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(Pre-)orders induced by uninorms

Our motivation for studying of uninorms

Problem

What is the behavior of non-representable uninorms, particularly, conjunctive uninorms in the rectangle $]0, e[\times]e, 1]$?

Why is it interesting? Part 1

This problem occurs, e.g., when constructing fuzzy implications of the form

$$I(x, y) = n_1(U(x, n_2(y))),$$

where n_1 and n_2 are suitably chosen negations, and studying especially the neutrality property ($I(x, 0) = x$) and the exchange principle ($I(x, I(y, z)) = I(y, I(x, z))$).

Why is it interesting? Part 2

The behaviour of uninorms in the rectangle $]0, e[\times]e, 1]$ is important also when studying the (pre-)order generated from uninorms

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Karaçal, F., Kesicioğlu, M.N.: *A t-partial order obtained from t-norms.* Kybernetika. **47** (2011) 300–314.

Definition

Let L be a bounded lattice, T be a t-norm on L. Then the order

$$x \preceq_T y \Leftrightarrow (\exists \ell \in L) T(\ell, y) = x$$

is called a t-order for the t-norm T.

$$x \preceq_T y \quad \Rightarrow \quad x \leq y$$

Yager and Rybalov, 1996

A uninorm U is a function $U : [0,1]^2 \to [0,1]$ that is increasing, commutative, associative and has a neutral element $e \in [0,1]$.
A uninorm U is said to be *conjunctive* if $U(x,0) = 0$, and U is said to be *disjunctive* if $U(1,x) = 1$, for all $x \in [0,1]$.

Yager and Rybalov, 1996

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Rewriting the definition of t-order we get:

Let U be a uninorm. We introduce the following relation

$$x \preceq_U y \Leftrightarrow (\exists \ell \in [0,1]) U(\ell, y) = x.$$

For an arbitrary uninorm U and arbitrary $(x, y) \in]0, e[\times]e, 1] \cup]e, 1] \times]0, e[$ we have

$$\min\{x, y\} \leq U(x, y) \leq \max\{x, y\}.$$

Let U be a uninorm with $e \in]0, 1[$. Then we denote

$$T_U = U \restriction [0, e]^2, \quad S_U = U \restriction [e, 1]^2.$$

$\mathcal{U}_{\{x,y\}}$ denotes those uninorms attaining at each point of $]0, e[\times]e, 1] \cup]e, 1] \times]0, e[$ the lower or the upper bound.

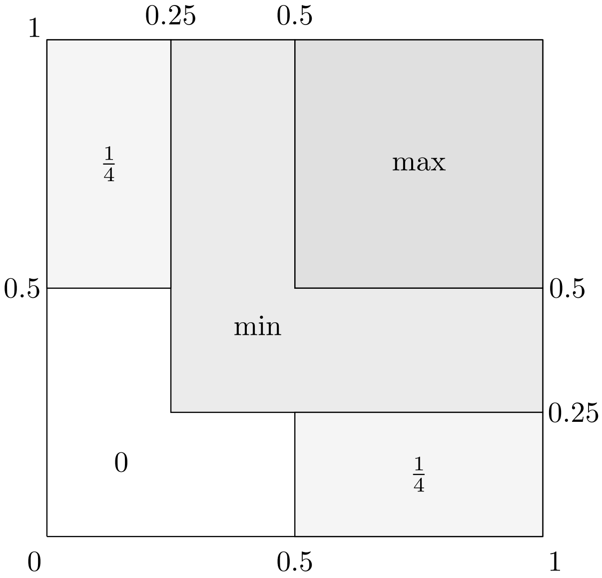
Lemma

Let $U \in \mathcal{U}_{\{x,y\}}$. Then:

1. $\preceq_U$ is an order,
2. $(\forall x, y \in [0, e])\,(x \preceq_U y \;\Leftrightarrow\; x \preceq_{T_U} y)$,
3. $(\forall x, y \in [e, 1])\,(x \preceq_U y \;\Leftrightarrow\; x \preceq_{S_U} y)$.

