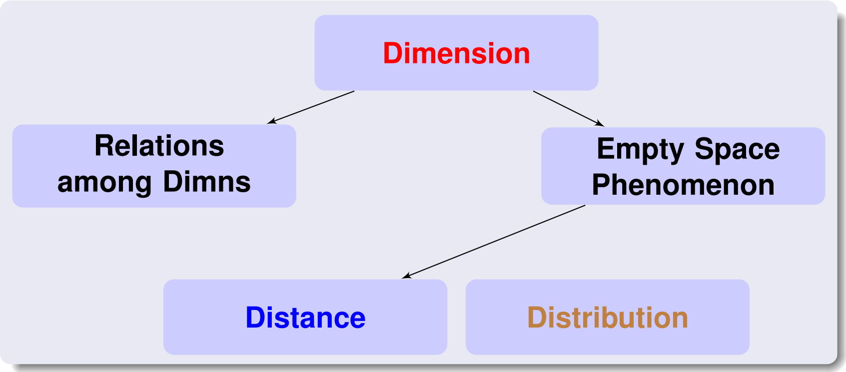

Meaningfulness of Distances

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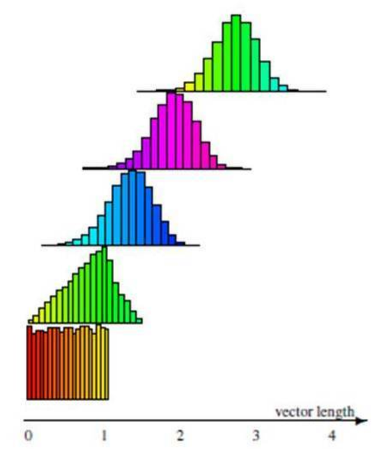
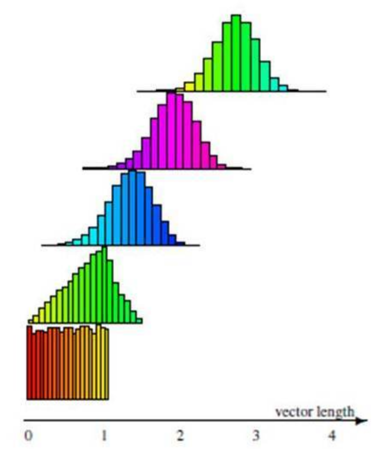


Distribution of Euclidean Distances

- $\|\bar{x}_i\|$ is small iff all m-components are small.
- When $m = 10$, $P\{\bar{x} \mid \|\bar{x}\| \leq 1\}$ is extremely small.



Distribution of lengths w.r.to any L_p Norm

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- Let $\bar{x} = (x_1, x_2, \ldots, x_m)$.

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Distribution of lengths of vectors in unit hypercube

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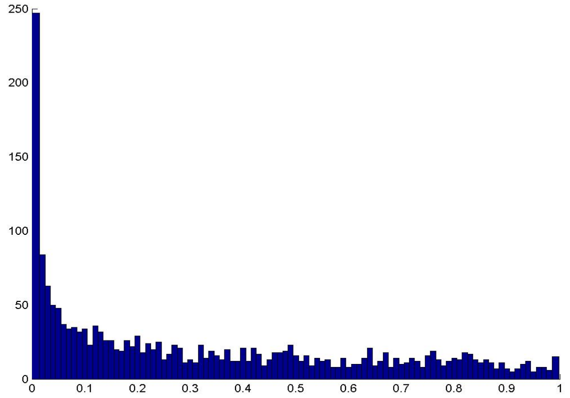
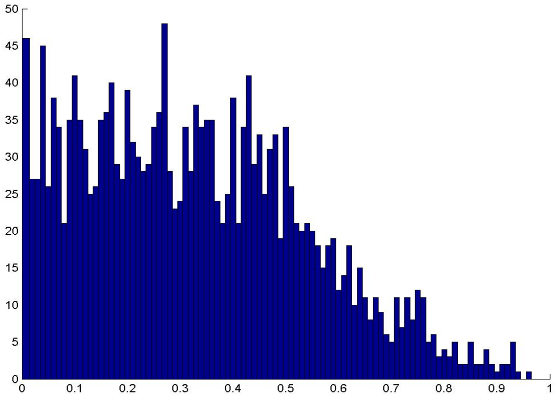
Distribution of distances to the Origin

$m = 1, p = 2$



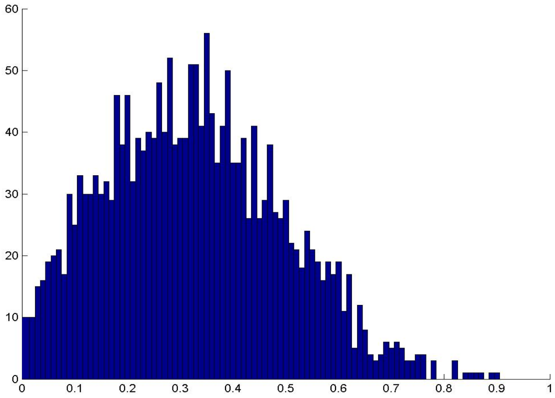
Distribution of distances to the Origin

$m = 2, p = 2$



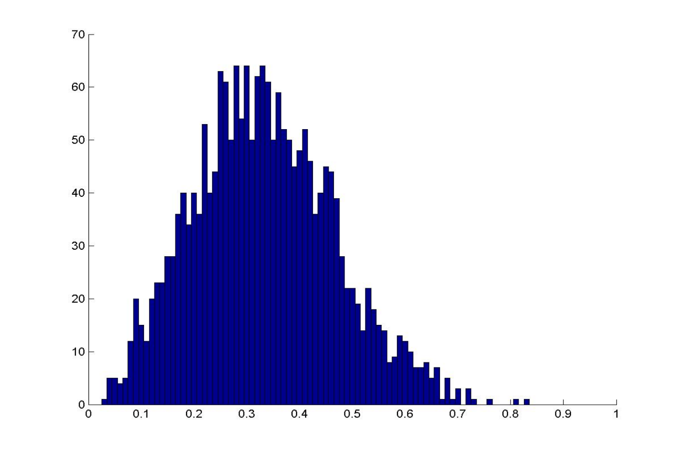
Distribution of distances to the Origin

$m = 3, p = 2$



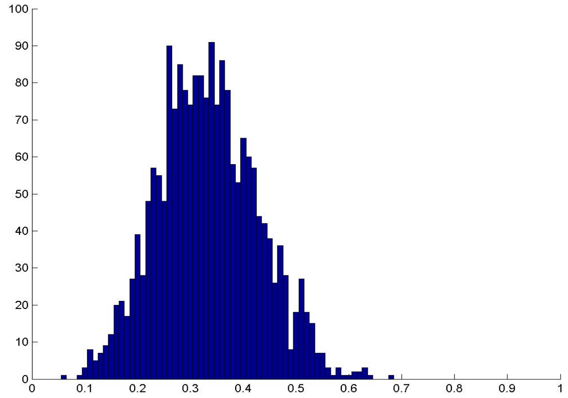
Distribution of distances to the Origin

$m = 5, p = 2$



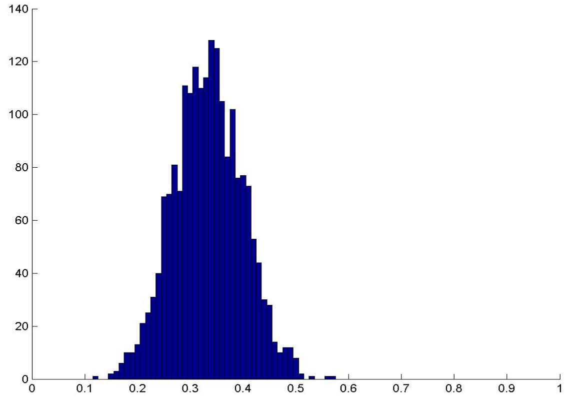
Distribution of distances to the Origin

$m = 10, p = 2$



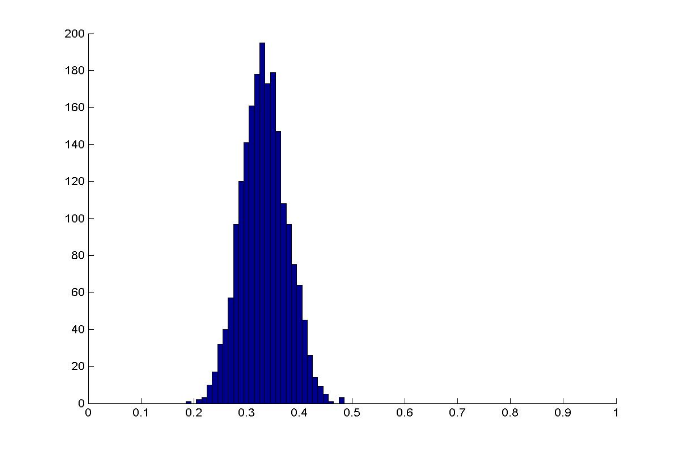
Distribution of distances to the Origin

$m = 20, p = 2$



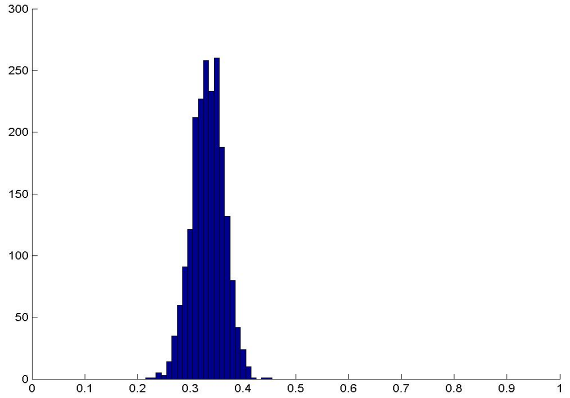
Distribution of distances to the Origin

$m = 50, p = 2$



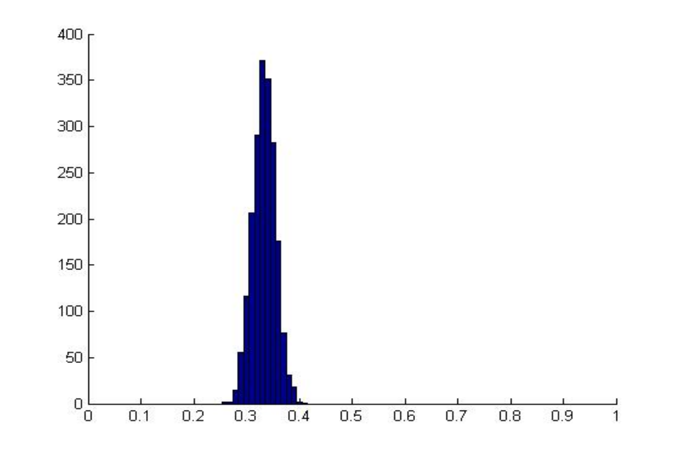
Distribution of distances to the Origin

$m = 100, p = 2$



Distribution of distances to the Origin

$m = 200, p = 2$



Distribution of lengths w.r.to any $\|\cdot\|$

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Meaningfulness of distance functions in HD

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- ... of the distances to the origin converges to 0, then ...

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The Empty Space phenomenon?

