Generalized weak pre-pseudo effect algebras

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Outline

- Summary of results - pre pseudo EAs
- Generalized pre pseudo EAs
- Results, Unitization
- RDP_0, RDP properties and proposed generalizations
- Attempt to define congruences

Summary of results - pre pseudo effect algebras

- improved method for searching models up to 11 elements
- found models and computer program are available at

`http://www.mat.savba.sk/~hycko/wprepea/`

Generalized weak pre-pseudo effect algebras

Let $(A; +, \backslash, /, 0)$ be a partial algebra of type $(2, 2, 2, 0)$ satisfying the following properties:

(GWPPEA1) $a \backslash a = 0 = a \,/\, a$;

(GWPPEA2) the relation $a \le b$, iff $b \backslash a$ is defined, iff $b \,/\, a$ is defined is a partial order;

(GWPPEA3) $a \backslash b$ is defined and $a \backslash b \ge c$, iff $c + b$ is defined and $a \ge c + b$. Moreover $(a \backslash b) \backslash c = a \backslash (c + b)$;

(GWPPEA4) $a \,/\, b$ is defined and $a \,/\, b \ge c$, iff $b + c$ is defined and $a \ge b + c$. Moreover $(a \,/\, b) \,/\, c = a \,/\, (b + c)$.

Then A is said to be a *generalized weak pre pseudo effect algebra*.

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interpretation: $a \backslash b \equiv a + (-b)$; $\quad a \,/\, b \equiv (-b) + a$.

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GPA5 if $a + b$ is defined then there are $d, e \in A$ such that $a + b = d + a = b + e$.

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Each generalized pseudo effect algebra is generalized pre-pseudo effect algebra.

Properties - GWPPEA

(i) $a + 0$, $0 + a$ are defined and $a + 0 = a = 0 + a$;
(ii) 0 is the bottom element in $(A; \leq)$;
(iii) $a \setminus 0$ and $a \,/\, 0$ are defined and $a \setminus 0 = a = a \,/\, 0$;
(iv) if $a + b$ is defined, then $a + b \geq a, b$ and $(a + b) \setminus b \geq a$, $(a + b) \,/\, a \geq b$;
(v) if $a \setminus b$ is defined, then $a \setminus b \leq a$ and $a \,/\, (a \setminus b) \geq b$;
(vi) if $a \,/\, b$ is defined, then $a \,/\, b \leq a$ and $a \setminus (a \,/\, b) \geq b$;
(vii) if $a + b = a$ (or $a + b = b$) then $b = 0$ ($a = 0$);
(viii) if $a + b = 0$, then $a = b = 0$;
(ix) $+$ is partially associative, i.e. $a + b$ and $(a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined. In such case $(a + b) + c = a + (b + c)$;

Properties - weak contd.

(x) if $a \geq b \geq c$ then $a \setminus c \geq b \setminus c$ and $a \,/\, c \geq b \,/\, c$;

(xi) if $a \geq b$ and $a + c$ is defined then $b + c$ is defined, $a + c \geq b + c$ and $(a + c) \setminus (b + c) \geq a \setminus b$;

(xii) if $a \geq b$ and $c + a$ is defined then $c + b$ is defined, $c + a \geq c + b$ and $(c + a) \,/\, (c + b) \geq a \,/\, b$;

(xiii) $a \geq b \geq c$ then $a \setminus c \geq (a \setminus b) + (b \setminus c)$ and $a \,/\, c \geq (b \,/\, c) + (a \,/\, b)$;

(xiv) $(b \setminus a) \,/\, c$ is defined, iff $(b \,/\, c) \setminus a$ is defined and in this case $(b \setminus a) \,/\, c = (b \,/\, c) \setminus a$;

(xv) relation $\sqsubseteq_L$ defined by $a \sqsubseteq_L b$, iff $\exists\, c$ such that $b = c + a$ is a partial order;

(xvi) relation $\sqsubseteq_R$ defined by $a \sqsubseteq_R b$, iff $\exists\, c$ such that $b = a + c$ is a partial order;

(xvii) $\sqsubseteq_L$, $\sqsubseteq_R$ implies $\leq$.