Using transformation of T_M in the square $[0, e]^2$ and S_M in the square $[e, 1]^2$

$$U_1(x,y) = \begin{cases} \max\{x,y\}, & \text{if } \min\{x,y\} \geq \frac{1}{2}, \\ \min\{x,y\}, & \text{if } \frac{1}{4} < \min\{x,y\} < \frac{1}{2}, \text{ or if } \max\{x,y\} = \frac{1}{2}, \\ \frac{1}{4}, & \text{if } 0 < \min\{x,y\} \leq \frac{1}{4} \text{ and } \max\{x,y\} > \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$


1
0.25
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max
0.5
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min
0.25
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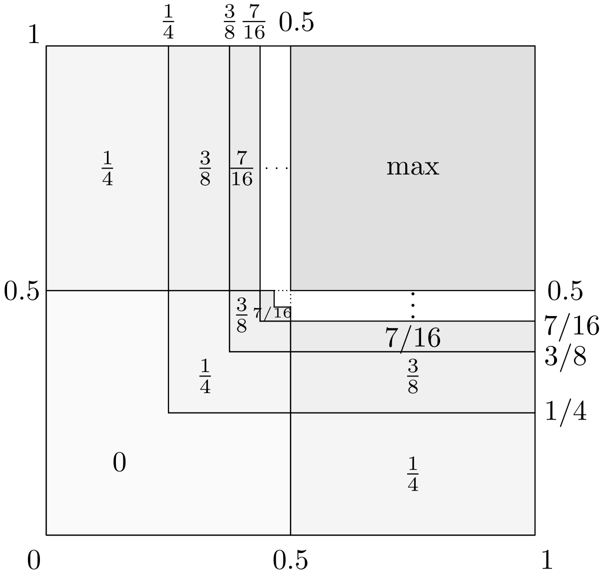
0.5
1

Using transformation of T_M in the square $[0, e]^2$ and S_M in the square $[e, 1]^2$

$$U_2(x,y) = \begin{cases} \max\{x,y\}, & \text{if } \min\{x,y\} \geq \frac{1}{2}, \\ \min\{x,y\}, & \text{if } \max\{x,y\} = e, \\ 0, & \text{if } \min\{x,y\} \leq \frac{1}{4} \text{ and } \max\{x,y\} < \frac{1}{2}, \\ & \text{or if } \min\{x,y\} = 0 \text{ and } \max\{x,y\} > \frac{1}{2}, \\ \frac{2^i-1}{2^{i+1}}, & \quad \text{for } i \in \{1,2,3,\dots\}, \\ & \text{if } \frac{2^{i-1}-1}{2^i} < \min\{x,y\} \leq \frac{2^i-1}{2^{i+1}} \text{ and } \max\{x,y\} > \frac{1}{2}, \\ & \text{or if } \frac{2^i-1}{2^{i+1}} < \min\{x,y\} \leq \frac{2^{i+1}-1}{2^{i+2}} \text{ and } \max\{x,y\} < \frac{1}{2}. \end{cases}$$



1
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0
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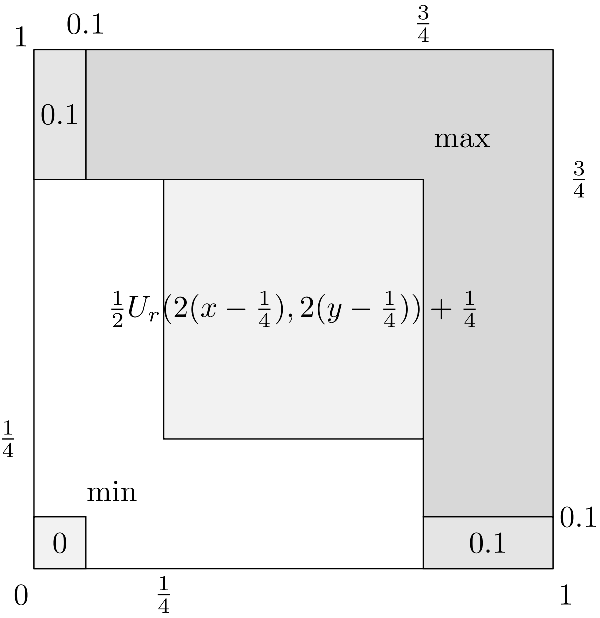
1

Let U_r be a representable uninorm.