Interplay between the 3D's - 1

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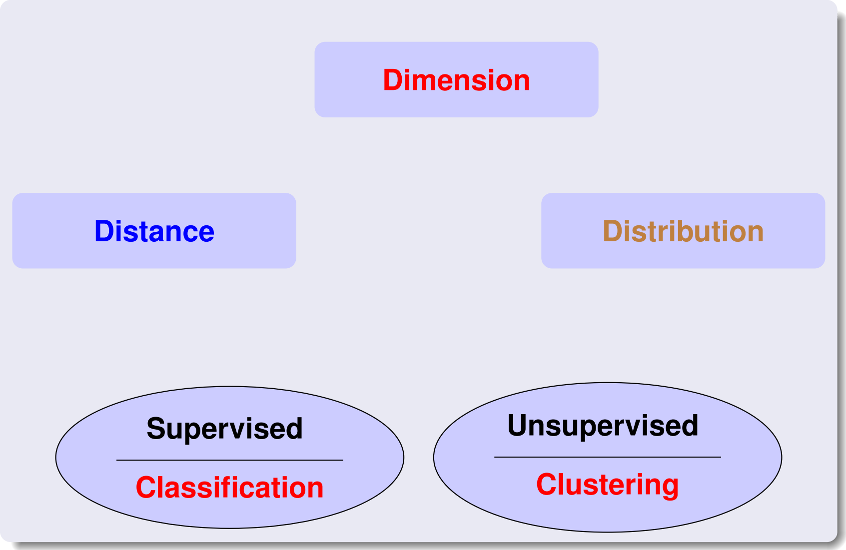
Dimension

Distance

Distribution

Interplay between the 3D's - 1



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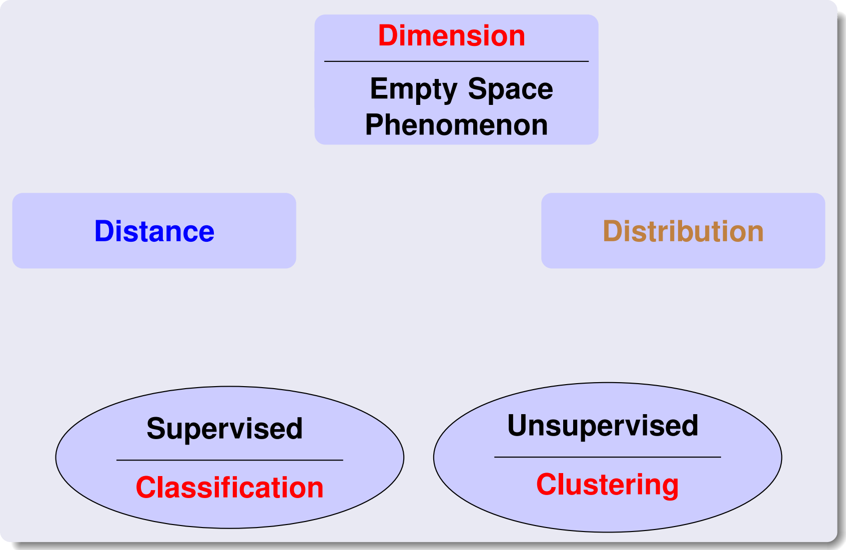

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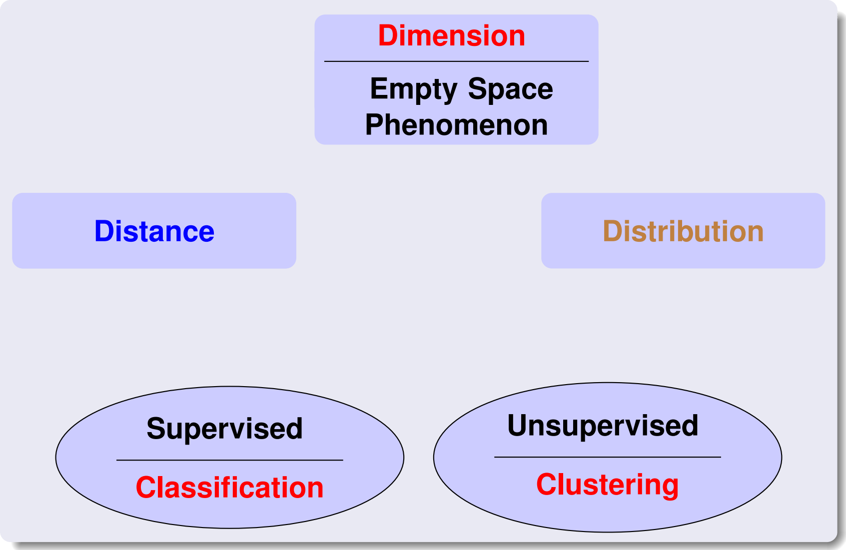

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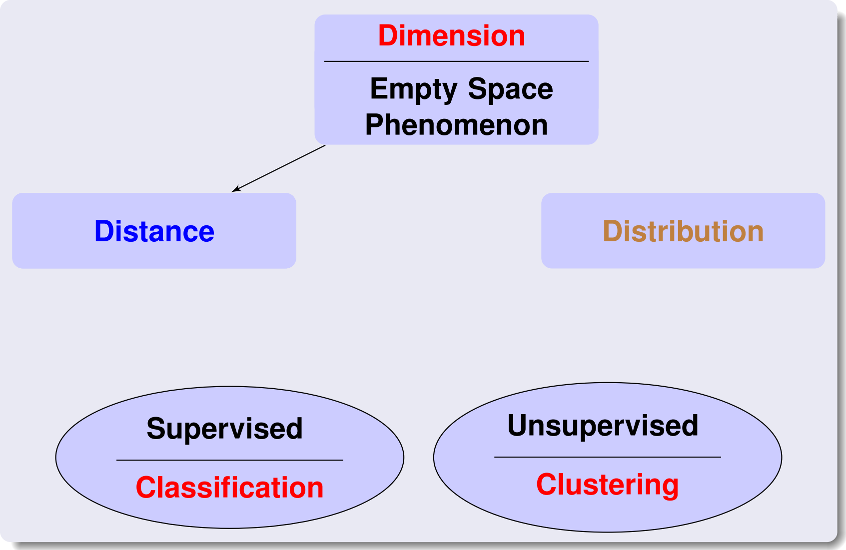

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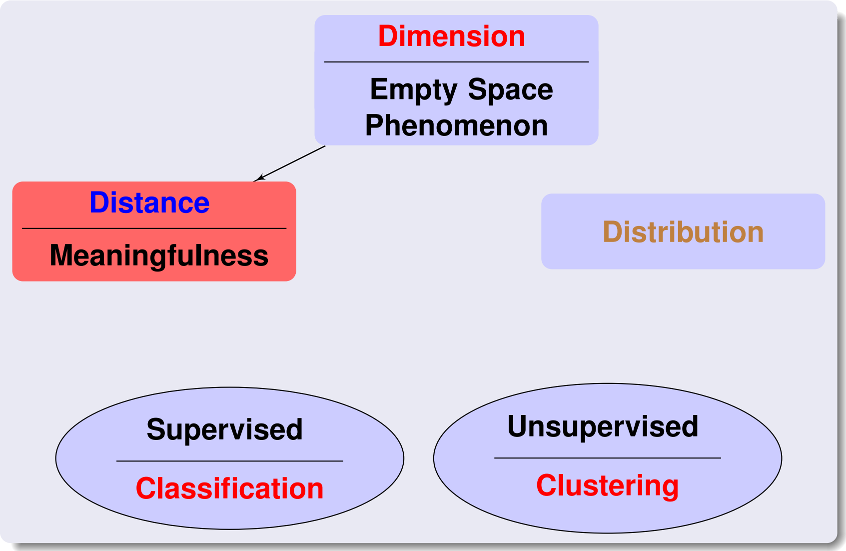

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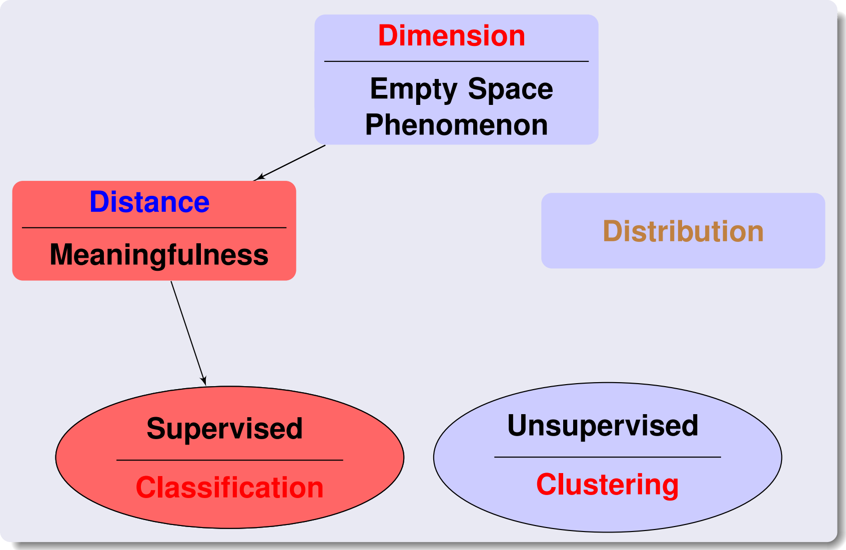

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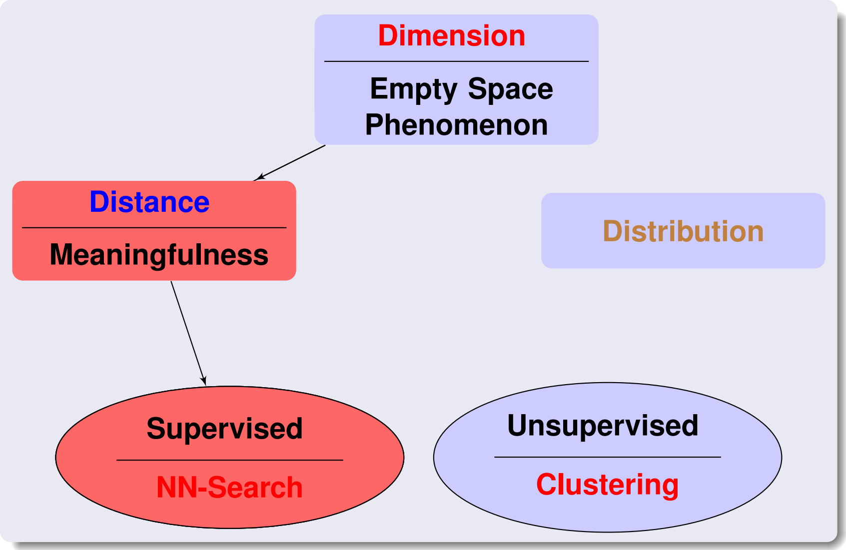

