On probabilistic-valued decomposable measures and integrals

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January 2014, Liptovský Ján

- Probabilistic (sub)measure
  - closely related to a numerical submeasure, i.e.

a mapping $\eta : \Sigma \to \overline{\mathbb{R}}_+$, where Σ be a ring of subsets of $\Omega \neq \emptyset$ such that
$\eta(\emptyset) = 0$,
$\eta(E) \leq \eta(F)$ for $E, F \in \Sigma$, $E \subset F$, (monotonicity)
$\eta(E \cup F) \leq \eta(E) + \eta(F)$ for $E, F \in \Sigma$. (subadditivity)

  - nonadditivity is useful in practical situations (decision making,...)
  - situations when we have only probabilistic information about measure of a set (e.g. lottery, a horse race,...)
  - closely related to a Probabilistic metric space
- Probabilistic integral

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Probabilistic metric space (K. Menger, 1942)

Problem: How to describe spaces, where we do not know exactly the distance between 2 points?

- idea: Fréchet metric $d(p, q) \Rightarrow$ distribution function $F_{p,q}(x)$

Definition [Šerstnev, 1962]

Let Ω be a non-empty set, $\mathcal{F} : \Omega \times \Omega \to \Delta^+$ and $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ a triangle function. If the following properties hold for all $p, q, r \in \Omega$

(i) $F_{p,q} = \varepsilon_0$ if and only if $p = q$;

(ii) $F_{p,q} = F_{q,p}$;

(iii) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$,

then the triple $(\Omega, \mathcal{F}, \tau)$ is called a ***probabilistic metric space***.

- Menger PM-space: $\tau_T(F_{p,q}, F_{q,r})(z) = \sup_{x+y=z} T(F_{p,q}(x), F_{q,r}(y))$

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Definition [Hutník, Mesiar, 2009]

Let $T : [0,1]^2 \to [0,1]$ be a t-norm, and Σ a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ (where $\gamma(E)$ is denoted by γ_E) such that

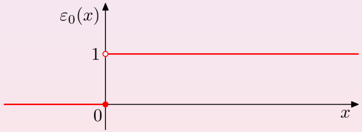
(a) if $E = \emptyset$, then $\gamma_\emptyset(x) = \varepsilon_0(x)$, $x > 0$;

(b) if $E \subset F$, then $\gamma_E(x) \geq \gamma_F(x)$, $x > 0$; (antimonotonicity)

(c) $\gamma_{E \cup F}(x + y) \geq T(\gamma_E(x), \gamma_F(y))$, $x, y > 0$, $E, F \in \Sigma$, (subadditivity)

is said to be a τ_T ***-submeasure***.

- where $\varepsilon_0 : \overline{\mathbb{R}} \to [0,1]$ is ”unit step”



- ”probabilistic version” of triangle inequality

$$F_{p,r}(x + y) \geq T(F_{p,q}(x), F_{q,r}(y))$$

Examples

- universal τ_T-submeasure corresponds to a distribution function of exponential distribution $E(\lambda)$ with parameter λ

$$\gamma_E(x) = 1 - \exp\left(-\left(\frac{cx}{\lambda\eta(E)}\right)\right), \ x > 0.$$

- other classes of τ_T-submeasures:

Family of t-norms	**Corresponding family of τ_T-submeasures**
Schweizer-Sklar t-norms T^{SS}_λ, $\lambda \in]-\infty, +\infty[$	$\gamma_E^{SS,\lambda}(x) = \min\left\{\sqrt[\lambda]{1+\lambda(x-\eta(E))},\ 1\right\}, \quad x > \max\left\{\eta(E) - \frac{1}{\lambda}, 0\right\}$ $\gamma_E^{SS,0}(x) = \min\{\exp(x-\eta(E)), 1\}, \quad x > 0$
Dombi t-norms T^D_λ, $\lambda \in]0, +\infty[$	$\gamma_E^{D,\lambda}(x) = \left(1 + \left[\max\{\eta(E) - x, 0\}\right]^{1/\lambda}\right)^{-1}$

⋮

Generalization of τ_T *-submeasure*

Let $T : [0,1]^2 \to [0,1]$ be a t-norm, and Σ a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ (where $\gamma(E)$ is denoted by γ_E) such that

(a) if $E = \emptyset$, then $\gamma_\emptyset(x) = \varepsilon_0(x)$, $x > 0$;

(b) if $E \subset F$, then $\gamma_E(x) \geq \gamma_F(x)$, $x > 0$; (antimonotonicity)

(c) $\gamma_{E \cup F}(L(x,y)) \geq T(\gamma_E(x), \gamma_F(y))$, $x, y > 0$, $E, F \in \Sigma$, (subadditivity)

is said to be a $\tau_{L,T}$ *-submeasure*.