Examples

For any partial order $\leq$ with bottom element 0 it is possible to construct at least one model of generalized weak pre-pseudo effect algebra.

- $+$ will be defined only for pairs $(0, x)$ and $(x, 0)$ with the result of x
- $/$ and $\backslash$ operations will be defined for pairs (b, a) such that $b \geq a$ with the result equal to 0.

Pre PEAs as Generalized Pre PEA

Can be any pre-pseudo effect algebra made to be a generalized pre-pseudo effect algebra?

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Answer: **No**

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Answer: **No**

Necessary condition: For any $a, b \in A$, such that $a \geq b$ the sets

$$L_{a,b} := \{k \in A : b + k \leq a\}$$

and

$$R_{a,b} = \{k \in A : k + b \leq a\}$$

are having the top element.

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Otherwise, there is not possible to define $a \,/\, b$ or $a \setminus b$, respectively.

It turns out to be also the sufficient confition.

Sufficient condition Pre PEA into Generalized Pre PEA

Let $(A; +, {}^L, {}^R, 0, 1)$ be a pre pseudo effect algebra.
[Thus $a \leq b$, iff $a + b^R$ is defined, iff $b^L + a$ is defined.]
Let us assume that for any $a, b \in A$, $a \geq b$ the sets $L_{a,b}$ and $R_{a,b}$ posses top elements denoted $l_{a,b}$ and $r_{a,b}$ respectively. Let us define partial operations $/$ and $\setminus$ for any $a \geq b$, $a \,/\, b = l_{a,b}$ and $a \setminus b = r_{a,b}$, otherwise undefined.
Then $(A; +, /, \setminus, 0)$ is a generalized pre-pseudo effect algebra.

Unitization

Let $(A;+,\backslash,/,0)$ be a generalized (weak) pre-pseudo effect algebra. Let us consider disjunctive copy of A, denoted as A^*, and let us denote its elements as a^* for each corresponding $a \in A$. Let us define operation $+_p$ as following:

- $a +_p b$ is defined, iff $a + b$ is defined and $a +_p b = a + b$;
- $a +_p b^*$ is defined, iff $b \geq a$ and $a +_p b^* = (b \setminus a)^*$;
- $b^* +_p a$ is defined, iff $b \geq a$ and $b^* +_p a = (b \,/\, a)^*$;
- $a^* +_p b^*$ is never defined.

For each element $a \in A$, let $a^R = a^L = a^*$ and
for each element $a^* \in A^*$ $(a^*)^R = (a^*)^L = a$.
Then $(A \cup A^*; +_p, {}^R, {}^L, 0, 0^*)$ is a weak pre-pseudo effect algebra.

Problems with non-weakness

There are 3 cases to be proved that previous construction of unitization performed on generalized pre-pseudo effect algebras would lead to pre-pseudo efect algebras.

We need to prove that if $a + b$ is defined then there are elements $d, e \in A \cup A^*$ such that $a + b = d + a = b + e$.

1. $a, b \in A$,
2. $a \in A,\ b^* \in A^*$
3. $a^* \in A^*,\ b \in A$

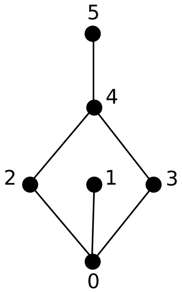
Unitization - non weak

Let us consider generalized pre-pseudo effect algebra:

+	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	-	-	-	-	-
2	2	-	5	-	-	-
3	3	-	5	5	5	-
4	4	-	5	-	-	-
5	5	-	-	-	-	-

/	**0**	**1**	**2**	**3**	**4**	**5**
0	0	-	-	-	-	-
1	1	0	-	-	-	-
2	2	-	0	-	-	-
3	3	-	-	0	-	-
4	4	-	0	0	0	-
5	5	-	2	4	2	0

\	**0**	**1**	**2**	**3**	**4**	**5**
0	0	-	-	-	-	-
1	1	0	-	-	-	-
2	2	-	0	-	-	-
3	3	-	-	0	-	-
4	4	-	0	0	0	-
5	5	-	4	3	3	0