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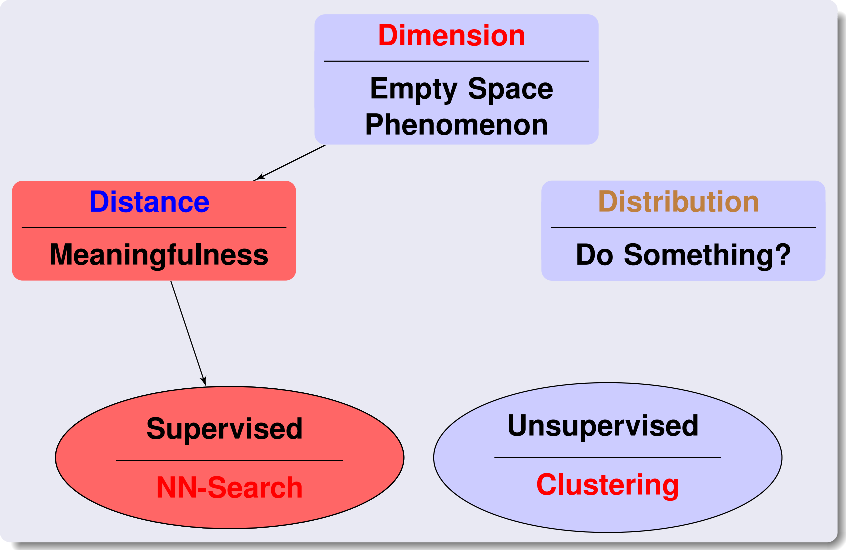

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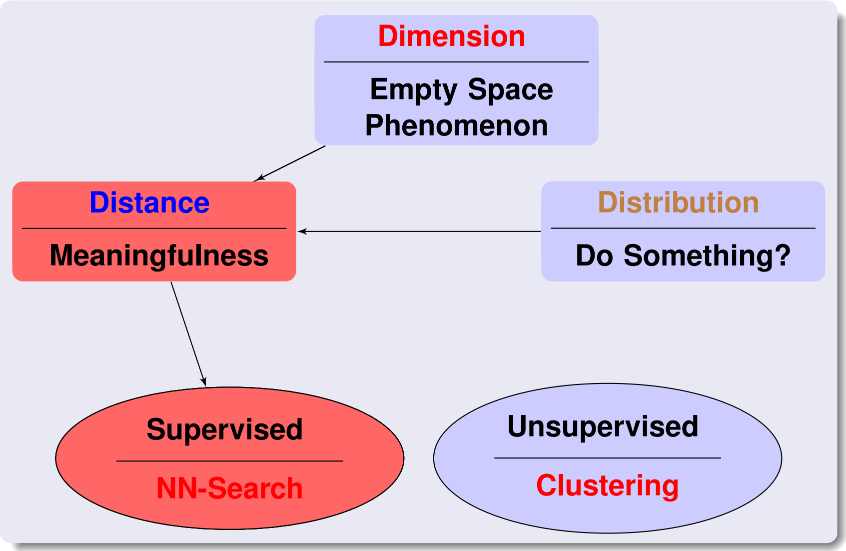

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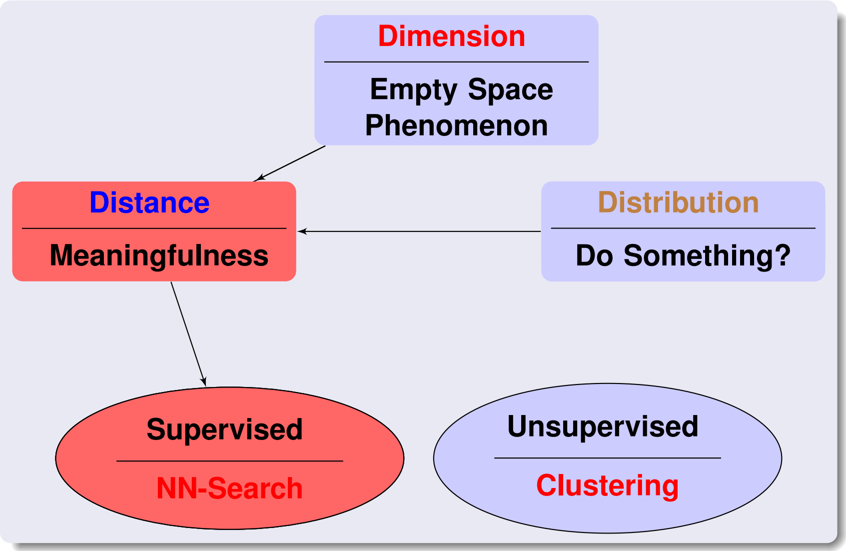

3-D’s in Data Analysis

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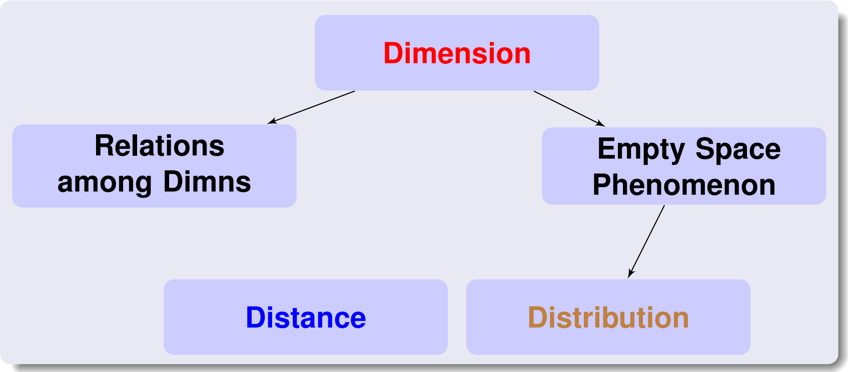
3. Distribution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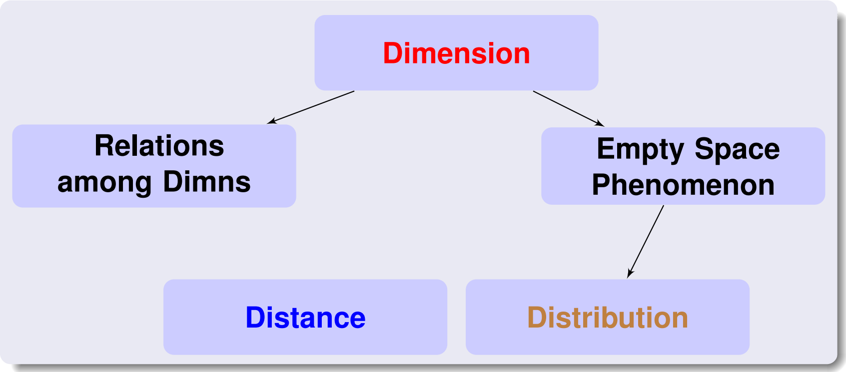

Meaningfulness of Results ?

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- So far, $\mathcal{X}$ contained r.$\bar{v}$ from a single distribution.

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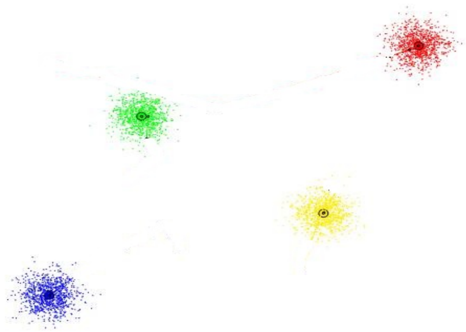
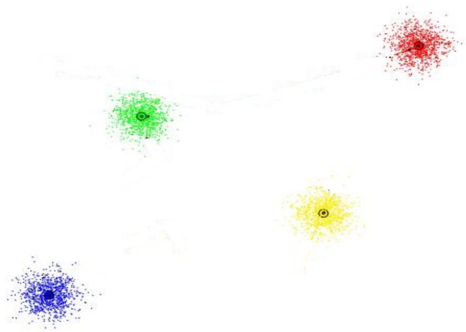


Data from a mixture of distributions

- Let there be c Gaussian clusters in m-dimensions.
-



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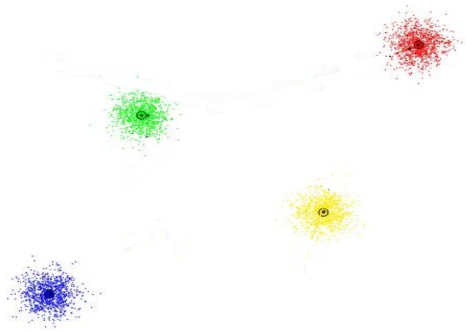

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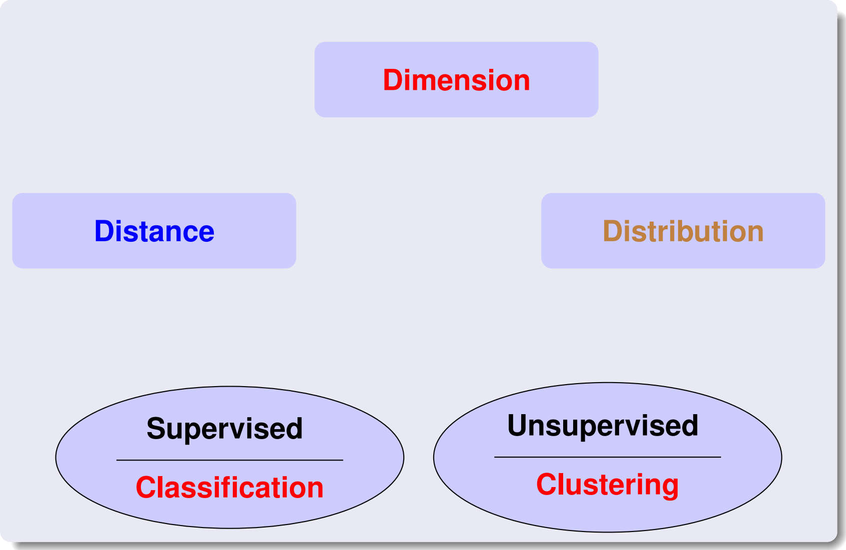

Interplay between the 3D's - 1

Dimension

Empty Space Phenomenon

Distance

Distribution

Supervised

Classification

Unsupervised

Clustering

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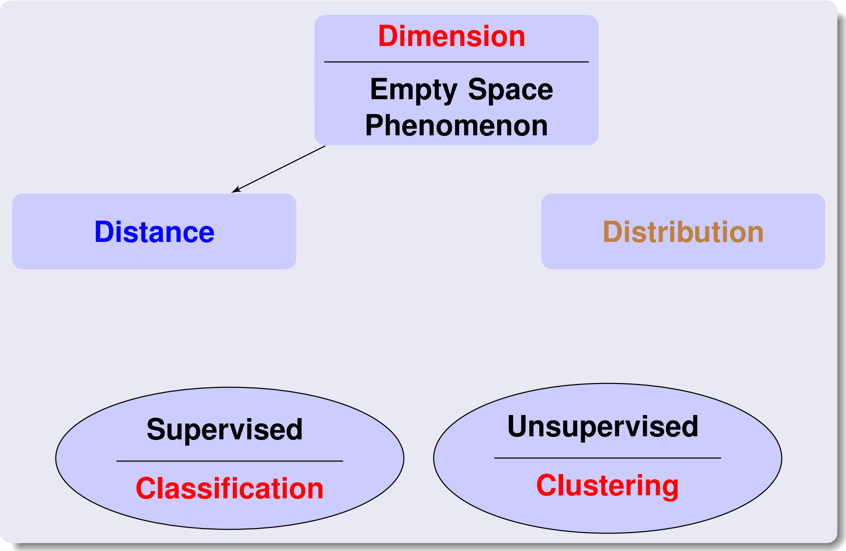