$$U_3(x,y) = \begin{cases} \frac{1}{2}U_r(2(x-\frac{1}{4}), 2(y-\frac{1}{4})) + \frac{1}{4}, & \text{if } (x,y) \in [\frac{1}{4}, \frac{3}{4}]^2, \\ \max\{x,y\}, & \text{if } \max\{x,y\} > \frac{3}{4} \text{ and } \min\{x,y\} > 0.1, \\ 0, & \text{if } \max\{x,y\} \leq 0.1 \text{ or if } \min\{x,y\} = 0, \\ 0.1, & \text{if } \max\{x,y\} > \frac{3}{4} \text{ and } 0 < \min\{x,y\} \leq 0.1, \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

In the uninorm U_3 we could change S_M for an arbitrary t-conorm on the square $[\frac{3}{4}, 1]^2$ without changing other values.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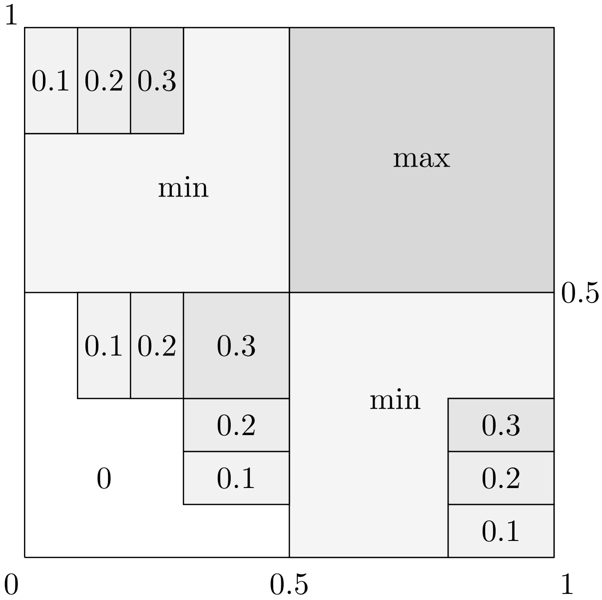
$\frac{1}{2}U_r(2(x-\frac{1}{4}),2(y-\frac{1}{4}))+\frac{1}{4}$
$\frac{1}{4}$
min
0.1
0
0.1
0
$\frac{1}{4}$
1

Using transformation of T_D in the square $[0,e]^2$, and S_M in the square $[e,1]^2$

$$U_4(x,y) = \begin{cases} \max\{x,y\}, & \text{if } \min\{x,y\} \geq 0.5, \\ \min\{x,y\}, & \text{if } 0.5 \leq \max\{x,y\} \leq 0.8 \text{ and } \min\{x,y\} < 0.5, \\ & \quad \text{or if } 0.3 < \min\{x,y\} < 0.5 \text{ and } \max\{x,y\} > 0.8, \\ 0.1, & \text{if } 0.1 \leq \min\{x,y\} < 0.2 \text{ and } 0.3 < \max\{x,y\} < 0.5, \\ & \quad \text{or if } 0 < \min\{x,y\} \leq 0.1 \text{ and } \max\{x,y\} > 0.8, \\ 0.2, & \text{if } 0.2 \leq \min\{x,y\} < 0.3 \text{ and } 0.3 < \max\{x,y\} < 0.5, \\ & \quad \text{or if } 0.1 < \min\{x,y\} \leq 0.2 \text{ and } \max\{x,y\} > 0.8, \\ 0.3, & \text{if } 0.3 \leq x < 0.5 \text{ and } 0.3 \leq y < 0.5, \\ & \quad \text{or if } 0.2 < \min\{x,y\} \leq 0.3 \text{ and } \max\{x,y\} > 0.8, \\ 0, & \quad \text{otherwise,} \end{cases}$$


1
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0.3
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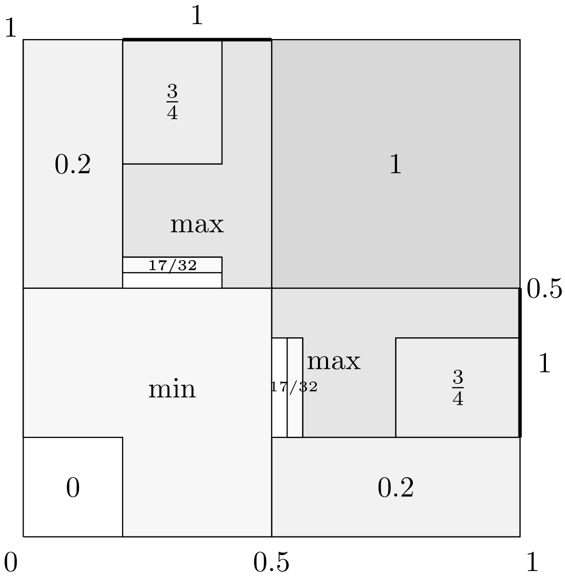
max
0.5
0.1
0.2
0.3
min
0.3
0.2
0.2
0
0.1
0.1
0
0.5
1