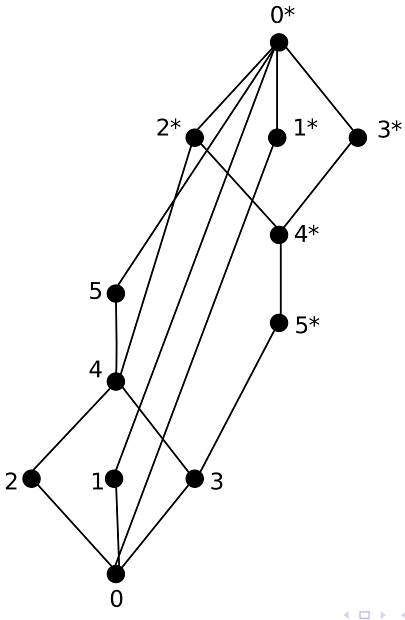


Unitization - non weak - contd.

$+_p$	**0**	**1**	**2**	**3**	**4**	**5**	**5****	**4***	**3***	**2***	**1***	**0***
0	0	1	2	3	4	5	5^*	4^*	3^*	2^*	1^*	0^*
1	1	-	-	-	-	-	-	-	-	-	0^*	-
2	2	-	5	-	-	-	4^*	0^*	-	0^*	-	-
3	3	-	5	5	5	-	3^*	0^*	0^*	-	-	-
4	4	-	5	-	-	-	3^*	0^*	-	-	-	-
5	5	-	-	-	-	-	0^*	-	-	-	-	-
5*	5^*	-	2^*	4^*	2^*	0^*	-	-	-	-	-	-
4*	4^*	-	0^*	0^*	0^*	-	-	-	-	-	-	-
3*	3^*	-	-	0^*	-	-	-	-	-	-	-	-
2*	2^*	-	0^*	-	-	-	-	-	-	-	-	-
1*	1^*	0^*	-	-	-	-	-	-	-	-	-	-
0*	0^*	-	-	-	-	-	-	-	-	-	-	-

$L = R$	**0**	**1**	**2**	**3**	**4**	**5**	**5***	**4***	**3***	**2***	**1***	**0***
	0^*	1^*	2^*	3^*	4^*	5^*	5	4	3	2	1	0

Unitization - non weak - contd.



Unitization - linear non weak

Even linearity of underlying generalized pre-pseudo effect algebra does not help.

$+$	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	4	4	4	-
2	2	4	4	-	-
3	3	4	4	-	-
4	4	-	-	-	-

$/$	**0**	**1**	**2**	**3**	**4**
0	0	-	-	-	-
1	1	0	-	-	-
2	2	0	0	-	-
3	3	0	0	0	-
4	4	3	2	2	0

$\backslash$	**0**	**1**	**2**	**3**	**4**
0	0	-	-	-	-
1	1	0	-	-	-
2	2	0	0	-	-
3	3	0	0	0	-
4	4	3	3	1	0

Unitization - linear non weak

$+_p$	**0**	**1**	**2**	**3**	**4**	$\mathbf{4}^*$	$\mathbf{3}^*$	$\mathbf{2}^*$	$\mathbf{1}^*$	$\mathbf{0}^*$
0	0	1	2	3	4	4^*	3^*	2^*	1^*	0^*
1	1	4	4	4	-	3^*	0^*	0^*	0^*	-
2	2	4	4	-	-	3^*	0^*	0^*	-	-
3	3	4	4	-	-	1^*	0^*	-	-	-
4	4	-	-	-	-	0^*	-	-	-	-
$\mathbf{4}^*$	4^*	3^*	2^*	2^*	0^*	-	-	-	-	-
$\mathbf{3}^*$	3^*	0^*	0^*	0^*	-	-	-	-	-	-
$\mathbf{2}^*$	2^*	0^*	0^*	-	-	-	-	-	-	-
$\mathbf{1}^*$	1^*	0^*	-	-	-	-	-	-	-	-
$\mathbf{0}^*$	0^*	-	-	-	-	-	-	-	-	-

$$3 + 4^* = 1^* = 4^* + ? = ? + 3$$

RDP_0 and RDP

Weak Riesz decomposition property - (RDP_0):

If for any $a, b_1, b_2 \in A$ such that $a \leq b_1 + b_2$, there are elements $a_1, a_2 \in A$ satisfying $a_1 \leq b_1$, $a_2 \leq b_2$ and $a = a_1 + a_2$.