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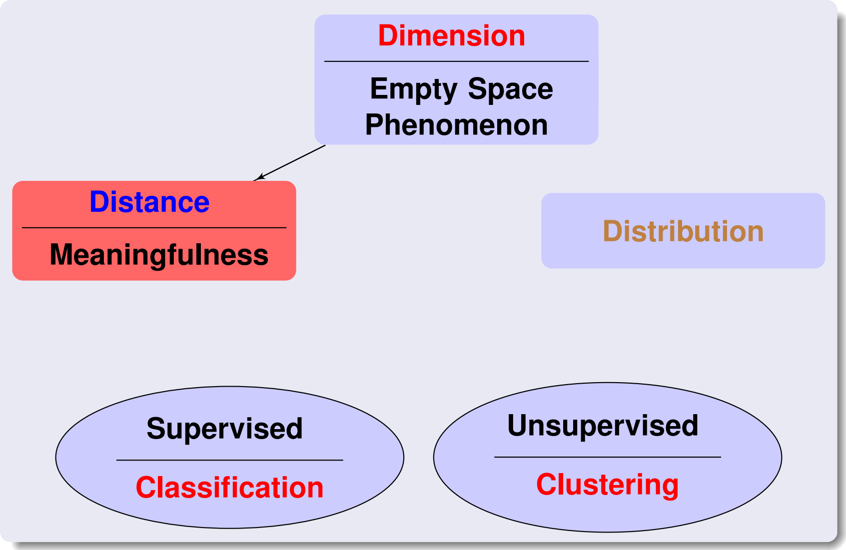

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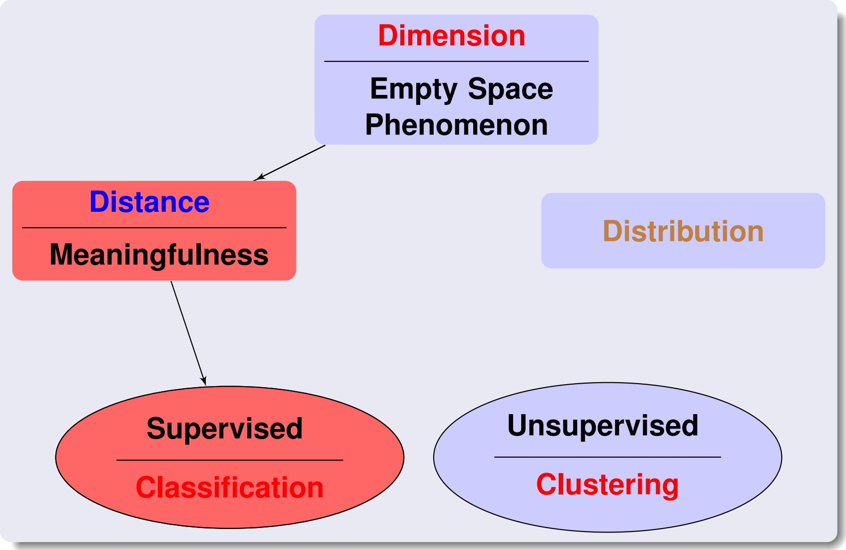

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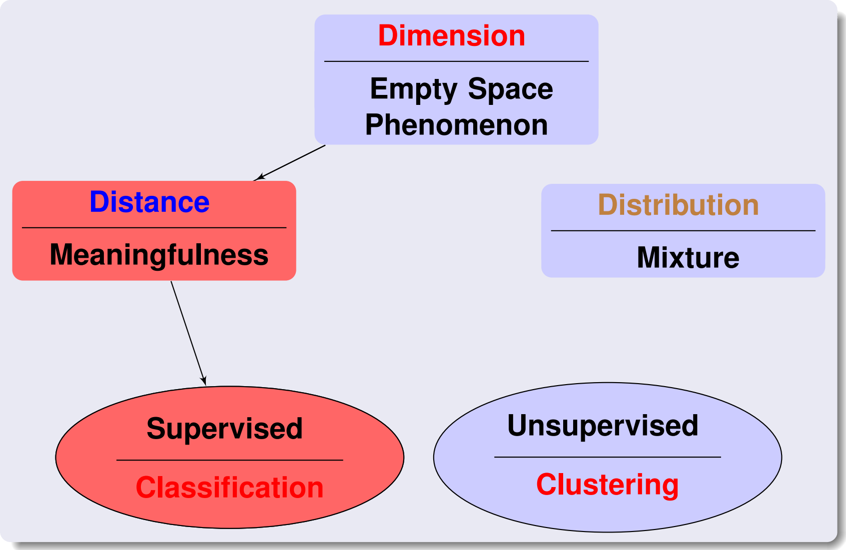

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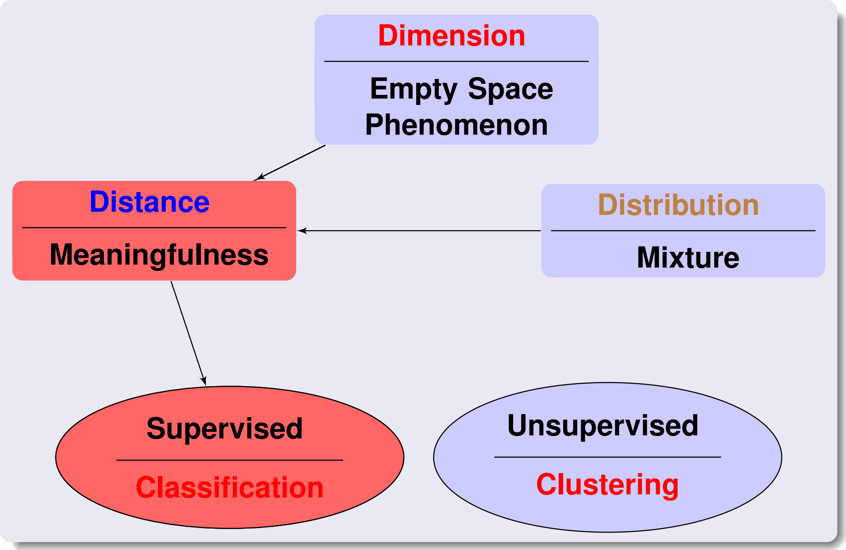

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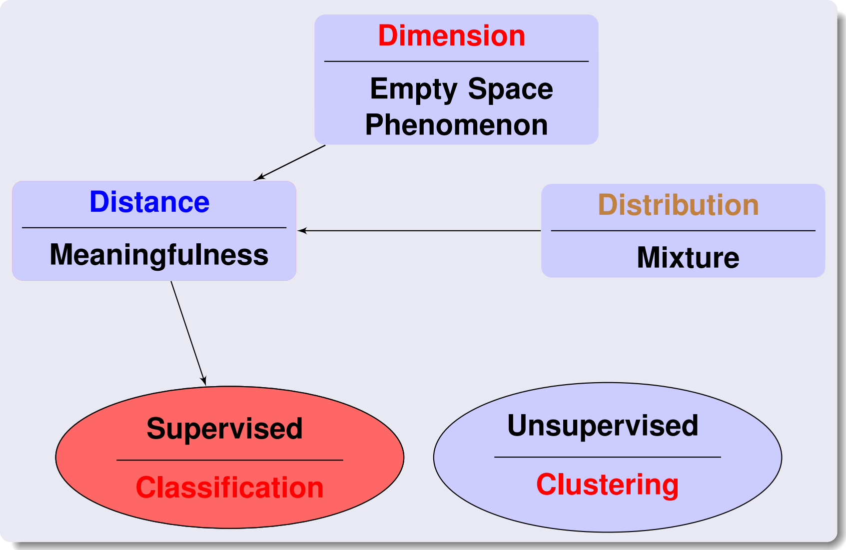

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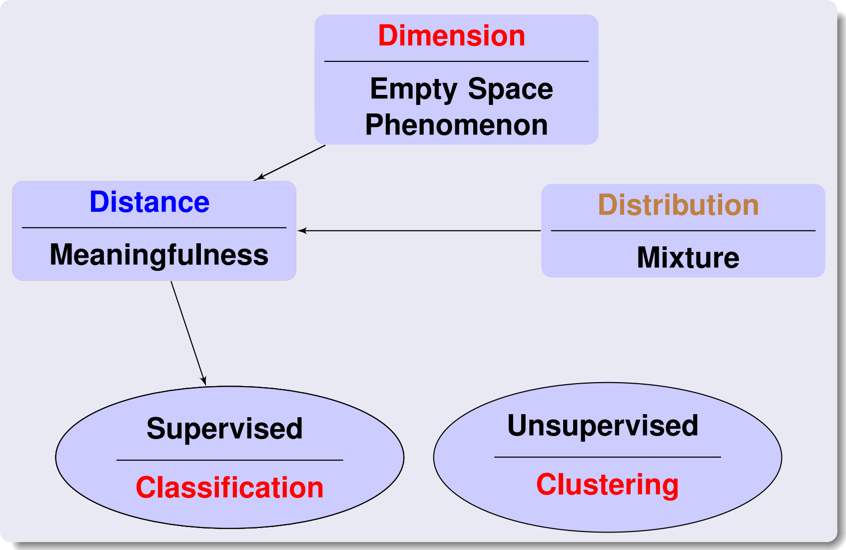

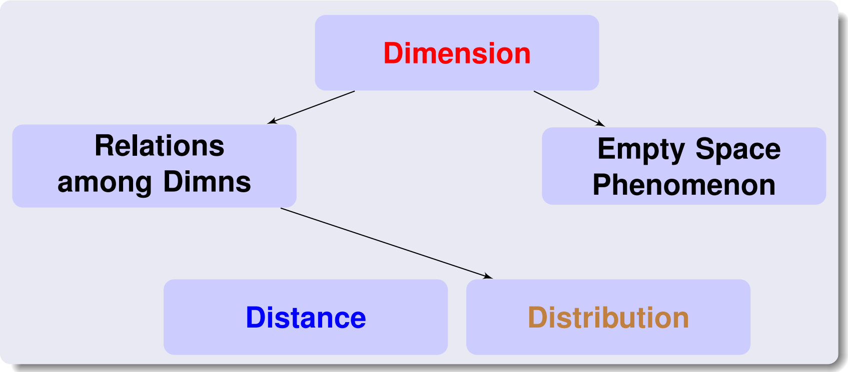
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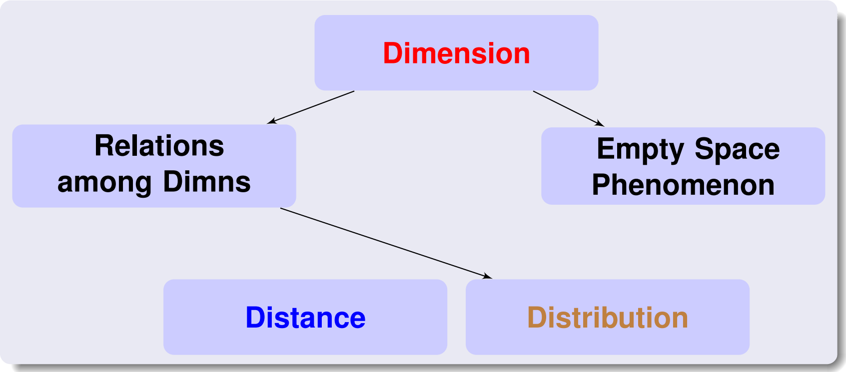