- L is binary operation on $\overline{\mathbb{R}}_+ = [0,\infty]$ such that
  - (a) L is commutative and associative;
  - (b) L is jointly strictly increasing, i.e., for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}_+$ with $u_1 < u_2$, $v_1 < v_2$ holds $L(u_1, v_1) < L(u_2, v_2)$;
  - (c) L is continuous on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$;
  - (d) L has 0 as its neutral element.
- ”probabilistic version” of triangle inequality

$$F_{p,r}(L(x,y)) \geq T(F_{p,q}(x), F_{q,r}(y))$$

- *Shen, Y.: On the probabilistic Hausdorff distance and a class of probabilistic decomposable measures Inform. Sci. (2013), in press*
- Shen studied the class of probabilistic (sub)measures:

Definition

Let $\top$ be a t-norm. A mapping $\mathfrak{M} : \Sigma \to \Delta^+$ with

(a) if $E = \emptyset$, then $\mathfrak{M}_\emptyset = \varepsilon_0$;

(b) $\mathfrak{M}_{E \cup F}(t) \geq \top(\mathfrak{M}_E(t), \mathfrak{M}_F(t)), \quad E, F \in \Sigma, t > 0,$ (subadditivity)

is called a probabilistic-valued $\top$-decomposable supmeasure.

- corresponds to the notion of $\tau_{\max, T}$-submeasure

What do these concepts have in common?

- τ_T *-submeasure*

$$\tau(G,H)(z) = \sup_{x+y=z} T(G(x),H(y)), \tag{1}$$

- $\tau_{L,T}$ *-submeasure*

$$\tau(G,H)(z) = \sup_{L(x,y)=z} T(G(x),H(y)), \tag{2}$$

- $\top$*-decomposable supmeasure*

$$\tau(G,H)(z) = T(G(z),H(z)). \tag{3}$$

Definition

Let τ be a triangle function on Δ^+ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ with

(a) $\gamma_\emptyset = \varepsilon_0$;

(b) $\gamma_{E \cup F} \geq \tau(\gamma_E, \gamma_F)$ for each disjoint sets $E, F \in \Sigma$,

is said to be a τ-*decomposable submeasure*.

A triangle function τ is a natural choice for "aggregation" of γ_E and γ_F:

- we expect $\gamma_{E \cup F} = \gamma_{F \cup E}$ for disjoint sets $E, F \in \Sigma$, from which follows that

$$\tau(\gamma_E, \gamma_F) = \tau(\gamma_F, \gamma_E),$$

- from $\gamma_{(E \cup F) \cup G} = \gamma_{E \cup (F \cup G)}$ we obtain $\tau(\tau(\gamma_E, \gamma_F), \gamma_G) = \tau(\gamma_E, \tau(\gamma_F, \gamma_G))$,
- since $\gamma_E = \gamma_{E \cup \emptyset} = \tau(\gamma_E, \gamma_\emptyset) = \tau(\gamma_E, \varepsilon_0)$, then ε_0 has to be neutral element of τ,
- $\gamma_E \geq \gamma_F$ whenever $E, F \in \Sigma$ such that $E \subseteq F$ follows from monotonicity of τ.

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Some properties of τ-decomposable measure

Theorem (characterization of τ-decomposable measures)

Let τ be a triangle function on Δ^+. Then γ is a τ-decomposable measure on Σ if and only if

$$\tau(\gamma_{E\cup F}, \gamma_{E\cap F}) = \tau(\gamma_E, \gamma_F), \quad \text{for each } \; E, F \in \Sigma.$$

Theorem (Construction of new decomposable (sub)measures)

Let τ, ϑ be two triangle functions on Δ^+ and $\gamma^1, \gamma^2 : \Sigma \to \Delta^+$ be τ-decomposable measures. Then

(i) if τ is distributive, the set function $\gamma := c \odot \gamma^1$ is a τ-decomposable measure for each $c \in \mathbb{R}_+$;

(ii) the set function $\zeta := \tau(\gamma^1, \gamma^2)$ is a τ-decomposable measure;

(iii) the set function $\lambda := \vartheta(\gamma^1, \gamma^2)$ is a τ-decomposable submeasure if and only if $\vartheta \gg \tau$.

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Multiplication on Δ^+ is defined

$$(c \odot F)(x) := \begin{cases} \varepsilon_0 & c = 0, \\ F\left(\frac{x}{c}\right), & \text{otherwise}, \end{cases} \qquad \text{for } F \in \Delta^+, c \in [0, \infty[.$$

Probabilistic integral with respect to a universal measure (Previous research)

$$\int_E f \, \mathrm{d}\vartheta,$$

where

- $E \in \Sigma$, where Σ is a ring of subsets of a non-empty set Ω;
- $f \in \Omega^{\mathbb{R}_+}$ is measurable with respect to Σ;
- $\vartheta : \Sigma \to \Delta^+$ is a ***universal probabilistic measure*** satisfying
  - (a) $\vartheta_\emptyset = \varepsilon_0$
  - (b) $\vartheta_{E \cup F}(x + y) = \mathbf{M}(\vartheta_E(x), \vartheta_F(y))$ for $x, y > 0$, $E, F \in \Sigma$, $E \cap F = \emptyset$.

