A fuzzy metric resulting from a set of metrics

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Basic definitions

Let X be an universal set (a universe).

Definiton

A mapping $A : X \to [0,1]$ is a fuzzy subset of X (a fuzzy set on the universe X).

The collection of all fuzzy subsets of X will be denoted by $F(X)$

Definition

A mapping $T : [0,1]^2 \to [0,1]$ is called *a triangular norm (a t-norm)*, if is fulfills the following conditions for all α, β and $\gamma \in [0,1]$:

1. $T(\alpha, T(\beta,\gamma)) = T(T(\alpha,\beta),\gamma)$,
2. $T(\alpha,\beta) = T(\beta,\alpha)$,
3. $if \alpha \leq \beta, then T(\alpha,\gamma) \leq T(\beta,\gamma)$,
4. $T(\alpha,1) = \alpha$.

Examples of t-norms

We will show the most important examples of t-norms.
Let $\alpha, \beta \in [0, 1]$:

1. $T_M(\alpha, \beta) = min\{\alpha, \beta\}$, called the *minimum t-norm*,
2. $T_P(\alpha, \beta) = a.b$, called the *product t-norm*,
3. $T_L(\alpha, \beta) = max\{0, \alpha + \beta - 1\}$, called the *Lukasiewicz t-norm*,
4. $$T_D(\alpha, \beta) = \begin{cases} min\{\alpha, \beta\}, & \text{if } max\{\alpha, \beta\} = 1 \\ 0, & otherwise \end{cases}$$

   called the *drastic product*.

A fuzzy metric

We recall the definition of a fuzzy metric and a fuzzy metric space, introduced by Kramosil and Michálek in 1975.

Fuzzy metric

Let T be a t-norm. The mapping $M : X^2 \times (0, \infty) \to [0, 1]$ satisfying the conditions (1)-(5) for all $x, y, z \in X; t, s > 0$

1. $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
2. $M(x, y, t) = M(y, x, t)$,
3. $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$,
4. $M(x, y, .) : (0, \infty) \to [0, 1]$ is left continuous,

is called a *fuzzy metric* on X.

The triple (X, M, T) is called a *fuzzy metric space*.

Alternative definition

A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

Fuzzy metric

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4. $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous

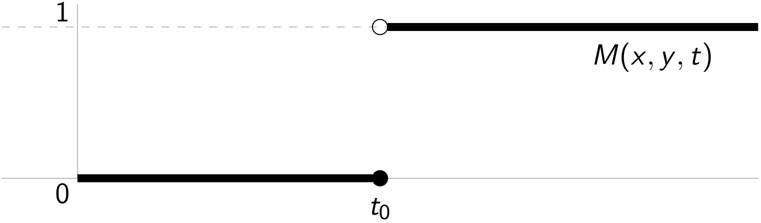
is called a *fuzzy metric* on X.

Fuzzy logical interpretation of $M(x, y, t) = \alpha$: The truth value of the statement "the distance of x, y does not exceed t" is α.

- the condition of continuity violates embedding the crisp case

The crisp case

Suppose that $d(x, y) = t_0$. Then the only possible function $M(x, y, t)$ fulfilling the above interpretation is not a continuous one:



Finite set of distances

Let $x, y \in X$. Suppose that mutual distance of the points $x, y \in X$ is evaluated by n observers. For the pair (x, y) we obtain a finite set of evaluations $\{a_1(x,y), a_2(x,y), ..., a_n(x,y)\}$.

We suppose that each $a_k, k = 1, 2, ..., n$ is a metric on X (it fulfills all properties of a standard metric).

Definition

Let $n \in N$, *let* $\{a_1(x,y), a_2(x,y), ..., a_n(x,y)\}$ be the set of values of the metrics $a_1, a_2, ..., a_n$. The mapping $M : X^2 \times (0, \infty) \to [0, 1]$ defined as

$$M(x, y, t) = \frac{c}{n}, \text{ where } c = card\,\{k \in \{1, 2, ..., n\}\,; a_k(x, y) < t\}$$

is called the *fuzzy distance* between x and y.