Intrinsic dimn vs. Embedding dimn

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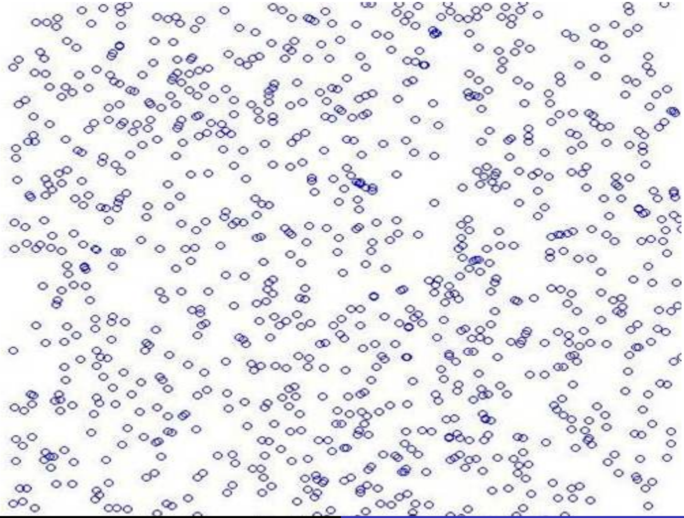

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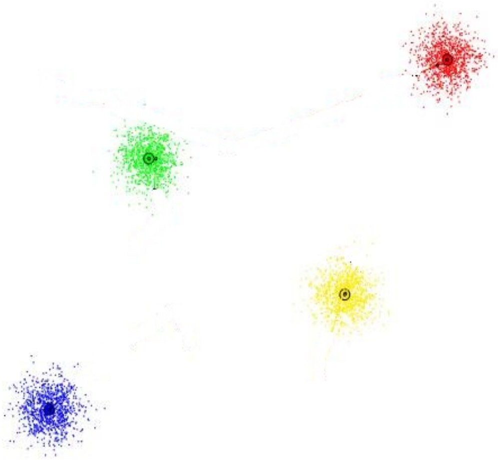

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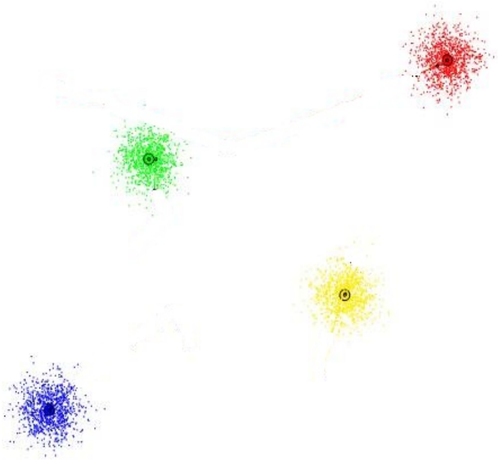

Intrinsic dimn far less than Embedding dimn - $\ell \ll m$

What is its effect on algorithms, say clustering?

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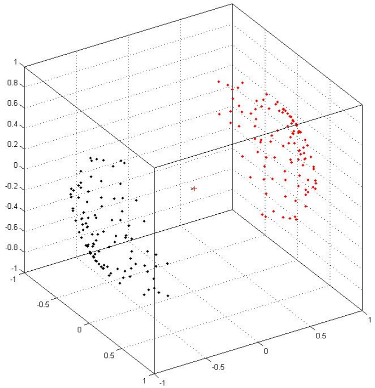
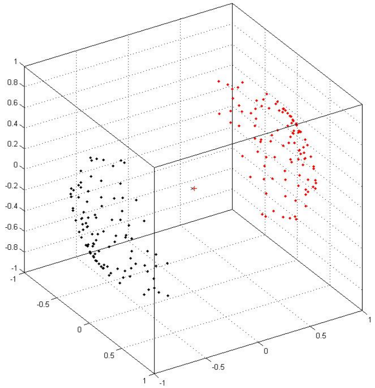


Intrinsic dimn far less than Embedding dimn - $\ell \ll m$

Data are distributed as two well-separated clusters.



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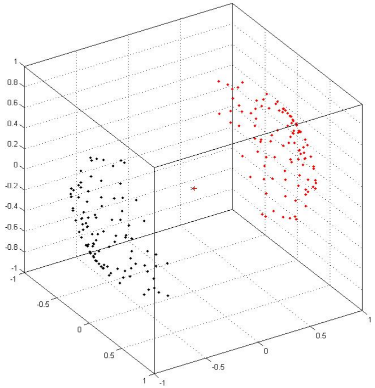
Data are distributed on a 3D-sphere $\Longrightarrow \ell = \mathbf{2} < \mathbf{3} = m$



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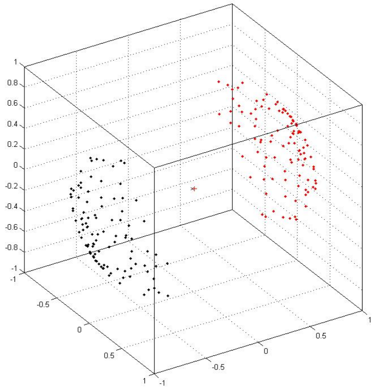

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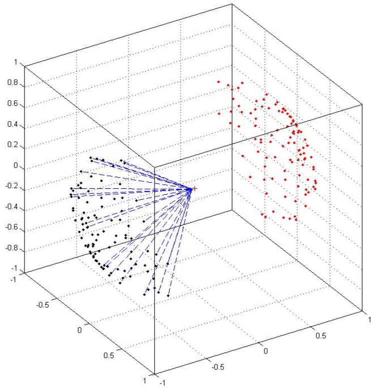

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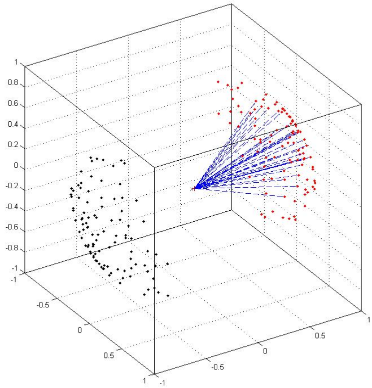
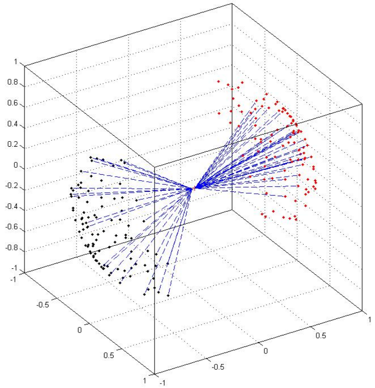


Intrinsic dimn far less than Embedding dimn - $\ell \ll m$

The centroids of both the clusters converge!!!



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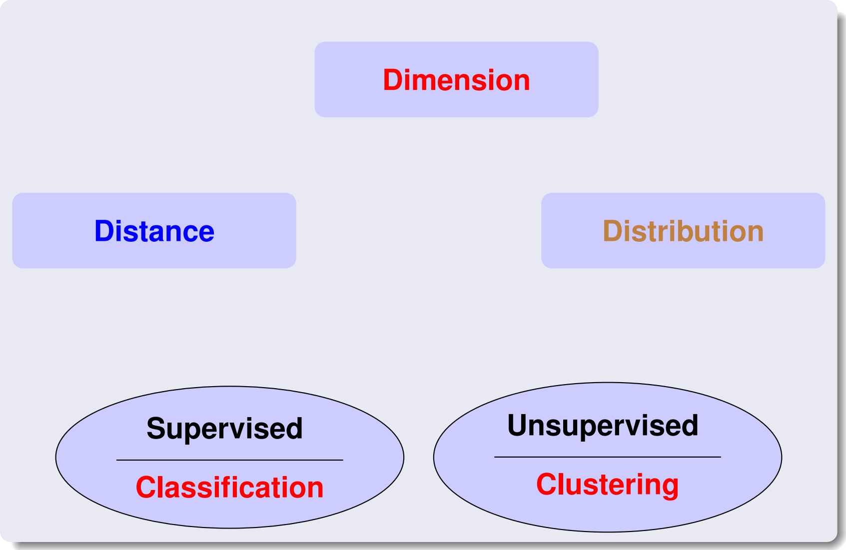

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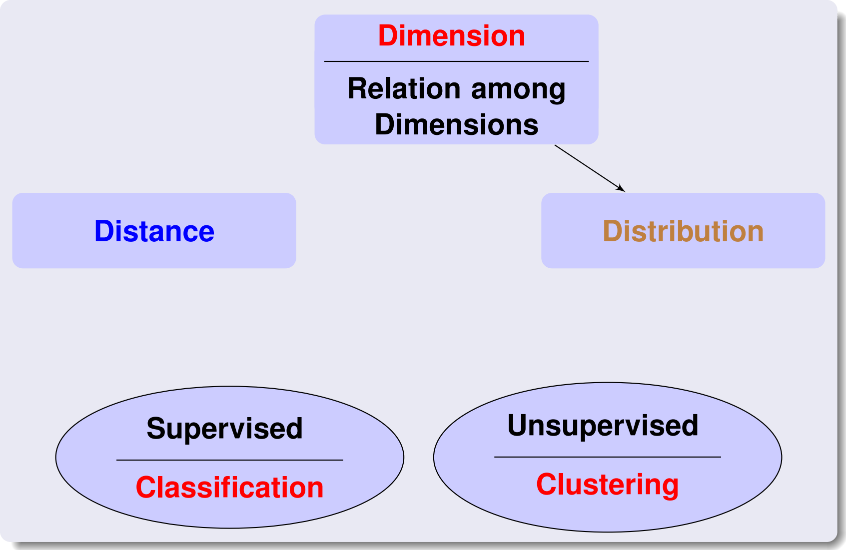

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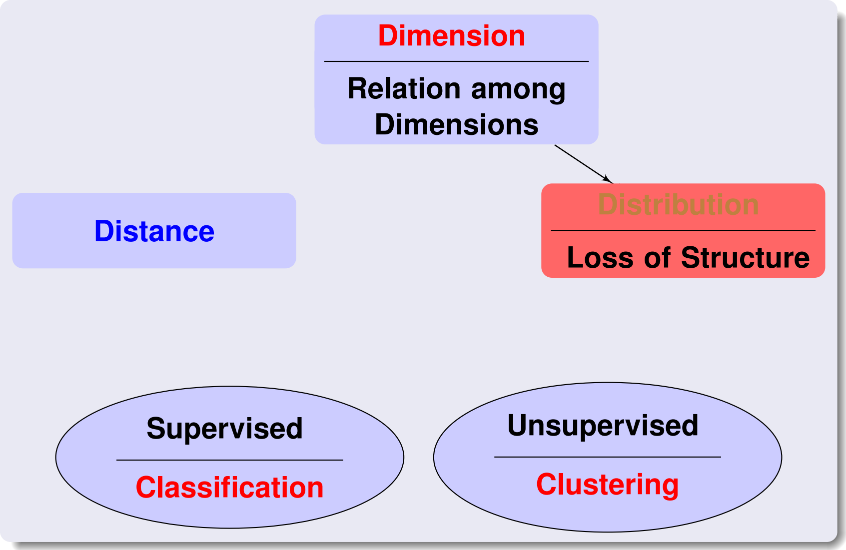