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Riesz decomposition property - (RDP):

Let for any $a_1, a_2, b_1, b_2 \in A$ holding $a_1 + a_2 = b_1 + b2$, there are elements $c_{11}, c_{12}, c_{21}, c_{22} \in A$ such that the sums in rows and columns equal to respective elements:

$$\begin{array}{ccc} & b_1 & b_2 \\ a_1 & c_{11} & c_{12} \\ a_2 & c_{21} & c_{22} \end{array}$$

That is $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

(RDP) does not imply (RDP$_0$)

Only trivial decompositions for $a_1 + a_2 = b_1 + b_2$.

$+$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	.	.	.
2	2	.	3	.
3	3	.	.	.

$/ = \backslash$	**0**	**1**	**2**	**3**
0	0	.	.	.
1	1	0	.	.
2	2	.	0	.
3	3	0	2	0

$1 \leq 2 + 2 = 3$, but no elements $a_1, a_2 \leq 2$ such that $1 = a_1 + a_2$.

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Only trivial decompositions for $a_1 + a_2 = b_1 + b_2$.

$+$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	.	.	.
2	2	.	3	.
3	3	.	.	.

$/ = \backslash$	**0**	**1**	**2**	**3**
0	0	.	.	.
1	1	0	.	.
2	2	.	0	.
3	3	0	2	0

$1 \leq 2 + 2 = 3$, but no elements $a_1, a_2 \leq 2$ such that $1 = a_1 + a_2$.
Linear:

$+$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	3	.	.
2	2	.	.	.
3	3	.	.	.

$/ = \backslash$	**0**	**1**	**2**	**3**
0	0	.	.	.
1	1	0	.	.
2	2	0	0	.
3	3	1	0	0

$2 \leq 1 + 1 = 3$, but no elements $a_1, a_2 \leq 1$ such that $2 = a_1 + a_2$.

(RDP_0) does not imply (RDP)

On the other hand, there is also the example of RDP_0, which does not satisfy RDP:

$+$	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	3	4	.	.
2	2	4	4	.	.
3	3	.	.	.	.
4	4	.	.	.	.

$/ = \backslash$	**0**	**1**	**2**	**3**	**4**
0	0	.	.	.	.
1	1	0	.	.	.
2	2	0	0	.	.
3	3	1	.	0	.
4	4	2	2	0	0

There is no decomposition for $1 + 2 = 4 = 2 + 2$.

RDP_0 generalizations

- $a = b$, implies $a \,/\, b = 0 = a \setminus b$
- The converse is not true in general
- Replace the equality with difference.

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Left modified RDP_0 - $LmodRDP_0$:
for any $b \le b_1 + b_2$ there are $a_1 \le b_1$, $a_2 \le b_2$ such that
$(b \,/\, a_1) \,/\, a_2 = 0 = b \,/\, (a_1 + a_2)$

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Right modified RDP_0 - $\mathrm{RmodRDP}_0$:
for any $b \leq b_1 + b_2$ there are $a_1 \leq b_1$, $a_2 \leq b_2$ such that $(b \setminus a_2) \setminus a_1 = 0 = b \setminus (a_1 + a_2)$

RDP_0 generalizations

- $a = b$, implies $a / b = 0 = a \setminus b$
- The converse is not true in general
- Replace the equality with difference.

Left modified RDP_0 - $\mathrm{LmodRDP}_0$:
for any $b \le b_1 + b_2$ there are $a_1 \le b_1$, $a_2 \le b_2$ such that $(b / a_1) / a_2 = 0 = b / (a_1 + a_2)$

Right modified RDP_0 - $\mathrm{RmodRDP}_0$:
for any $b \le b_1 + b_2$ there are $a_1 \le b_1$, $a_2 \le b_2$ such that $(b \setminus a_2) \setminus a_1 = 0 = b \setminus (a_1 + a_2)$