Probabilistic integral with respect to a τ-decomposable measure

$$\int_E f \,\mathrm{d}\gamma,$$

where

- $E \in \Sigma$, where Σ is a ring of subsets of a non-empty set Ω;
- $f \in \Omega^{\mathbb{R}_+}$ is measurable with respect to Σ;
- $\gamma : \Sigma \to \Delta^+$ is a τ-decomposable measure, where τ is a distributive triangle function, i.e. for each $c \in \mathbb{R}_+$, $G, H \in \Delta^+$ holds

$$c \odot (G \oplus_\tau H) = (c \odot G) \oplus_\tau (c \odot H).$$

Operations on Δ^+:

- addition of d.d.f. may be defined $(G \oplus_\tau H)(x) := \tau(G, H)(x)$.
- multiplication of d.d.f., for $F \in \Delta^+$, $c \in [0, \infty[$

$$(c \odot F)(x) := \begin{cases} \varepsilon_0 & c = 0, \\ F\left(\frac{x}{c}\right), & \text{otherwise.} \end{cases}$$

Probabilistic integral with respect to a τ-decomposable measure

$$\int_E f \,\mathrm{d}\gamma,$$

where

- $E \in \Sigma$, where Σ is a ring of subsets of a non-empty set Ω;
- $f \in \Omega^{\mathbb{R}_+}$ is measurable with respect to Σ;
- $\gamma : \Sigma \to \Delta^+$ is a τ-decomposable measure, where τ is a distributive triangle function, i.e. for each $c \in \mathbb{R}_+$, $G, H \in \Delta^+$ holds

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$$(c \odot F)(x) := \begin{cases} \varepsilon_0 & c = 0, \\ F\left(\frac{x}{c}\right), & \text{otherwise.} \end{cases}$$

(A) If f is a characteristic function $f(x) = \begin{cases} 1, & x \in C, \\ 0, & x \notin C, \end{cases}$ where $C \in \Sigma$, their γ-integral we define as follows

$$\int_E f(x)\,\mathrm{d}\gamma := \gamma_{E \cap C}(x).$$

(B) For a simple non-negative measurable function $f \in \Omega^{\mathbb{R}_+}$

$$f(x) = \sum_{i=1}^{n} x_i \chi_{E_i}(x)$$

we put

$$\int_E f(x)\,\mathrm{d}\gamma := \bigoplus_{i=1}^{n} x_i \odot \gamma_{E \cap E_i}(x).$$

(C) For a non-negative measurable function $f \in \Omega^{\mathbb{R}_+}$?

Definition

Let $\gamma : \Sigma \to \Delta^+$ be a τ-decomposable measure, τ is a distributive triangle function. We say that measurable function $f \in \Omega^{\mathbb{R}_+}$ is *γ-integrable* on the set $E \in \Sigma$, if there exists a distribution function $H \in \Delta^+$ such that $\int_E g \, \mathrm{d}\gamma \geq H$ for all $g \in \mathcal{S}_f$. In this case we put

$$\int_E f \, \mathrm{d}\gamma := \inf \left\{ \int_E g \, \mathrm{d}\gamma ; \; g \in \mathcal{S}_f \right\}$$

and is said to be γ*-integral function f on $E \in \Sigma$*.

- $\mathcal{S}_f$- is the set of all SNMF such that $g \leq f$ $\quad \left(\int_E g \, \mathrm{d}\gamma \geq \int_E f \, \mathrm{d}\gamma \right)$
- it is sufficient to consider monotonic SNMF i.e. $(f_n)_1^\infty$, where $f_n \in \mathcal{S}_f$, such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $f_n \to f$ (pointwise)

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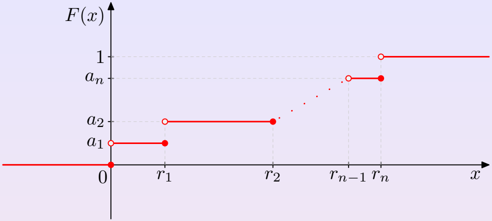

Fig. Integral of a constant function with respect to a γ^{a_i} measures.

- triangle function τ_M: $\quad \tau(G,H)(z) = \sup\limits_{x+y=z} M(G(x),H(y))$;
- $f(x) = x_0\chi_E(x)$;
- $\gamma^{a_i} \in \Delta^+$, $a_1 \le a_2 \le \cdots \le a_n$, $a_i \in [0,1]$ are particular τ_M(universal)-decomposable measures

$$\bigoplus_{i=1}^{n} \int_{E_i} f \,\mathrm{d}\gamma^{a_i} = a_1\chi_{]0,r_1]} + a_2\chi_{]r_1,r_2]} + \ldots a_n\chi_{]r_{n-1},r_n]} + \chi_{]r_n,\infty]}.$$

Thank you for your attention!