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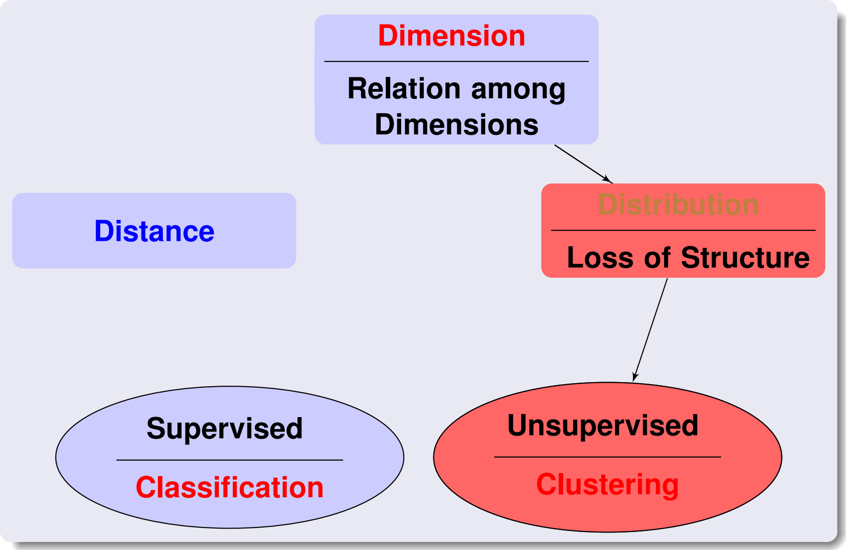

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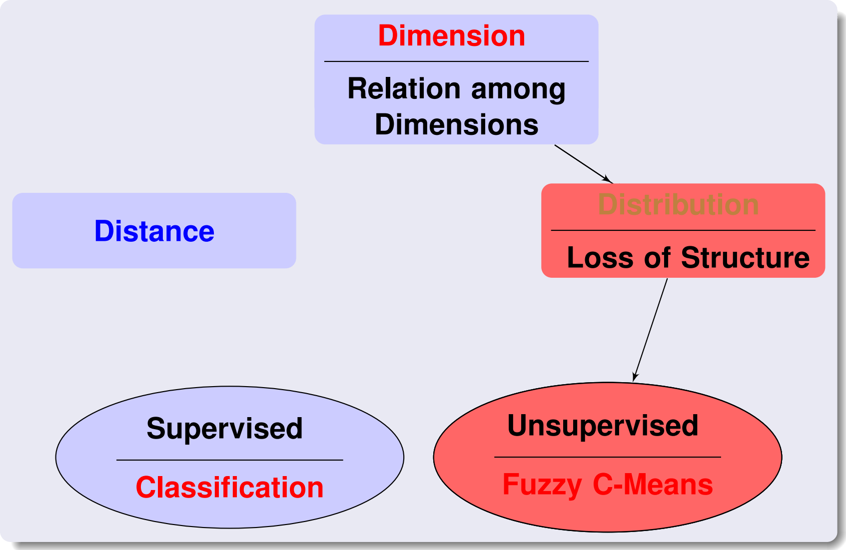

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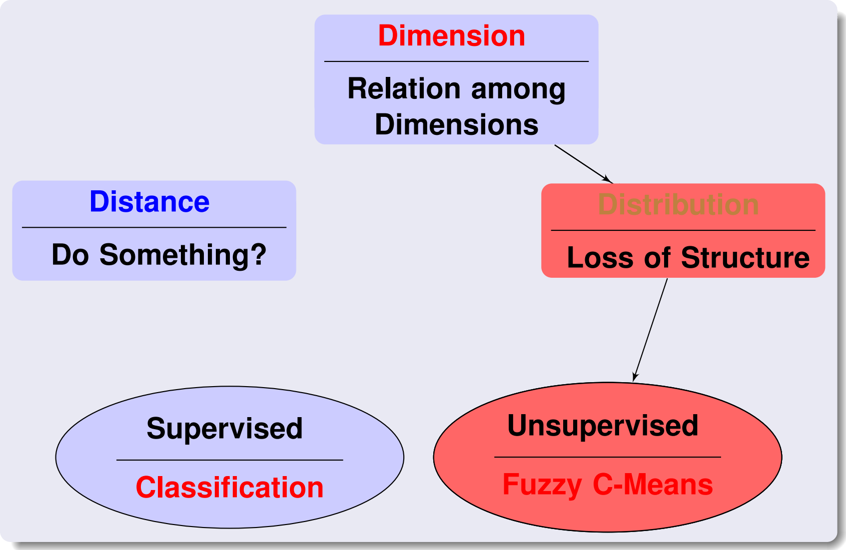

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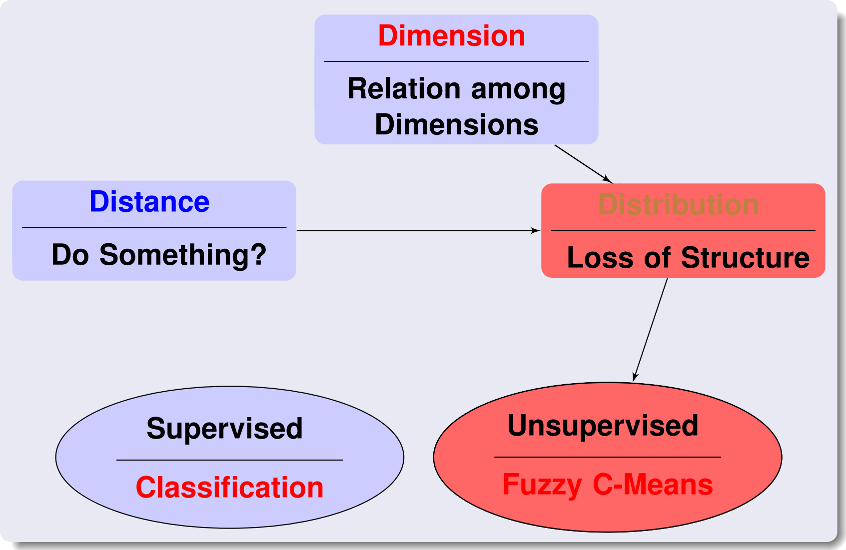

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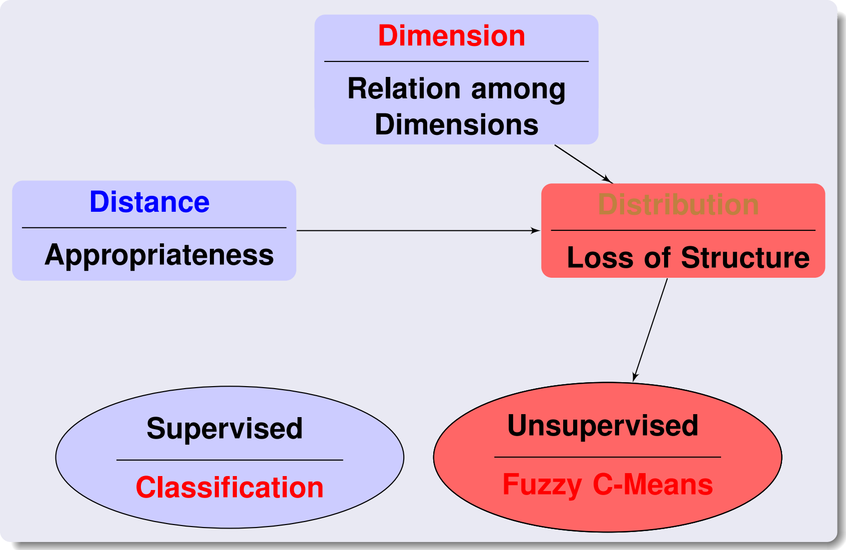

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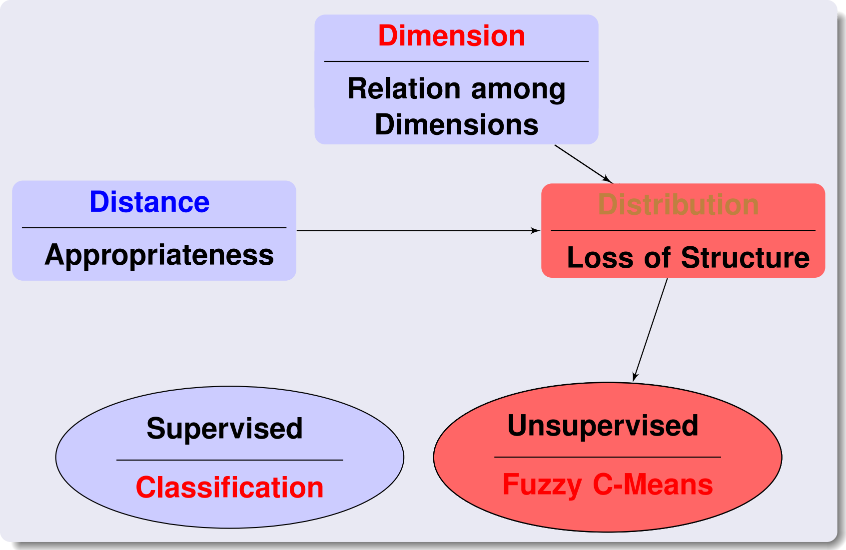

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Take Home Message!!

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Take Home Message!!

Fuzzy Set Theory (still) has its role to play!!

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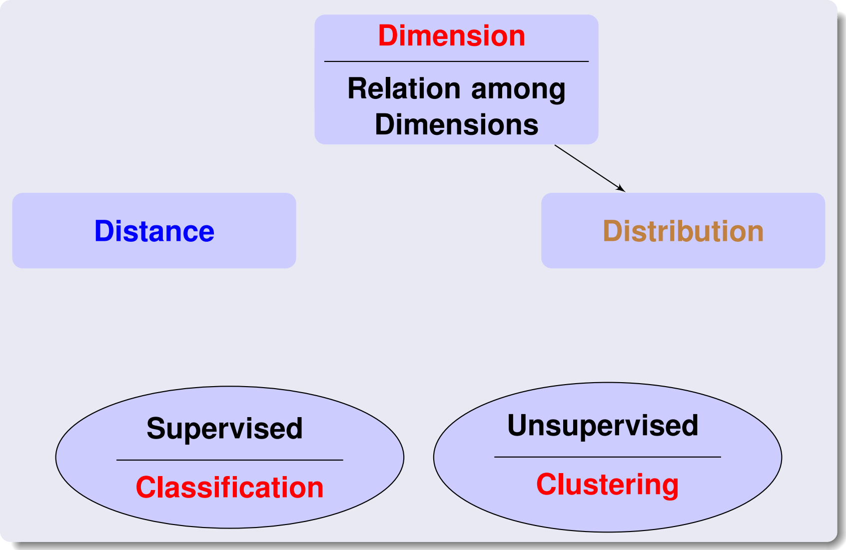