Using transformation of T_D in the square $[0, e]^2$, and S_D in the square $[e, 1]^2$

$$U_5(x, y) = \begin{cases} 1, & \text{if } \min\{x, y\} > 0.5, \text{ or if } \max\{x, y\} = 1 \text{ and } \min\{x, y\} > 0.2, \\ \max\{x, y\}, & \text{if } \max\{x, y\} > 0.5 \text{ and } 0.4 < \min\{x, y\} \le 0.5, \\ & \quad \text{or if } \frac{9}{16} \le \max\{x, y\} < \frac{3}{4} \text{ and } 0.2 < \min\{x, y\} \le 0.4, \\ \frac{3}{4}, & \text{if } \frac{3}{4} \le \max\{x, y\} < 1 \text{ and } 0.2 < \min\{x, y\} \le 0.4, \\ \frac{2^i+1}{2^{i+1}}, & \text{if } \frac{2^i+1}{2^{i+1}} \le \max\{x, y\} < \frac{2^{i-1}+1}{2^i} \text{ and } 0.2 < \min\{x, y\} \le 0.4, \\ & \quad \text{for } i \in \{4, 5, 6, \dots\}, \\ 0.2, & \text{if } \max\{x, y\} > 0.5 \text{ and } 0 < \min\{x, y\} \le 0.2, \\ \min\{x, y\}, & \text{if } 0.2 < \max\{x, y\} \le 0.5 \text{ and } 0 < \min\{x, y\} \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$


1
1
0.2
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1
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17/32
0.5
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1
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17/32
$\frac{3}{4}$
0
0.2
0
0.5
1

Proposition 1

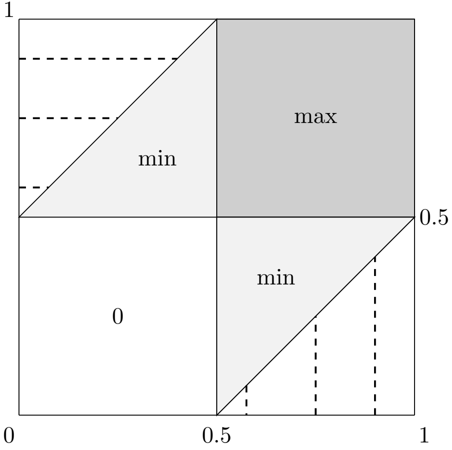
If U is a representable uninorm or if there exists a square $[a,b]^2$ with a representable uninorm, then the relation $\preceq_U$ is not an ordering.

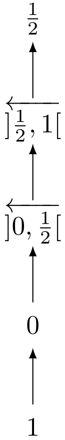
Proposition 2

Let $U \notin \mathcal{U}_{x,y}$. Then at least one of the following equivalences is violated:

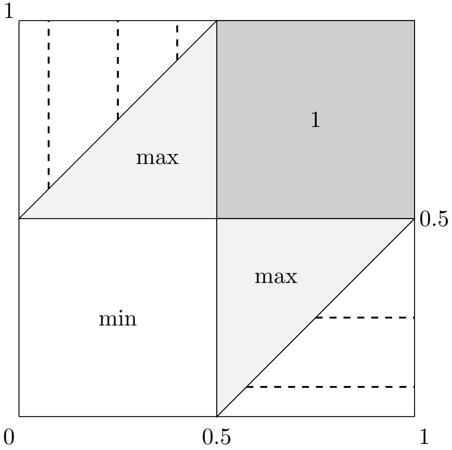
- $(\forall x, y \in [0,e])\,(x \preceq_U y \;\Leftrightarrow\; x \preceq_{T_U} y)$,
- $(\forall x, y \in [e,1])\,(x \preceq_U y \;\Leftrightarrow\; x \preceq_{S_U} y)$.