Infinite set of distances

Let $x, y \in X$. Suppose that mutual distance of the points $x, y \in X$ is evaluated by a sequence of observers. For the pair (x, y) we obtain a sequence of evaluations $\{a_i(x, y)\}_{i=1}^{\infty}$.

We suppose that each $a_i, i = 1, 2, \ldots$ is a metric on X (it fulfills all properties of a standard metric).

Definition

Let $n \in N$, *let* $\{a_i(x, y)\}_{i=1}^{\infty}$ be the set of values of the metrics $a_1, a_2, \ldots$. Let $\{z_i\}_{i=0}^{\infty}$ be an increasing sequence such that $z_0 = 0, \lim_{n\to\infty} z_n = 1$. The mapping $M : X^2 \times (0, \infty) \to [0, 1]$ defined as

$$M(x, y, t) = z_n, \text{ where } n = card\,\{k; a_k(x, y) < t\} \text{ if the set } \{k; a_k(x, y) < t\} \text{ is finite}$$

and $M(x, y, t) = 1$ if the set $\{k; a_k(x, y) < t\}$ is infinite is called the *fuzzy distance* between x and y.

Example for T_M and T_P

The fuzzy distance need not be a fuzzy metric. We consider the minimum t-norm (denoted by T_M) and the product t-norm (denoted by T_P).
Let the sets of evaluations are given as follows: (1,1,3,1) for (x,y), (1,1,1,3) for (y,z) and (2,2,3,3) for (x,z). Take $t = s = 1.1$.
Then for the minimum t-norm T_M we obtain

$$T_M(M(x,y,1.1), M(y,z,1.1)) = T_M(\tfrac{3}{4}, \tfrac{3}{4}) = min\left\{\tfrac{3}{4}, \tfrac{3}{4}\right\} = \\ = \tfrac{3}{4} > \tfrac{1}{2} = M(x,z,2.2).$$

Similarly for the product t-norm T_P we obtain

$$T_P(M(x,y,1.1), M(y,z,1.1)) = T_P(\tfrac{3}{4}, \tfrac{3}{4}) = \tfrac{3}{4}.\tfrac{3}{4} = \\ = \tfrac{9}{16} > \tfrac{1}{2} = M(x,z,2.2).$$

We can see that the condition (4) from the definition of a fuzzy metric does not hold.

Fuzzy distance as a fuzzy metric

Fuzzy metric space

Let M be a fuzzy distance from the previous definition, let T be a t-norm. Then (X, M, T) is a fuzzy metric space if and only if $T \leq T_L$.

Convergence

Denote by M_0 the mapping $M(x, x, t)$, where
$M(x, x, 0) = 0, M(x, x, t) = 1$ for all $t > 0$.

Suppose $x_n \to x_0$ in all the metrics a_i, i.e.

$$a_i(x_n, x_0) \to 0 \text{ for all } i = 1, 2, \ldots.$$

The corresponding convergence in the constructed fuzzy metric space is the convergence of the functions $M(x_n, x_0, t)$ to the function M_0. There are different conditions for pointwise and uniform convergence.

Convergence

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X, let $x_0 \in X$.

Pointwise convergence

If for all $i = 1, 2, \ldots$ there is $x_n \to x_0$ for all the metrics $a_1, a_2, \ldots$, then the sequence $M(x_n, x_0, t)$ converges to M_0 pointwisely.

Uniform convergence

If for all $\varepsilon > 0$ there is an integer n_0 such that for all the metrics $a_1, a_2, \ldots$ and for all $N \geq n_0$ there is $a_k(x_n, x_0) < \varepsilon$, then the sequence $M(x_n, x_0, t)$ converges to M_0 uniformly.

Conclusion

1. the sequence of metrics on a universe enables to define a single fuzzy metric on this universe (under assumption of a t-norm in the interval $[T_D, T_L]$
2. another argument for **not** restricting to continuous functions in the definition of a fuzzy metric space