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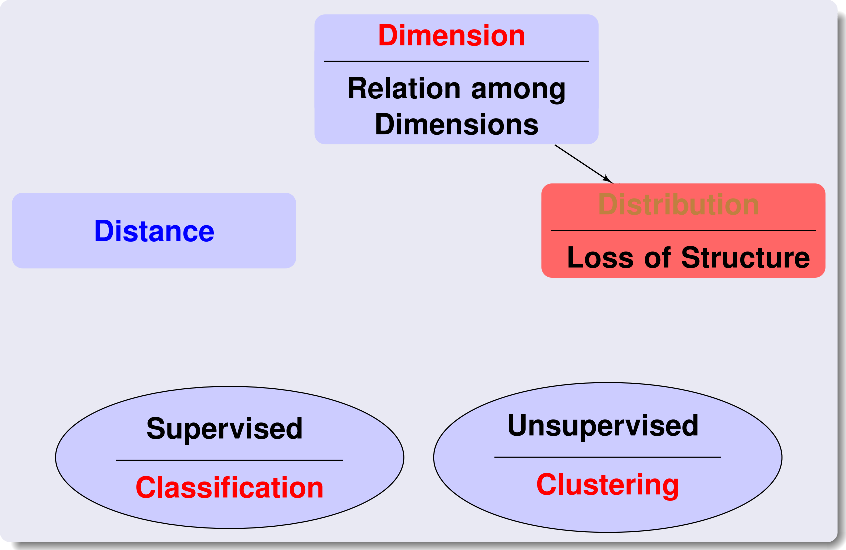

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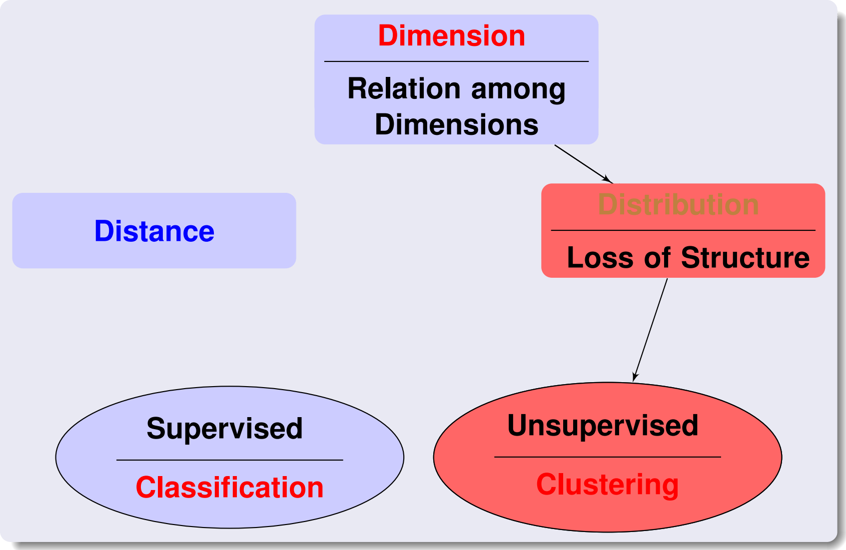

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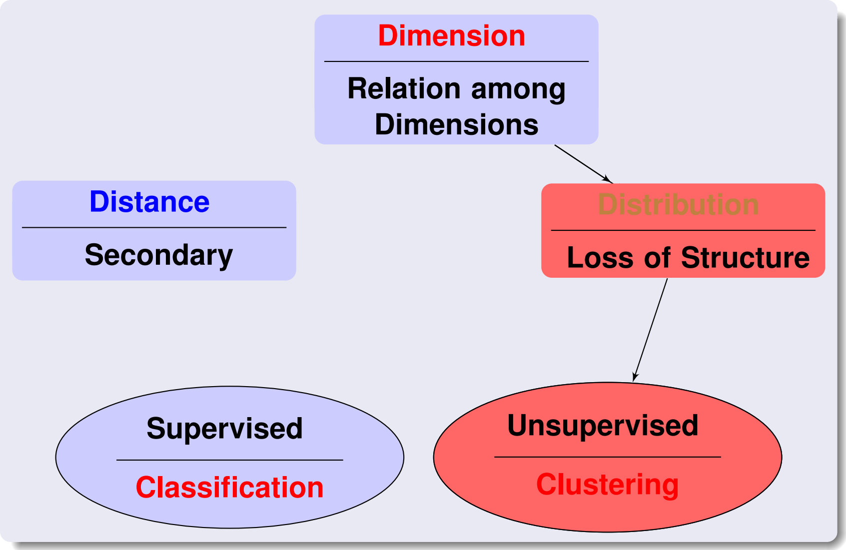

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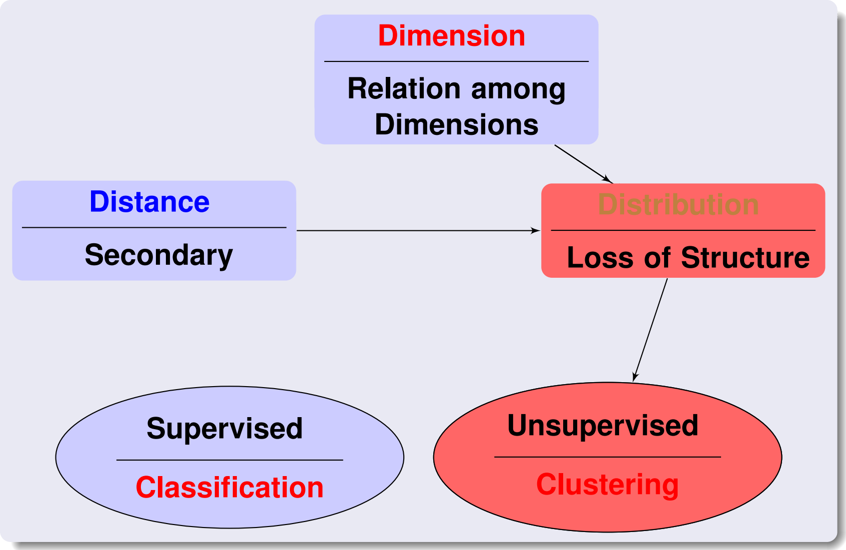

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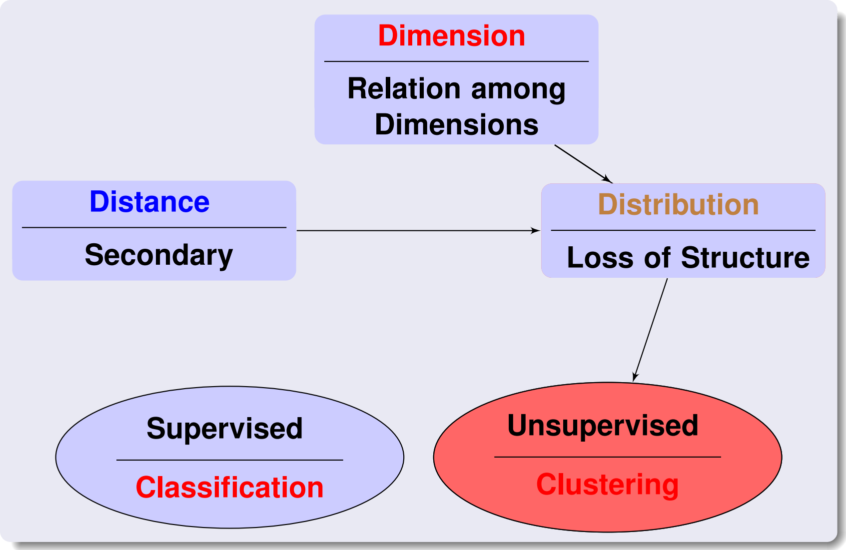

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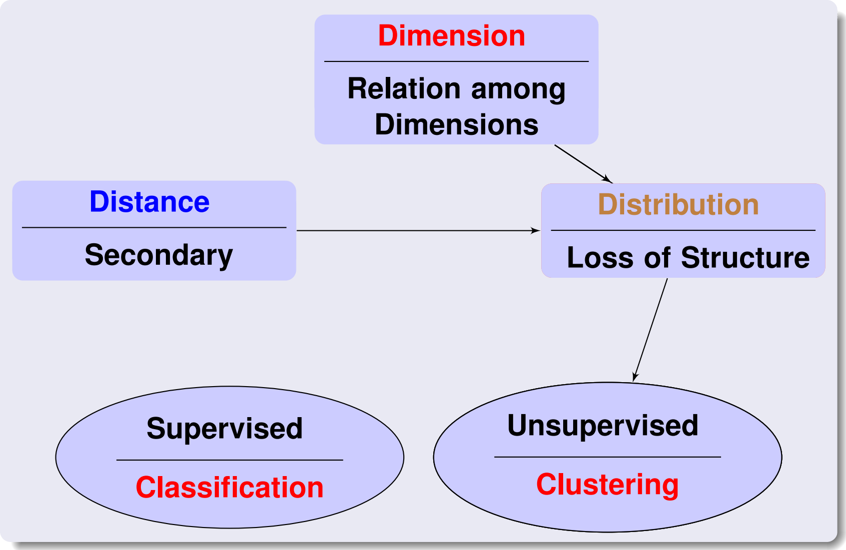

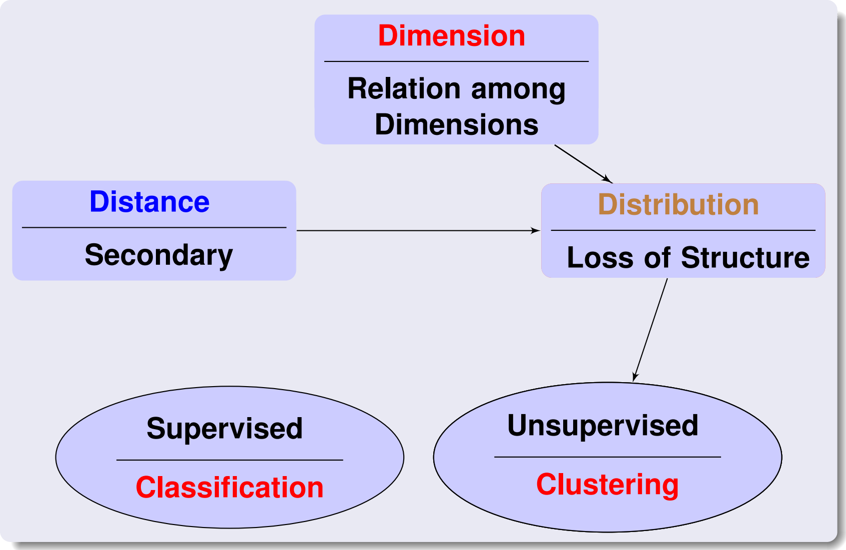
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Summary of the talk

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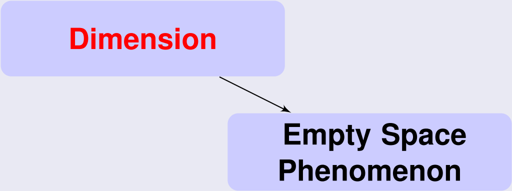

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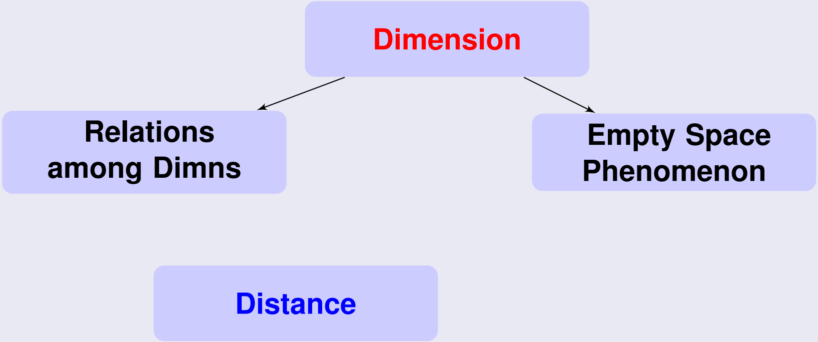