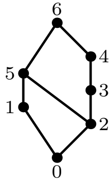
Left-Right modified RDP_0 - $\mathrm{LRmodRDP}_0$: equivalent to
Right-Left modified RDP_0 - $\mathrm{RLmodRDP}_0$:
for any $b \le b_1 + b_2$ there are $a_1 \le b_1$, $a_2 \le b_2$ such that $(b / a_1) \setminus a_2 = 0$ $[= (b \setminus a_2) / a_1]$

$\mathrm{RmodRDP}_0$ which is not $\mathrm{LmodRDP}_0$, $\mathrm{LRmodRDP}_0$

$+$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	1	2	3	4	5	6
1	1	.	.	.	.	.	.
2	2	5	4	.	.	6	.
3	3	6	4	.	.	6	.
4	4	6	.	.	.	.	.
5	5	.	.	.	.	.	.
6	6	.	.	.	.	.	.

$/$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	.	.	.	.	.	.
1	1	0	.	.	.	.	.
2	2	.	0	.	.	.	.
3	3	.	0	0	.	.	.
4	4	.	2	2	0	.	.
5	5	0	1	.	.	0	.
6	6	0	5	5	1	0	0

$\backslash$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	.	.	.	.	.	.
1	1	0	.	.	.	.	.
2	2	.	0	.	.	.	.
3	3	.	0	0	.	.	.
4	4	.	3	0	0	.	.
5	5	2	0	.	.	0	.
6	6	4	3	0	0	3	0



$$4 \leq 3 + 1$$

$\mathrm{LmodRDP}_0$ which is not $\mathrm{LmodRDP}_0$, $\mathrm{LRmodRDP}_0$

$+$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	1	2	3	4	5	6
1	1	.	5	6	6	.	.
2	2	.	4	4	.	.	.
3	3	.	.	.	.	.	.
4	4	.	.	.	.	.	.
5	5	.	6	6	.	.	.
6	6	.	.	.	.	.	.

$/$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	.	.	.	.	.	.
1	1	0	.	.	.	.	.
2	2	.	0	.	.	.	.
3	3	.	0	0	.	.	.
4	4	.	3	0	0	.	.
5	5	2	0	.	.	0	.
6	6	4	3	0	0	3	0

$\backslash$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	.	.	.	.	.	.
1	1	0	.	.	.	.	.
2	2	.	0	.	.	.	.
3	3	.	0	0	.	.	.
4	4	.	2	2	0	.	.
5	5	0	1	.	.	0	.
6	6	0	5	5	1	0	0

$4 \leq 1 + 3$

Generalizations of RDP

In the definition of RDP there are 5 equalities. Each equality can be modified in the similar way as in the case of RDP_0:

- unmodified,
- Lmod,
- Rmod,
- LRmod.

Thus there is $4^5 - 1 = 1023$ possibilities to modify the definition of RDP.

Congruences

Let $A = (A; +, /, \backslash, 0)$ be a generalized (weak) pre pseudo effect algebra and let $\sim$ be a relation of equivalence on A.

Weak congruence:

Let $a_1 \sim b_1$ and $a_2 \sim b_2$ and

- if $a_1 + a_2$ and $b_1 + b_2$ are defined, then $a_1 + a_2 \sim b_1 \sim b_2$;
- if $a_1 \geq a_2$, $b_1 \geq b_2$, then $a_1 \mathbin{/} a_2 \sim b_1 \mathbin{/} b_2$ and $a_1 \setminus a_2 \sim b_1 \setminus b_2$.

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- $[a] + [b] = \{m = a' + b' : a' \in [a], b' \in [b]\}$
- $[a] \,/\, [b] = \{m = a' \,/\, b' : a' \in [a], b' \in [b]\}$
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In general $[a]$ op $[b] \subseteq [t]$.

Congruences - contd.

Congruence:

for any op $\in \{+, /, \backslash\}$ if $[a]$ op $[b] \subseteq [t]$ is non-empty, then for any $t' \in [t]$ there are $a' \in [a]$, $b' \in [b]$ such that $t' = a'$ op b'.

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The problem: $[a]\ /\ [b] \neq \emptyset$ and $[b]\ /\ [a] \neq \emptyset$?implies? $[a] = [b]$.

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