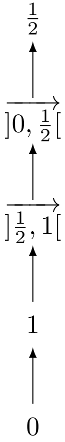
Disjunctive uninorm



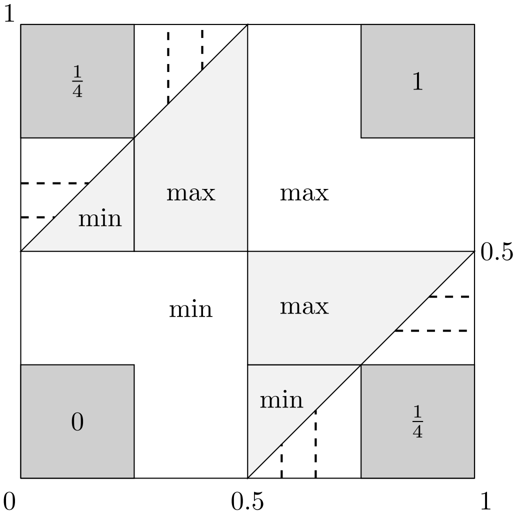


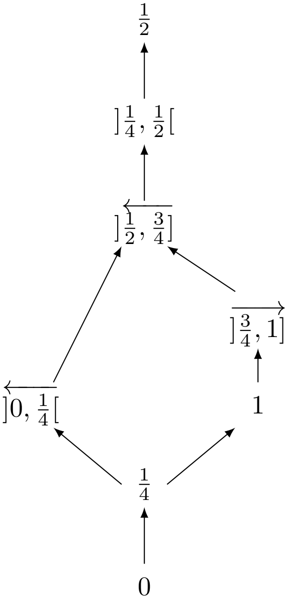
Conjunctive uninorm





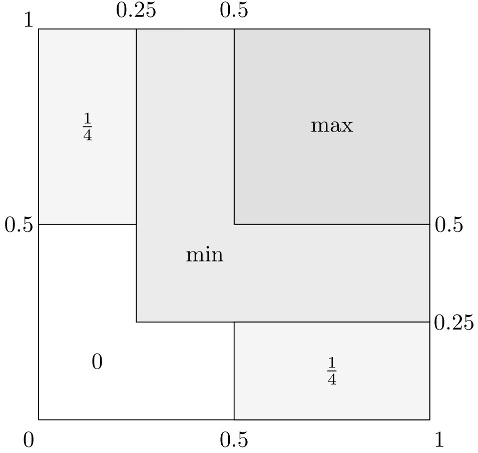
Conjunctive uninorm

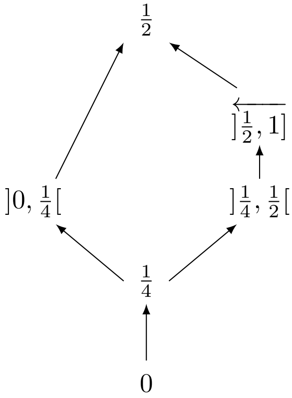




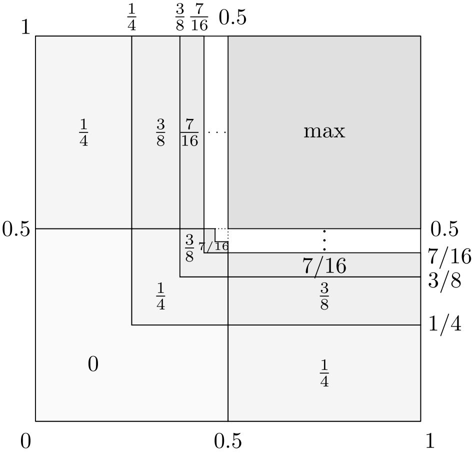
(Pre-)orderings generated by uninorms U_1–U_3

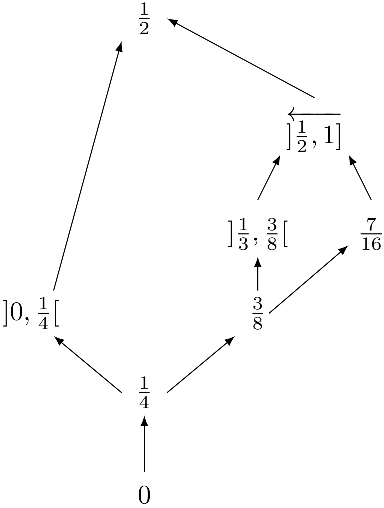
Conjunctive uninorm





Conjunctive uninorm





Conjunctive uninorm, U_r is conjunctive