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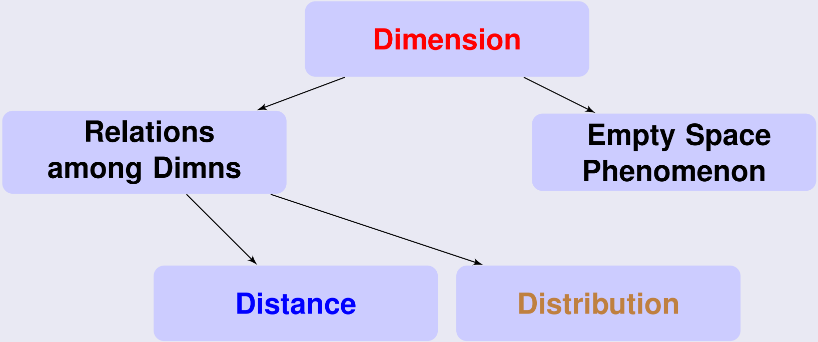

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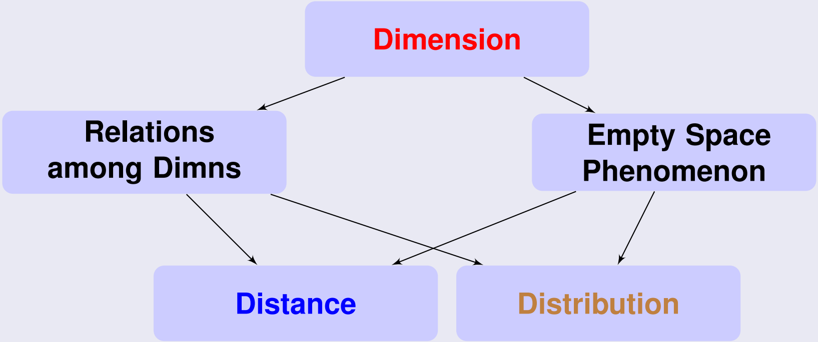

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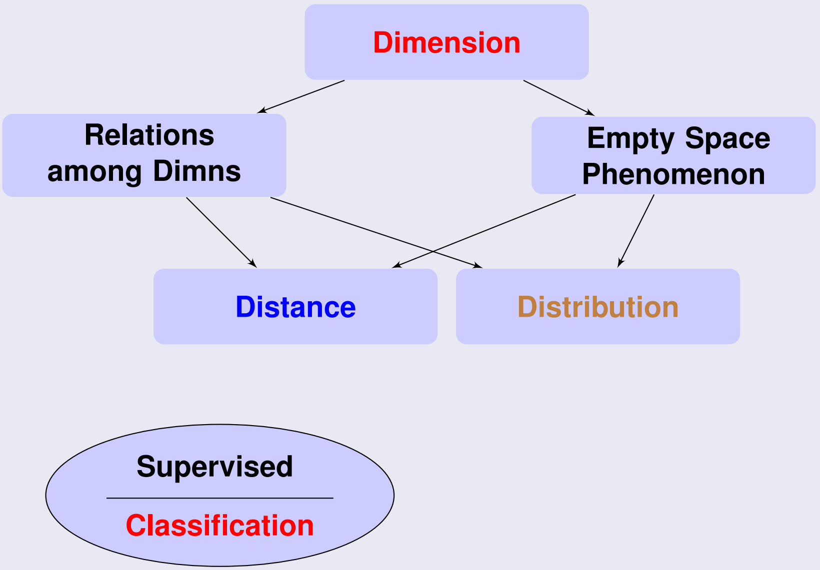

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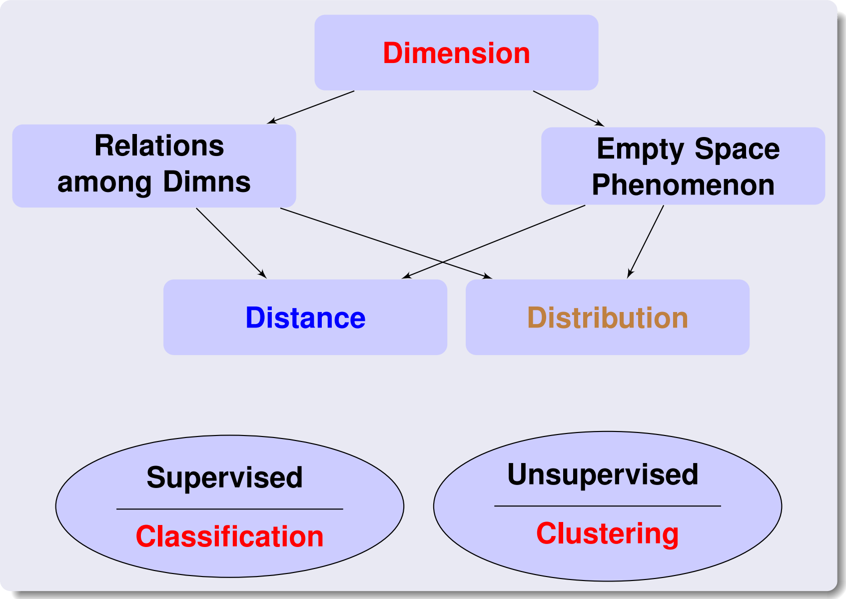

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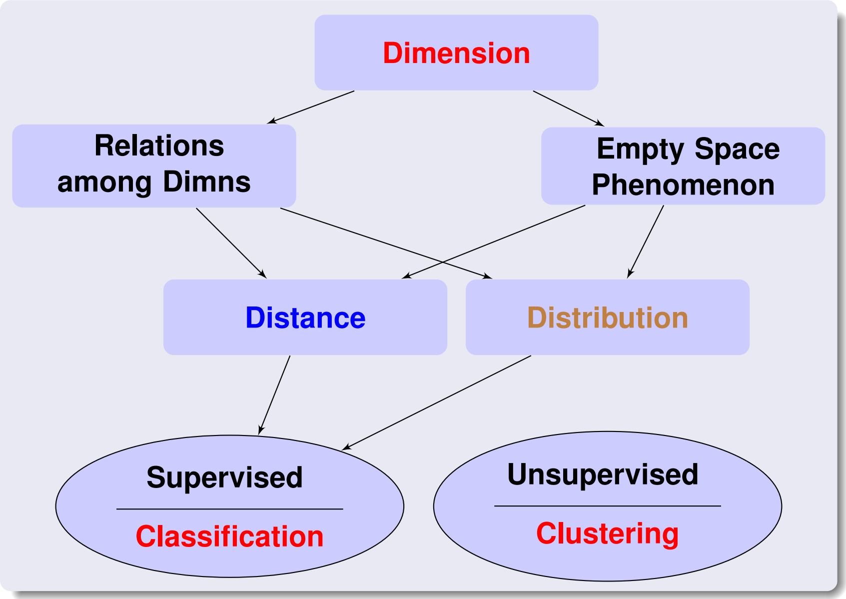

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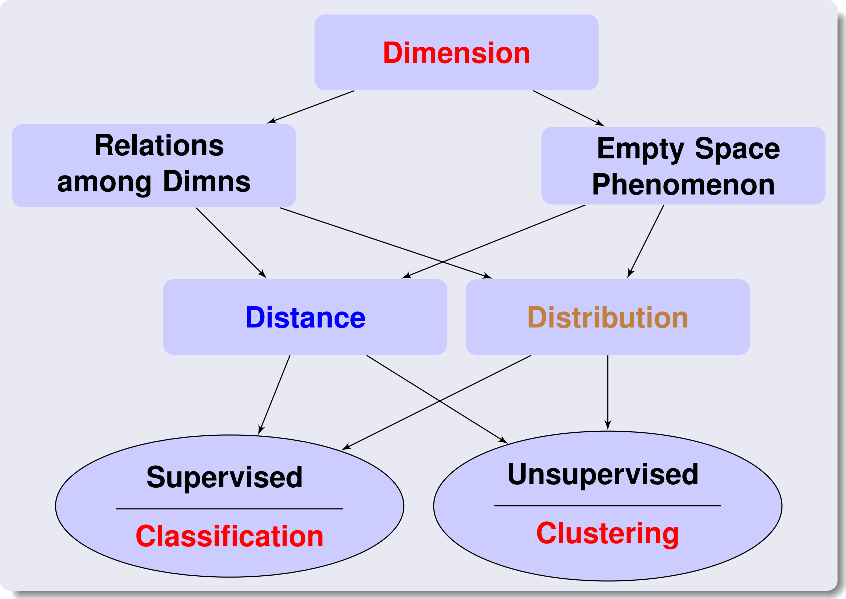

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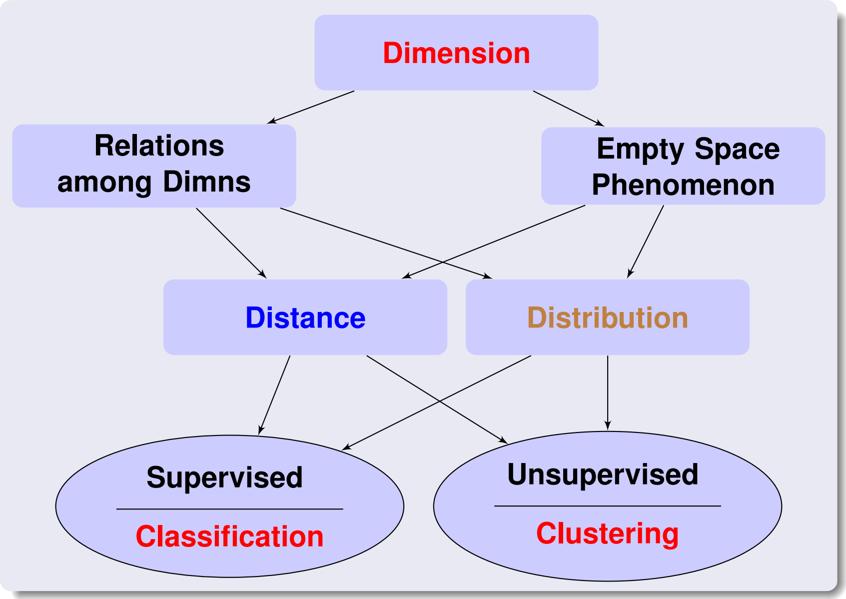

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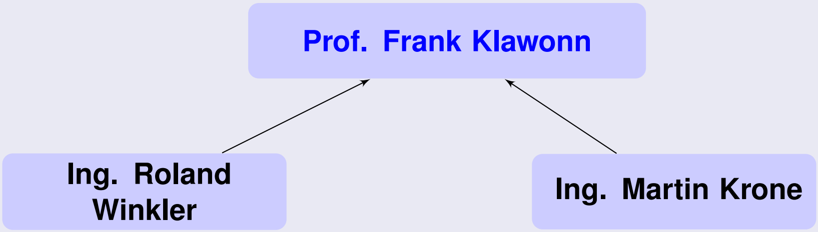

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Any Questions ???